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Topo-distance: Measuring the Difference between Spatial Patterns

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Abstract. A framework to deal with spatial patterns at the qualitative level of mereotopology is proposed. The main contribution is to provide formal tools for issues of model equivalence and model similarity. The framework uses a multi-modal language $S4_u$ interpreted on topological spaces (rather than Kripke semantics) to describe the spatial patterns. Model theoretic notions such as topological bisimulations and topological model comparison games are introduced to define a distance on the space of all topological models for the language $S4_u$. In the process, a new take on mereotopology is given, prompting for a comparison with prominent systems, such as RCC.

Keywords: qualitative spatial reasoning, RCC, mereotopology, model comparison games

1 Introduction

There are various ways to take space qualitatively. Topology, orientation or distance have been investigated in a non-quantitative manner. The literature especially is abundant in mereotopological theories, i.e. theories of parthood `P` and connection `C`. Even though the two primitives can be axiomatized independently, the definition of part in terms of connection suffices for AI applications. Usually, some fragment of topology is axiomatized and set inclusion is used to interpret parthood (see the first four chapters of [9] for a complete overview).

Most of the efforts in mereotopology have gone into the axiomatization of the specific theories, disregarding important model theoretic questions. Issues such as model equivalence are seldom (if ever) addressed. Seeing an old friend from high-school yields an immediate comparison with the image one had from the school days. Most often, one immediately notices how many aesthetic features have changed. Recognizing a place as one already visited involves comparing the present sensory input against memories of the past sensory inputs. "Are these trees the same as I saw six hours ago, or are they arranged differently?" An image retrieval system seldom yields an exact match, more often it yields a series of 'close' matches. In computer vision, object occlusion cannot be disregarded. One 'sees' a number of features of an object and compares them with other sets of features to perform object recognition. Vision is not a matter of precise matching, it is more closely related to similarity. The core of the problem lies in the precise definition of 'close' match, thus the question shall be: *How similar are two spatial patterns?*

In this paper, a general framework for mereotopology is presented, providing a language that subsumes many of the previously proposed ones, and then model theoretic questions are addressed. Not only a notion of model equivalence is provided, but also a precise definition of distance between models.

2 A general framework for Mereotopology

2.1 The language $\mathbf{S4}_u$

The proposed framework takes the beaten road of mereotopology by extending topology with a mereological theory based on the interpretation of set inclusion as parthood. Hence, a brief recall here of the basic topological definitions is in order.

A *topological space* is a couple $\langle X, O \rangle$, where X is a set and $O \subseteq \mathcal{P}(X)$ such that: $\emptyset \in O$, $X \in O$, O is closed under arbitrary union, O is closed under finite intersection. An element of O is called an *open*. A subset A of X is called *closed* if $X - A$ is open. The *interior* of a set $A \subseteq X$ is the union of all open sets contained in A. The *closure* of a set $A \subseteq X$ is the intersection of all closed sets containing A.

To capture a considerable fragment of topological notions a multi-modal language $\mathrm{S4}_u$ interpreted on topological spaces (à la Tarski [17]) is used. A *topological model* $M = \langle X, O, \nu \rangle$ is a topological space $\langle X, O \rangle$ equipped with a valuation function $\nu : P \to \mathcal{P}(X)$, where P is the set of proposition letters of the language.

The definition and interpretation of $\mathrm{S4}_u$ follows that given in [2]. In that paper though, emphasis is given to the topological expressivity of the language rather than the mereotopological implications. Every formula of $\mathrm{S4}_u$ represents a region. Two modalities are available. $\Box\varphi$ to be interpreted as "interior of the region φ", and $U\varphi$ to be interpreted as "it is the case everywhere that φ." The truth definition can now be given. Consider a topological model $M = \langle X, O, \nu \rangle$ and a point $x \in X$:

$$
\begin{array}{lll}
M, x \models p & \text{iff} & x \in \nu(p) (\text{with } p \in P) \\
M, x \models \neg\varphi & \text{iff} & \text{not } M, x \models \varphi \\
M, x \models \varphi \to \psi & \text{iff} & \text{not } M, x \models \varphi \text{ or } M, x \models \psi \\
M, x \models \Box\varphi & \text{iff} & \exists o \in O :\ x \in o \wedge \\
 & & \forall y \in o :\ M, y \models \varphi \\
M, x \models U\varphi & \text{iff} & \forall y \in X :\ M, y \models \varphi
\end{array}
$$

Since $\Box$ is interpreted as interior and $\Diamond$ (defined dually as $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$, for all φ) as closure, it is not a surprise that these modalities obey the following axioms[1], [17]:

$$\Box A \to A \tag{T}$$
$$\Box A \to \Box\Box A \tag{4}$$
$$\Box \top \tag{N}$$
$$\Box A \wedge \Box B \leftrightarrow \Box(A \wedge B) \tag{R}$$

(4) is idempotence, while (N) and (R) are immediately identifiable in the definition of topological space. For the universal—existential modalities U and E (defined dually: $E\varphi \leftrightarrow \neg U \neg \varphi$) the axioms are those of S5:

$$U(\varphi \to \psi) \to (U\varphi \to U\psi) \tag{K}$$
$$U\varphi \to \varphi \tag{T}$$
$$U\varphi \to UU\varphi \tag{4}$$
$$\varphi \to UE\varphi \tag{B}$$

In addition, the following 'connecting' principle is part of the axioms:

$$\Diamond\varphi \to E\varphi$$

The language $S4_u$ is thus a multi-modal S4*S5 logic interpreted on topological spaces. Extending S4 with universal and existential operators to get rid of its intrinsic 'locality' is a known technique used in modal logic, [12]. In the spatial context, similar settings have been used initially in [7] to encode decidable fragments of the region connection calculus RCC (the fundamental and most widely used qualitative spatial reasoning calculi in the field of AI, [14]), then by [15] to identify maximal tractable fragments of RCC and, recently, by [16]. Even though the logical technique is similar to that of [7,15], there are two important differences. First, in the proposed use of $S4_u$ there is no commitment to a specific definition of connection (as RCC does by forcing the intersection of two regions to be non-empty). Second, the stress is on model equivalence and model comparison issues, not only spatial representation. On the other hand, there is no treatment here of consistency checking problems, leaving them for future investigation.

2.2 Expressivity

The language $S4_u$ is perfectly suited to express mereotopological concepts. Parthood $\mathtt{P}$: a region A is part of another region B if it is the case everywhere that A implies B:

$$\mathtt{P(A,B)} := U(A \to B)$$

[1] The axiomatization of $\Box$ given is known as S4. Usually thought S4's axiomatization is given replacing axioms (N) and (R) by (K), see [7].

This captures exactly the set-inclusion relation of the models. As for connection $\mathtt{C}$, two regions A and B are connected if there exists a point where both A and B are true:

$$\mathtt{C(A,B)} := E(A \wedge B)$$

From here it is immediate to define all the usual mereotopological predicates such as proper part, tangential part, overlap, external connection, and so on. Notice that the choice made in defining $\mathtt{P}$ and $\mathtt{C}$ is arbitrary. So, why not take a more restrictive definition of parthood? Say, A is part of B whenever the closure of A is contained in the interior of B?

$$\mathtt{P(A,B)} := U(\Diamond A \to \Box B)$$

As this formula shows, $\mathrm{S4}_u$ is expressive enough to capture also this definition of parthood. In [10], the logical space of mereotopological theories is systematized. Based on the intended interpretation of the connection predicate $\mathtt{C}$, and the consequent interpretation of $\mathtt{P}$ (and fusion operation), a type is assigned to mereotopological theories. More precisely, a *type* is a triple $\tau = \langle i, j, k\rangle$, where the first i refers to the adopted definition of $\mathtt{C_i}$, j to that of $\mathtt{P_j}$ and k to the sort of fusion. The index i, referring to the connection predicate $\mathtt{C}$, accounts for the different definition of connection at the topological level. Using $\mathrm{S4}_u$ one can repeat here the three types of connection:

$$\begin{aligned}
\mathtt{C_1(A,B)} &:= E(A \wedge B)\\
\mathtt{C_2(A,B)} &:= E(A \wedge \Diamond B) \vee E(\Diamond A \wedge B)\\
\mathtt{C_3(A,B)} &:= E(\Diamond A \wedge \Diamond B)
\end{aligned}$$

Looking at previous mereotopological literature, one remarks that RCC uses a $\mathtt{C_3}$ definition, while the system proposed in [4] uses a $\mathtt{C_1}$. Similarly to connectedness, one can distinguish the various types of parthood, again in terms of $\mathrm{S4}_u$:

$$\begin{aligned}
\mathtt{P_1(A,B)} &:= U(A \to B)\\
\mathtt{P_2(A,B)} &:= U(A \to \Diamond B)\\
\mathtt{P_3(A,B)} &:= U(\Diamond A \to \Diamond B)
\end{aligned}$$

In [10], the definitions of the $\mathtt{C_i}$ are given directly in terms of topology, and the definitions of $\mathtt{P_j}$ in terms of a first order language with the addition of a predicate $\mathtt{C_i}$. Finally, a general fusion $\phi_{\mathtt{k}}$ is defined in terms of a first order language with a $\mathtt{C_i}$ predicate. Fusion operations are like algebraic operations on regions, such as adding two regions (product), or subtracting two regions. One cannot repeat the general definition given in [10] at the $\mathrm{S4}_u$ level. Though, one can show that various instances of fusion operations are expressible in $\mathrm{S4}_u$. For example, the product $A \times_k B$:

$$\begin{aligned}
\mathtt{A} \times_1 \mathtt{B} &:= A \wedge B\\
\mathtt{A} \times_2 \mathtt{B} &:= (\Diamond A \wedge B) \vee (A \wedge \Diamond B)\\
\mathtt{A} \times_3 \mathtt{B} &:= (\Diamond A \wedge \Diamond B)
\end{aligned}$$

The above discussion has shown that $S4_u$ is a general language for mereotopology. All the different types $\tau = \langle i, j, k \rangle$ of mereotopological theories are expressible within $S4_u$.

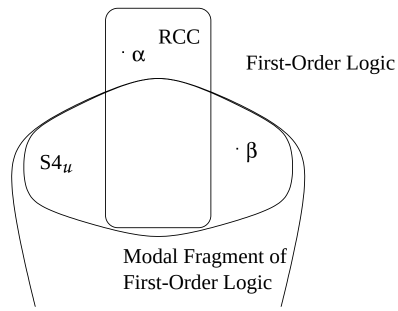

Fig. 1. The positioning of $S4_u$ and RCC with respect to well-known logics.

Before diving into the similarity results of this paper a remark is in order. The language $S4_u$ is a multi-modal language with nice computational properties. It is complete with respect to topological models, it is decidable, it has the finite model property (see [3] for the proofs of these facts). It captures a large and "well-behaved" fragment of mereotopology, though it is not a first-order language. In other words, it is not possible to quantify over regions. A comparison with the best-known RCC is in order.

Comparison with RCC RCC is a first order language with a distinguished connection predicate $\mathtt{C}_3$. The driving idea behind this qualitative theory of space is that regions of space are primitive objects and connection is the basic predicate. This reflects in the main difference between RCC and the proposed system, which instead builds on traditional point-based topology.

RCC and $S4_u$ *capture different portions of mereotopology.*

To show this, two formulas are given: an RCC formula which is not expressible in $S4_u$ and, vice-versa, one expressible in $S4_u$, but not in RCC. The situation is depicted in Figure **??**. In RCC, one can write:

$$\forall A \exists B : \mathtt{P}(A, B) \qquad (\alpha)$$

meaning that every region is part of another one (think of the entire space). On the other hand, one can write a $S4_u$ formula such as:

$$\neg E(p \wedge \Diamond \Box \neg p) \qquad (\beta)$$

which expresses the regularity of the region p. It is easy to see that α is not expressible in $S4_u$ and that β is not in RCC.

This fact may though be misleading. It is not the motivations, nor the core philosophical intuitions that draw the line between RCC and $S4_u$. Rather, it is the logical apparatus which makes the difference. To boost the similarities, next it is shown how the main predicates of RCC can be expressed within $S4_u$. Consider the case of RCC8:

RCC8	$S4_u$	Interpretation
$\mathtt{DC}(A, B)$	$\neg E(A \wedge B)$	A is DisConnected from B
$\mathtt{EC}(A, B)$	$E(\Diamond A \wedge \Diamond B) \wedge$ $\neg E(\Box A \wedge \Box B)$	A and B are Externally Connected
$\mathtt{PO}(A, B)$	$E(A \wedge B) \wedge E(A \wedge \neg B) \wedge$ $E(\neg A \wedge B)$	A and B Properly Overlap
$\mathtt{TPP}(A, B)$	$U(A \rightarrow B) \wedge$ $E(\Diamond A \wedge \Diamond B \wedge \Diamond \neg A \wedge \Diamond \neg B)$	A is a Tangential Proper Part of B
$\mathtt{NTPP}(A, B)$	$U(\Diamond A \rightarrow \Box B)$	A is a Non Tangential Proper Part of B
$\mathtt{TPPi}(A, B)$	$U(B \rightarrow A) \wedge$ $E(\Diamond B \wedge \Diamond A \wedge \Diamond \neg B \wedge \Diamond \neg A)$	The inverse of the `TTP` predicate
$\mathtt{NTPPi}(A, B)$	$U(\Diamond B \rightarrow \Box A)$	The inverse of the `NTTP` predicate
$\mathtt{EQ}(A, B)$	$U(A \leftrightarrow B)$	A and B are EQual

Indeed one can define the same predicates as RCC8, but as remarked before the nature of the approach is quite different. Take for instance the non tangential part predicate. In RCC it is defined by means of the non existence of a third entity C:

$$\mathtt{NTTP}(A, B) \text{ iff } \mathtt{P}(A, B) \wedge \neg \mathtt{P}(B, A) \wedge \neg \exists C[\mathtt{EC}(C, A) \wedge \mathtt{EC}(C, B)]$$

On the other hand, in $S4_u$ it is simply a matter of topological operations. As in the previous table, for $\mathtt{NTTP}(A, B)$ it is sufficient to take the interior of the containing region $\Box B$, the closure of the contained region $\Diamond A$ and check if all points that satisfy the latter $\Diamond A$ also satisfy the former $\Box B$.

The RCC and $S4_u$ are even more similar if one takes the perspective of looking at RCC's modal decidable encoding of Bennett, [7]. Bennett's approach is to start from Tarski's original interpretation of modal logic in terms of topological spaces (Tarski proves S4 to be the complete logic of all topological spaces) and then to increase the expressive power of the language by means of a universal modality. The positive side effect is that the languages obtained in this manner usually maintain nice computational properties. The road to $S4_u$ has followed the same path and was inspired by Bennett's original work.

Here is the most important difference of the two approaches: the motivation for the work of Bennett comes from RCC, the one for the proposed framework from topology. $S4_u$ keeps a general topological view on spatial reasoning, it gives means to express more of the topological intricacy of the regions in comparison with RCC. For example regularity is not enforced by axioms (like in RCC), but it is expressible directly by a $S4_u$ formula (β). More on the 'topological expressive power' of S4 and its universal extension can be found in [2].

3 When are two spatial patterns the same?

One is now ready to address questions such as: *When are two spatial patterns the same?* or *When is a pattern a sub-pattern of another one?* More formally, one wants to define a notion of equivalence adequate for $S4_u$ and the topological models. In first-order logic the notion of 'partial isomorphism' is the building block of model equivalence. Since $S4_u$ is multi-modal language, one resorts to bisimulation, which is the modal analogue of partial isomorphism. Bisimulations compare models in a structured sense, 'just enough' to ensure the truth of the same modal formulas [8,13].

Definition 1 (Topological bisimulation). Given two topological models $\langle X, O, \nu \rangle$, $\langle X', O', \nu' \rangle$, a *total topological bisimulation* is a non-empty relation $\leftrightharpoons \subseteq X \times X'$ defined for all $x \in X$ and for all $x' \in X'$ such that if $x \leftrightharpoons x'$:

(base): $x \in \nu(p)$ iff $x' \in \nu'(p)$ (for any proposition letter p)

(forth condition): if $x \in o \in O$ then
$\exists o' \in O' : x' \in o'$ and $\forall y' \in o' : \exists y \in o : y \leftrightharpoons y'$

(back condition): if $x' \in o' \in O'$ then
$\exists o \in O : x \in o$ and $\forall y \in o : \exists y' \in o' : y \leftrightharpoons y'$

If only conditions (i) and (ii) hold, the second model *simulates* the first one.

The notion of bisimulation is used to answer questions of 'sameness' of models, while simulation will serve the purpose of identifying sub-patterns. Though, one must show that the above definition is adequate with respect to the mereotopological framework provided in this paper.

Theorem 1. *Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two models, $x \in X$, and $x' \in X'$ bisimilar points. Then, for any modal formula φ in $S4_u$, $M, x \models \varphi$ iff $M', x' \models \varphi$.*

Theorem 2. *Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two models with finite O, O', $x \in X$, and $x' \in X'$ such that for every φ in $S4_u$, $M, x \models \varphi$ iff $M', x' \models \varphi$. Then there exists a total bisimulation between M and M' connecting x and x'.*

In words, extended modal formulas are invariant under total bisimulations, while finite modally equivalent models are totally bisimilar. The proofs are straightforward extensions of those of Theorem 1 and Theorem 2 in [2], respectively. In the case of Theorem 1, the inductive step must be extended also to consider the universal and existential modalities; while for Theorem 2, one needs to add an universal quantification over all points of the two equivalent models. One may notice, that in Theorem 2 a finiteness restriction is posed on the open sets. This will not surprise the modal logician, since the same kind of restriction holds for Kripke semantics and does not affect the proposed use for bisimulations in the mereotopological framework.

4 How different are two spatial patterns?

If topological bisimulation is satisfactory from the formal point of view, one needs more to address qualitative spatial reasoning problems and computer vision issues. If two models are not bisimilar, or one does not simulate the other, one must be able to quantify the difference between the two models. Furthermore, this difference should behave in a coherent manner across the class of all models. Informally, one needs to answer questions like: *How different are two spatial patterns?*

To this end, the game theoretic definition of topo-games as in [2] is recalled, and the prove of the main result of this paper follows, namely the fact that topo-games induce a distance on the space of all topological models for $S4_u$. First, the definition and the theorem that ties together the topo-games, $S4_u$ and topological models is given.

Definition 2 (Topo-game). Consider two topological models $\langle X, O, \nu \rangle$, $\langle X', O', \nu' \rangle$ and a natural number n. A *topo-game* of length n, notation $TG(X, X', n)$, consists of n rounds between two players, Spoiler and Duplicator, who move alternatively. Spoiler is granted the first move and always the choice of which type of round to engage, either global or local. The two sorts of rounds are defined as follows:

- **global**
  - (i) Spoiler chooses a model X_s and picks a point $\bar{x}_s$ anywhere in X_s
  - (ii) Duplicator chooses a point $\bar{x}_d$ anywhere in the other model X_d
- **local**
  - (i) Spoiler chooses a model X_s and an open o_s containing the current point x_s of that model
  - (ii) Duplicator chooses an open o_d in the other model X_d containing the current point x_d of that model
  - (iii) Spoiler picks a point $\bar{x}_d$ in Duplicator's open o_d in the X_d model
  - (iv) Duplicator replies by picking a point $\bar{x}_s$ in Spoiler's open o_s in X_s

The points $\bar{x}_s$ and $\bar{x}_d$ become the new current points. A game always starts by a global round. By this succession of actions, two sequences are built. The form after n rounds is:

$$\{x_1, x_2, x_3, \ldots, x_n\}$$
$$\{x'_1, x'_2, x'_3, \ldots, x'_n\}$$

After n rounds, if x_i and x'_i (with $i \in [1, n]$) satisfy the same propositional atoms, Duplicator *wins*, otherwise, Spoiler wins. A *winning strategy (w.s.)* for Duplicator is a function from any sequence of moves by Spoiler to appropriate responses which always end in a win for him. Spoiler's winning strategies are defined dually.

The *multi-modal rank* of a $S4_u$ formula is the maximum number of nested modal operators appearing in it (i.e. $\Box$, $\Diamond$, U and E modalities). The following adequacy of the games with respect to the mereotopological language holds.

Theorem 3 (Adequacy). Duplicator has a winning strategy for n rounds in $TG(X, X', n)$ iff X and X' satisfy the same formulas of multi-modal rank at most n.

The reader is referred to [2] for a proof, various examples of plays and a discussion of winning strategies.

The interesting result is that of having a game theoretic tool to compare topological models. Given any two models, they can be played upon. If Spoiler has a winning strategy in a certain number of rounds, then the two models are different up to a certain degree. The degree is exactly the minimal number of rounds needed by Spoiler to win. On the other hand, one knows (see [2]) that if Spoiler has no w.s. in any number of rounds, and therefore Duplicator has in all games, including the infinite round game, then the two models are bisimilar.

A way of comparing any two given models is not of great use by itself. It is essential instead to have some kind of measure. It turns out that topo-games can be used to define a distance measure.

Definition 3 (isosceles topo-distance). Consider the space of all topological models T. *Spoiler's shortest possible win* is the function $spw : T \times T \to I\!N \cup \{\infty\}$, defined as:

$$spw(X_1, X_2) = \begin{cases} n & \text{if Spoiler has a winning strategy in } TG(X_1, X_2, n), \text{ but not in } TG(X_1, X_2, n-1) \\ \infty & \text{if Spoiler does not have a winning strategy in } TG(X_1, X_2, \infty) \end{cases}$$

The *isosceles topo-model distance (topo-distance,* for short*)* between X_1 and X_2 is the function $tmd : T \times T \to [0, 1]$ defined as:

$$tmd(X_1, X_2) = \frac{1}{spw(X_1, X_2)}$$

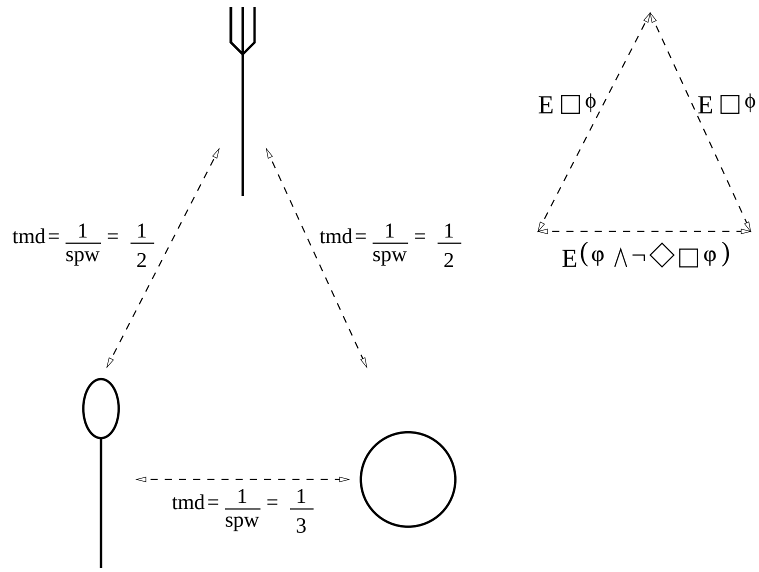

Fig. 2. On the left, three models and their relative distance. On the right, the distinguishing formulas.

The distance was named 'isosceles' since it satisfies the triangular property in a peculiar manner. Given three models, two of the distances among them (two sides of the triangle) are always the same and the remaining distance (the other side of the triangle) is smaller or equal. On the left of Figure 1, three models are displayed: a spoon, a fork and a plate. Think these cutlery objects as subsets of a dense space, such as the real plane, which evaluate to ϕ, while the background of the items evaluates to $\neg\phi$. The isosceles topo-distance is displayed on the left next to the arrow connecting two models. For instance, the distance between the fork and the spoon is $\frac{1}{2}$ since the minimum number of rounds that Spoiler needs to win the game is 2. To see this, consider the formula $E\Box\phi$, which is true on the spoon (there exists an interior point of the region ϕ associated with the spoon) but not on the fork (which has no interior points). On the right of the figure, the formulas used by spoiler to win the three games between the fork, the spoon and the plate are shown. Next the proof that *tmd* is really a distance, in particular the triangular property, exemplified in Figure 1, is always satisfied by any three topological models.

Theorem 4 (isosceles topo-model distance). *tmd* is a distance measure on the space of all topological models.

Proof. *tmd* satisfies the three properties of distances; i.e., for all X_1, $X_2 \in T$:

(i) $tmd(X_1, X_2) \geq 0$ and $tmd(X_1, X_2) = 0$ iff $X_1 = X_2$

(ii) $tmd(X_1, X_2) = tmd(X_2, X_1)$
(iii) $tmd(X_1, X_2) + tmd(X_2, X_3) \geq tmd(X_1, X_3)$

As for (i), from the definition of topo-games it follows that the amount of rounds that can be played is a positive quantity. Furthermore, the interpretation of $X_1 = X_2$ is that the spaces X_1, X_2 satisfy the same modal formulas. If Spoiler does not have a w.s. in $\lim_{n\to\infty} TG(X_1, X_2, n)$ then X_1, X_2 satisfy the same modal formulas. Thus, one correctly gets

$$tmd(X_1, X_2) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Equation (ii) is immediate by noting that, for all X_1, X_2, $TG(X_1, X_2, n) = TG(X_2, X_1, n)$.

As for (iii), the triangular property, consider any three models X_1, X_2, X_3 and the three games playable on them,

$$TG(X_1, X_2, n),\ TG(X_2, X_3, n),\ TG(X_1, X_3, n) \tag{1}$$

Two cases are possible. Either Spoiler does not have a winning strategy in all three games (1) for any amount of rounds, or he has a winning strategy in at least one of them.

If Spoiler does not have a winning strategy in all the games (1) for any number of rounds n, then Duplicator has a winning strategy in all games (1). Therefore, the three models satisfy the same modal formulas, $spw \to \infty$, and $tmd \to 0$. Trivially, the triangular property (iii) is satisfied.

Suppose Spoiler has a winning strategy in one of the games (1). Via Theorem 3 (adequacy), one can shift the reasoning from games to formulas: there exists a modal formula γ of multi-modal rank m such that $X_i \models \gamma$ and $X_j \models \neg\gamma$. Without loss of generality, one can think of γ as being in normal form:

$$\gamma = \bigvee \bigwedge [\neg] U(\varphi_{S4}) \tag{2}$$

This last step is granted by the fact that every formula φ of $S4_u$ has an equivalent one in normal form whose modal rank is equivalent or smaller to that of φ.[2] Let γ^* be the formula with minimal multi-modal depth m^* with the property: $X_i \models \gamma^*$ and $X_j \models \neg\gamma^*$. Now, the other model X_k either satisfies γ^* or its negation. Without loss of generality, $X_k \models \gamma^*$ and therefore X_j and X_k are distinguished by a formula of depth m^*. Suppose X_j and X_k to be distinguished by a formula β of multi-modal rank $h < m^*$: $X_j \models \beta$ and $X_k \models \neg\beta$. By the minimality of m^*, one has that $X_i \models \beta$, and hence, X_i and X_k can be distinguished at depth h. As this argument is symmetric, it shows that either

[2] In the proof, the availability of the normal form is not strictly necessary, but it gives gives a better impression of the behavior of the language and it has important implementation consequences, [2].

– one model is at distance $\frac{1}{m*}$ from the other two models, which are at distance $\frac{1}{l}$ ($\leq \frac{1}{m*}$), or
– one model is at distance $\frac{1}{h}$ from the other two models, which are at distance $\frac{1}{m^*}$ ($\leq \frac{1}{h}$) one from the other.

It is a simple matter of algebraic manipulation to check that m^*, l and h, m^* (as in the two cases above), always satisfy the triangular inequality.

The nature of the isosceles topo-distance triggers a question. Why, given three spatial models, the distance between two couples of them is always the same?

First an example, consider a spoon, a chop-stick and a sculpture from Henry Moore. It is immediate to distinguish the Moore's sculpture from the spoon and from the chop-stick. The distance between them is high and the same. On the other hand, the spoon and the chop-stick look much more similar, thus, their distance is much smaller. Mereotopologically, it may even be impossible to distinguish them, i.e., the distance may be null.

In fact one is dealing with models of a qualitative spatial reasoning language of mereotopology. Given three models, via the isosceles topo-distance, one can easily distinguish the very different patterns. In some sense they are far apart as if they were belonging to different equivalence classes. Then, to distinguish the remaining two can only be harder, or equivalently, the distance can only be smaller.

5 Concluding Remarks

In this paper, a new perspective on mereotopology is taken, addressing issues of model equivalence and especially of model comparison. Defining a distance that encodes the mereotopological difference between spatial models has important theoretical and application implications. In addition, the use of model comparison games is novel. Model comparison games have been used only to compare two given models, but the issue of setting a distance among a whole class of models has not been addressed. The technique employed in Theorem 4 for the language $S4_u$ is more general, as it can be used for all Ehrenfeucht-Fraïssé style model comparison games[3] adequate for modal and first-order languages equipped with negation. A question interesting *per se*, but out of the scope of the present paper, is: which is the class of games (over which languages) for which a notion of isosceles distance holds? (E.g. are pebble games suited too?)

Another question open for further investigation is the computability of the topo-distance. First, there is a general issue on how to calculate the distance for any topological space. One may be pessimistic at a first glance, since the definition and the proof of the Theorem 4 are not constructive, but actually the proof of the adequacy theorem for topo-games given in [2] is. Furthermore, decidability results for the logic $S4_u$ on the usual Kripke semantics (cf. [12]) should extend to the topological interpretation. Second, in usual applications

[3] For an introduction to Ehrenfeucht-Fraïssé games see, for instance, [11].

the topological spaces at hand are much more structured and tractable. For example in a typical geographical information system, regions are represented as a finite number of open and/or closed polygons. With these structures, it is known that finiteness results apply (cf. [3]) and one should be able to compute the topo-distance by checking a finite number of points of the topological spaces. Currently, an image retrieval system based on spatial relationships where the indexing parameter is the topo-distance is being built, [1]. The aim is twofold, on the one hand one wants to build a system effectively computing the topo-distance, on the other one wants to check with the average user whether and how much the topo-distance is an intuitive and meaningful notion.

Broadening the view, another important issue is that of increasing the expressive power of the spatial language, then considering how and if the notion of isosceles distance extends. The most useful extensions are those capturing geometrical properties of regions, e.g. orientation, distance or shape. Again one can start by Tarski's ideas, who fell for the fascinating topic of axiomatizing geometry, [18], but can also follow different paths. For example, staying on the ground of modal logics, one can look at languages for incidence geometries. In this approach, one distinguishes the sorts of elements that populate space and considers the incidence relation between elements of the different sorts (see [6,5,19]).

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