Logical Patterns in Space

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Abstract

In this paper, we revive the topological interpretation of modal logic, turning it into a general language of patterns in space. In particular, we define a notion of bisimulation for topological models that compares different visual scenes. We refine the comparison by introducing Ehrenfeucht-Fraïssé style games between patterns in space. Finally, we consider spatial languages of increased logical power in the direction of geometry.

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Contents

1 Reasoning about Space

Spatial structures and spatial reasoning are essential to perception and cognition. Much day-to-day practical information is about what happens at certain spatial locations. But also, spatial representation is a powerful source of geometric intuitions ('vision' in the literal sense) which can be harnessed to more general cognitive tasks. Even so, spatial reasoning has not attracted much attention from logicians. Those with physical interests, besides the usual mathematical and linguistic ones, seem to put their efforts into temporal reasoning, which pervades action and the flow of communication. Perhaps, people have also felt that logics of space would be similar qua format and themes, and therefore not really innovative. Nevertheless, there is a recent trickle of interesting publications on logics of space, motivated by such diverse concerns as spatial representation and vision in AI [Shanahan, 1995], semantics of spatial prepositions in linguistics [Winter and Zwarts, 1997], or more instrumentally, reasoning with diagrams [Hammer, 1995, Gurr, 1998].

But do we need a formal logic of space? After all, many successful accounts already exist. One of the oldest parts of mathematics, geometry, is the theory of space par excellence. And in the early part of this century, mathematics spawned another successful branch devoted to general spatial structure, viz. topology. Around the same time, philosophers developed mereology, as another foundational account of part-whole relationships, continuing themes found already in Aristotle. And finally, linguists have been studying spatial semantics in great depth, charting the vocabulary of spatial expressions found across natural languages. Of course, these accounts do not all address the same issues. One must distinguish between objective spatial structure in the world, spatial representations used in cognition, or the spatial expressions of natural languages. Here, logic may provide a unifying viewpoint, incorporating both structural and linguistic aspects.

Nevertheless, logics of space are very diverse, too. Theories differ in their accounts of the primitive objects: points, lines, polygons, regions. For a sample of the contrast, compare the two publications [Tarski, 1938, Tarski, 1959]. Likewise, theories differ in their choice of primitive spatial relations: which include inclusion, overlap, touching, 'space' versus 'place'. Mereological theories (meros, Greek for part), topological theories (limit points and connection play a major role) and mereotopological ones (Connection, Parthood and External Connection). Systematic accounts of the genesis of spatial vocabulary have been around since Helmholtz' work on invariances of movement, but no generally agreed primitive relations have emerged. Moreover, axioms differ across theories: [Clarke, 1981, Clarke, 1985] versus [Pratt and Schoop, 1997] versus [Casati and Varzi, 1999]. It is not our aim to survey, let alone to resolve, this multitude of approaches. Instead, we join the fray, proposing one more perspective—viz. a modal logic of space. Even this approach is not new, witness the previous publications [Segerberg, 1970, Segerberg, 1976, Shehtman, 1983, Venema, 1992]. But we hope our particular take on the subject is an advance toward generality.

2 Topological Structure: a Modal Approach

In this paper, we consider regions as our primitive objects, in line with much of the literature. Most logics of space agree on some basic relations: overlap, inclusion, touching. Some also have operations, such as interiors or boundaries of regions, or their convex closures. These determine the richness of visual scenes. Here is our paradigm.

Consider setting a table. We start with a clean linen sheet, forming one region as our visual field. Then we put a circular plate, which gives us several more: the interior of the plate, its boundary, and its complement: the rest of the sheet. Next, we put cutlery: e.g. a fork. It defines a new region, a boundary, a convex closure (things can be in the region 'in between its prongs'), but also convex closure of two regions, such as the zone 'in between the fork and the plate'. Moreover, we may also want to say that the fork defines a direction, viz. a line cutting the plane into half. Finally, there may be metric, or semi-metric structure: the fork may be close to the plate, but not close to the knife. Thus, we encounter all the familiar structures of mathematical (and common-sense) geometry, which are also reflected in our use of natural language to describe visual scenes. "Setting the table" will serve as a running example in what follows, as it demonstrates quite a few complexities of visual representation and reasoning.

Our aim is to propose a modeling of the basic visual structure of situations like this, look at the properties of the resulting spatial logic, and show how this analysis works in computational practice. Now, the literature on spatial reasoning contains many different approaches, as noticed above. Our proposal uses modal logic, interpreted, not with its standard possible worlds semantics, but in topological spaces. This so-called 'topological interpretation' dates back to the 30s [Tarski, 1938]. It is only used marginally nowadays, e.g., as a side-light on the interpretation of intuitionistic logic as a theory of 'open sets', viewed as information pieces [van Dalen and Troelstra, 1988]. Our aim is to revive and extend it for the specific business of spatial reasoning.

2.1 The topological view of space

Topology has a very abstract view of space. It slightly enriches set theory by introducing connectedness. This is obtained by adding to the concepts of ordinary set theory, i.e. the ability of talking about sets and membership relations between elements and sets, a family of subsets with specific structure: the open sets. We will follow topology's paramount role in perception, concentrating on qualitative spatial reasoning and at times on the semantics of spatial fragments of natural language.

Let us consider through a simple (and simplified) example the *forma mentis* of topology. To fully understand a theory one has to understand what is equivalent and what is different looking at the world with its 'glasses'. In topology, spaces are considered equivalent up to elastic deformations. If you take a balloon and start inflating it, at any instant the balloon is geometrically different. Its volume is increasing. But from a topological point of view it is the very same balloon. Quantitatively during the act of inflation with air the balloon changes, but qualitatively you always have an inflated balloon. If at a certain moment, a naughty kid comes with a needle and explodes the balloon, topology registers a difference. Punching the needle in the balloon provokes a non-elastic transformation, not topologically safe. Now, our one piece spheric shaped object has become a lower dimensional surface, possibly composed of more pieces. At a very abstract and qualitative level, topology provides a theory close to the way humans think. Roughly, an inflated balloon, before, and little disconnected chunks of rubber after. Topology 'thinks' the same way.

In the following two subsections we give the basic definitions related to topological spaces and the actions that preserve topological properties, the above mentioned 'elastic' transformations.

2.1.1 Topological spaces

Definition 1 A *topological space* is a couple $\langle X, O\rangle$, where X is a set and $O \subseteq \mathcal{P}(X)$ such that:

1. $\emptyset \in O$ and $X \in O$,
2. O is closed under arbitrary unions,
3. O is closed under finite intersections.

Related to the definition of topological space, we have:

(i) An element of O is called an *open*. A subset A of X is called *closed* if $X - A$ is open.

(ii) A point $s \in X$ is a *limit point* of a subset A of X if for each $o \in O$ such that $s \in o$, $(o - \{s\}) \cap A$ is not empty.

(iii) The *interior* of a set $A \subseteq X$ is the union of all open sets contained in A.

(iv) The *closure* of a set $A \subseteq X$ is the intersection of all closed sets containing A or, equivalently, the union of the set A with all its limit points.

(v) Given a set A, the set of points y such that for any open set o containing y both $o \cap A \neq \emptyset$ and $o \cap (X - A) \neq \emptyset$ hold, is called the *frontier*, or *boundary*, of A.

(vi) A family of open sets B is a *base* of the space X if all open sets are unions of members of B. Such a family is a *subbase* of X, if the collection of all finite intersections of elements of B is a base for X.

Often when referring to a topological space $\langle X, O\rangle$, the set of opens is omitted.

A topological space can be defined equivalently in terms of closed sets, the dual notion of open set. In this case, the closed sets C satisfy:

1. $\emptyset \in C$ and $X \in C$,
2. C is closed under finite unions,
3. C is closed under arbitrary intersections.

Special open and closed sets can also be important.

Definition 2 A set is called *closed regular* if it coincides with the closure of its interior. It is called *open regular* if it coincides with the interior of its closure.

Intuitively, thinking of sets as regions, regular sets are regions with no lower dimension cracks or spikes.

Examples of topological spaces are:

Example 1 (i) **indiscrete topology** $\langle X, \{\emptyset, X\}\rangle$

(ii) **discrete topology** $\langle X, \mathcal{P}(X)\rangle$

(iii) **metric spaces** every metric space is a topological space. A base that builds up the topology is the family of sets $\{x : \text{distance}(x, p) < r\}$ for arbitrary points p of the space and nonnegative r. This is called the *standard* topology.

(iv) **Cantor space** all infinite sequences of $0, 1$. A base that builds up the topology is the family of sets A_σ, consisting of all the sequences extending the finite initial segment σ.

Our attention will focus in the rest of the paper on metric spaces. In particular we will look at one-, two- and three-dimensional spaces. As a point of notation, when considering intervals in one dimensional metric spaces we adopt the following (a, b) for $\{x : a < x < b\}$. Square brackets denote that the frontier point belongs to the interval, e.g. $(a, b]$ stands for $\{x : a < x \leq b\}$.

2.1.2 Special properties of topological spaces

Definition 3 A topological space X is *connected* if the only sets which are both open and closed are $\emptyset$ and X.

Example 2 Examples of connected spaces are the metric spaces $\mathbb{R}^n$ with the standard topology, for any positive integer n. Non-connected spaces are the rationals $\mathbb{Q}$. E.g., consider the two non-empty open and closed sets $(-\infty, \sqrt{2})$ and $(\sqrt{2}, \infty)$.

Definition 4 Let X be a topological space. A collection $V_i \in \mathcal{P}(X)$ is a *covering* of X if $\bigcup_i V_i = X$. It is an *open covering* if all the V_i are open. A topological space X is said to be *compact* if every open covering has a finite subcovering.

Example 3 No space $\mathbb{R}^n$ is compact. But all (and only) their *bounded* subsets are compact.

Definition 5 A set A in a topological space X, is said to be *dense* in X, if all points of X are a point or a limit point of A. A topological space is said to be *dense* if all its points are limit points for itself.

Another interesting way to discern topological spaces uses their richness in terms of points and open sets. If there are enough of them one can 'separate' points. This formally shows in so-called 'separation axioms':

Definition 6 A topological space X is called

(i) T_0 if for any two distinct points x_1 and x_2 $(\in X)$, there exists an open set $o \in X$ containing one but not the other,

(ii) T_1 if for any two distinct points x_1 and x_2 ($\in X$), there exist an open set $o_1 \in X$ containing x_1 but not x_2 and there exists an open set $o_2 \in X$ containing x_2 but not x_1,

(iii) T_2 *(Hausdorff)* as T1 with the additional requirement that $o_1 \cap o_2 = \emptyset$,

(iv) T_3 *(regular)* as T1 and for every closed set and point not contained in it there exist two disjoint open sets containing the point and the closed set respectively,

(v) T_4 *(normal)* as T1 and for every two closed disjoint sets there exists two disjoint open sets each containing one of the closed sets.

2.1.3 Structure preserving mappings

We now give the definitions of mappings between topological spaces with their associated properties.

Definition 7 (Continuity) Given two topological spaces $\langle X, O\rangle$ $\langle X', O'\rangle$, and a mapping $f : X \to X'$, f is *continuous* iff for all open sets $o' \in O'$ we have that $f^{-1}(o')$ is in O, i.e. inverse images of open sets are open.

Definition 8 (Homeomorphism) Two topological spaces $\langle X, O\rangle$, $\langle X', O'\rangle$ are *homeomorphic* iff there are two continuous mappings $f : X \to X'$ and $g : X' \to X$ such that $f \circ g$, $g \circ f$ are the identity maps of the respective spaces.

If two topological spaces $\langle X, O\rangle$, $\langle X', O'\rangle$ are homeomorphic, then X and X' are in 1-1 correspondence. In other words, a necessary condition for two spaces to be homeomorphic is their having the same cardinality.

Example 4 The two subsets $(0, 1)$ and $(1, \infty)$ of the metric space $I\!R$ with the standard topology are homeomorphic. The two inverse functions $f(x) = g(x) = \frac{1}{x}$ are continuous and compose to identity maps both ways. By a similar construction of homeomorphisms, the real plane $I\!R^2$ and a unit circle $x \in I\!R^2 : d(x, 0) < 1$ are homeomorphic. Also Cantor space is homeomorphic to $[0, 1]$.

Two non-homeomorphic spaces are the real plane $I\!R^2$ and a three dimensional unit ball $x \in I\!R^3 : d(x, 0) < 1$, our inflated and exploded balloon example.

A more general notion than homeomorphism is also available in topology.

Definition 9 (Homotopy) Let X and X' be topological spaces, and let f_0 and f_1 be continuous maps from X to X'. f_0 is *homotopic* to f_1 (notation $f_0 \simeq f_1$) if there exists a continuous map $F : X \times I \to X'$ such that for all x $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$, where I is $[0, 1]$. F is called an *homotopy* from f_0 to f_1.

Definition 10 (Homotopy type) Two topological spaces X, X' are of the same *homotopy type* if there exists two continuous maps $f : X \to X'$ and $g : X' \to X$ such that $g \circ f$ is homotopic to the identity mapping on X and $f \circ g$ is homotopic to the identity mapping on X'.

Example 5 Homeomorphic spaces are also homotopic. So an example of homotopic spaces is the real line and the real unit interval. A more interesting example is the homotopy between a single point and any real metric space $\mathbb{R}^n$.

The real plane without its origin $\mathbb{R}^2 - (0,0)$ and the unit circle are an example of non-homotopic spaces.

Equipped with these topological notions we can begin our exploration of visual scenes. We consider 'elastic' and 'pseudo-elastic' transformations that preserve visual structure. Visual scenes lie mostly in dense mono-dimensional, bidimensional and tridimensional metric spaces. We will consider both the generic spaces, and the peculiarities of the latter.

But why topology to begin with? Why not start right away with, say, geometry? As we mentioned before, we think topology is at the right abstract level for most cognitive tasks. Furthermore, being much rougher than geometry, it provides simple characterizations of spatial arrangements and usually more tractable theories.

Eventually, we will not be satisfied with this level either: considering 'rougher' transformations on the one hand, but also richer features in the end. Our point of departure for this is a modal language for talking about visual patterns.

From this first simple topological language, we then increase the expressive power towards geometry, without ever reaching it ...

3 Basic Modal Logic of Space

3.1 Topological language and semantics

In this section we introduce the simple language that forms the basis of this article. It is a propositional modal formalism with a topological interpretation for the modal operators: diamond as closure, and box as interior. Each formula represents a region, i.e. a subset of some topological space

A brief historical note. This topological interpretation was originally given by Tarski in the thirties with various completeness theorems [McKinsey and Tarski, 1944]. The interpretation of the same language in terms of the dominant Kripke models came only two decades later. Kripke models are nowadays the most used interpretation for modal logics. In the appendix, we give a comparison between Kripke models and topological models via so-called neighbourhood semantics. But it will help a lot in what follows if the reader knows some basics of standard modal logic.

Consider the following vocabulary:

- a set of proposition letters P,
- the usual propositional connectives $\top, \bot, \neg, \wedge, \vee, \rightarrow$,
- unary modal operators $\Box, \Diamond$.

Definition 11 (The basic language $\mathcal{L}$) The formulas of the language $\mathcal{L}$ are obtained inductively using the following rules:

- **Atomic.** $\top$ ('true'), $\bot$ ('false') and all elements in P are formulas.
- **Propositional.** If φ, ψ are formulas, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi$ are formulas.
- **Modal.** If φ is a formula then $\Box\varphi$ and $\Diamond\varphi$ are formulas.

Each formula φ of the language $\mathcal{L}$ is intended to represent a region, as exemplified in the following table and in Figure 1 with the case of a region shaped as a spoon.

$\top$	the universe
$\bot$	the empty region
$\neg\varphi$	the complement of a region
$\varphi \wedge \psi$	intersections of the regions φ and ψ
$\varphi \vee \psi$	union of the regions φ and ψ
$\Box\varphi$	interior of the region φ
$\Diamond\varphi$	closure of the region φ

These intuitions about our language $\mathcal{L}$ are reflected in its semantics. Topological models $M = \langle X, O, \nu \rangle$ will be topological spaces (X, O) plus a valuation function $\nu : P \rightarrow \mathcal{P}(X)$.

Definition 12 (The topological semantics of $\mathcal{L}$) Truth of modal formulas is defined at points x in topological models M:

$$
\begin{array}{rll}
M, x \models \bot & \text{iff} & \text{never} \\
M, x \models \top & \text{iff} & \text{always} \\
M, x \models p & \text{iff} & x \in \nu(p) \text{ (with } p \in P) \\
M, x \models \neg\varphi & \text{iff} & \text{not } M, x \models \varphi \\
M, x \models \varphi \wedge \psi & \text{iff} & M, x \models \varphi \text{ and } M, x \models \psi \\
M, x \models \varphi \vee \psi & \text{iff} & M, x \models \varphi \text{ or } M, x \models \psi \\
M, x \models \varphi \rightarrow \psi & \text{iff} & \text{not } M, x \models \varphi \text{ or } M, x \models \psi \\
M, x \models \Box\varphi & \text{iff} & \exists o \in O : x \in o \wedge \forall y \in o : M, y \models \varphi \\
M, x \models \Diamond\varphi & \text{iff} & \forall o \in O : x \notin o \vee \exists y \in o : M, y \models \varphi
\end{array}
$$

As usual we can economize a bit, by defining $\varphi \vee \psi$ as $\neg\varphi \rightarrow \psi$, and $\Diamond\varphi$ as $\neg\Box\neg\varphi$. We will do this when convenient—e.g. in proofs of 'system properties'.

Following Tarski's ideas, we show in the sequel that the language $\mathcal{L}$ has the well known modal logic S4 as a sound and complete proof system with respect to topological models. Here are axioms of S4:

$$
\begin{array}{lc}
\Diamond A \leftrightarrow \neg\Box\neg A & \text{(Dual.)} \\
\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) & \text{(K)} \\
\Box A \rightarrow A & \text{(T)} \\
\Box A \rightarrow \Box\Box A & \text{(4)}
\end{array}
$$

Modus Ponens and Necessitation are the rules of inference which complete $\mathcal{L}$:

$$\frac{\varphi \rightarrow \psi \qquad \varphi}{\psi} \tag{MP}$$

$$\frac{\varphi}{\Box\varphi} \tag{N}$$

The following derived theorems of S4 are also relevant:

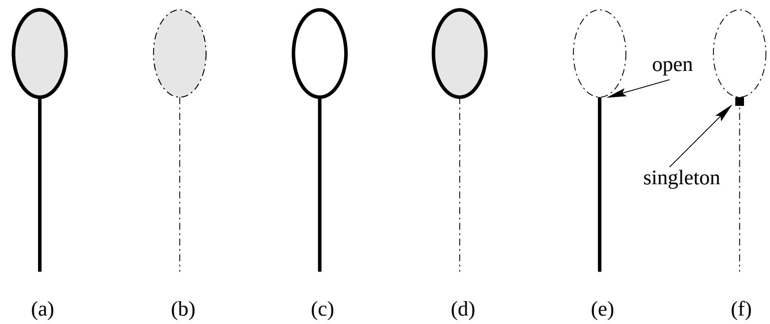

Figure 1: A formula of the language $\mathcal{L}$ identifies a region in a topological space. (a) a spoon, p. (b) the container part of the spoon, $\Box p$. (c) the boundary of the spoon, $p \wedge \neg\Box p$. (d) the container part of the spoon with its boundary, $\Diamond\Box p$. (e) the handle of the spoon, $p \wedge \neg\Diamond\Box p$. In this case the handle does not contain the junction point handle-container. (f) the joint point handle-container of the spoon, $\Diamond\Box p \wedge \Diamond(p \wedge \neg\Diamond\Box p)$: a singleton in the topological space.

$$\Box\top \qquad \text{(N)}$$
$$\Box A \wedge \Box B \leftrightarrow \Box(A \wedge B) \qquad \text{(R)}$$
$$\Box A \vee \Box B \leftrightarrow \Box(\Box A \vee \Box B) \qquad \text{(or)}$$

The first axiom expresses the modal duality of $\Diamond$ and $\Box$, which reflects the topological duality of the interior and closure operators. The axiom (K) does not have an immediate intuitive interpretation in topology, but it is equivalent to the axioms (N) and (R), which do (cf. [Bennett, 1995]). Using (N),(R) in place of (K), helps to develop the topological intuitions. (N) corresponds to the condition that the whole space is open. (R) corresponds to the third condition in the definition of topological space. Finally, (or) expresses that open sets are closed under finite unions. Next, axiom (T) states that the interior of any set is contained in the set. (4) expresses the idempotence of the interior operator.

Even when not valid, modal formulas can define important notions in topology. For instance, consider the following principle derivable in S4:

$$\text{if } \Box(\varphi \leftrightarrow \Diamond\Box\varphi), \text{ then } \Box(\Box\neg\varphi \leftrightarrow \Box\Diamond\Box\neg\varphi)$$

This says that if a set is closed regular, so is its 'open complement'. Thus, modal laws encode properties of regular sets.

Remark 6 In this section, we have introduced topological models as topological spaces plus a valuation function. In the remainder of the paper we often refer to just the underlying topological space looking at its properties from a modal point of view. This is like considering *frame* properties when interpreting modal logics with Kripke semantics.

3.2 Topological bisimulation

3.2.1 Definition and examples

In modal logics the role covered by 'potential isomorphism' in first-order logic is played by the concept of bisimulation. It compares models in a structured sense,

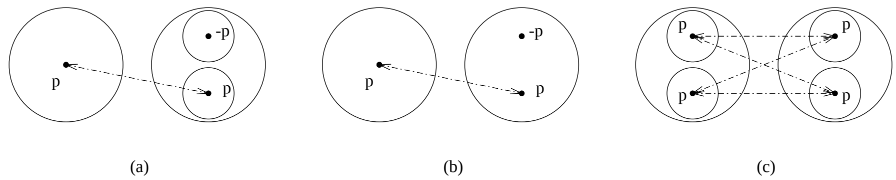

Figure 2: Three comparisons between topological models. Cases (a) and (c) are of bisimilar points, while (b) is an example of two non bisimilar points.

'just enough' to ensure truth of the same modal formulas ([van Benthem, 1976, Park, 1981]). We now formulate our topological version.

Definition 13 (Topological Bisimulation) Consider a language $\mathcal{L}$ and two topological models $\langle X, O, \nu\rangle$, $\langle X', O', \nu'\rangle$. A *topological bisimulation* is a non-empty relation $\leftrightarrows \subseteq X \times X'$ such that if $x \leftrightarrows x'$ then:

(i) $x \in \nu(p) \Leftrightarrow x' \in \nu'(p)$ (for any proposition letter p)

(ii) *(forth condition):* $x \in o \in O \Rightarrow \exists o' \in O' : x' \in o'$ and $\forall y' \in o' : \exists y \in o : y \leftrightarrows y'$

(iii) *(back condition):* $x' \in o' \in O' \Rightarrow \exists o \in O : x \in o$ and $\forall y \in o : \exists y' \in o' : y \leftrightarrows y'$

We call a bisimulation *total* if it is defined for all elements of X and of X'. We overload the $\leftrightarrows$ symbol extending it to models with points: $\langle X, O, \nu\rangle, x \leftrightarrows \langle X', O', \nu'\rangle, x'$, where we require that $x \leftrightarrows x'$. If only the conditions (i) and (ii) hold, we say that the second model *simulates* the first one.

If $x \leftrightarrows x'$, then these points are in the same 'modal setting'. E.g., if x lies in the open neighbourhood o, then x' must lie in some corresponding neighbourhood o', where we test the quality of the latter correspondence by taking $y' \in o'$, and requiring a matching y in o.

First, let us look at some simple and somewhat artificial examples of bisimulations. Consider Figure 2, we have topological spaces and valuations in terms of the proposition letter p. In case (a) we have two topological spaces with the discrete topology. Relating the points that valuate to p gives us a bisimulation. In case (b), the two models do not bisimulate, the reason being that if one takes an open set (the only one available) containing the p point on the right hand side it is not possible to find a point on the left hand side that relates to $\neg p$. Note however, that the model on the left simulates the one on the right. (c) is an example of a 'universal relation'. More generally, we note that, given two topological spaces with a valuation function that assigns the same letter to all the points, one can always relate all points of one space with all points of the other, thus obtaining a rather trivial bisimulation.

In this paper, most of the visual examples deal with cutlery. We intend such 'pictures' as subsets of $\mathbb{R}^2$. Closed contours indicate that the set is not only the contour, but also all the points inside. Often the examples are also applicable to other metric spaces. Anyhow, we only use such examples to ground intuitions,

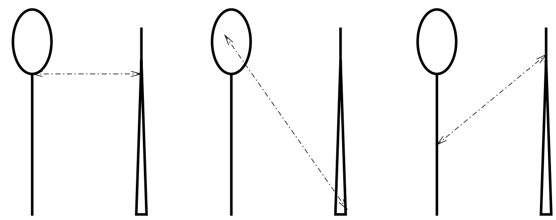

Figure 3: A fork is bisimilar to a chop-stick with a wide handle. In the figure the relation among points that match is highlighted via the double headed arrows.

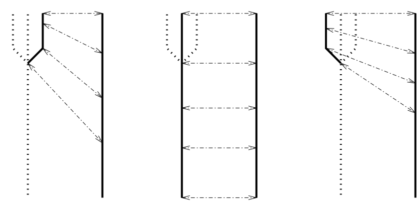

Figure 4: A fork is also bisimilar to a very thin chop-stick. See the text for an explanation.

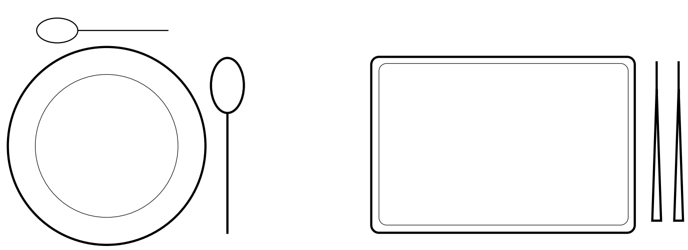

Figure 5: Two bisimilar visual scenes. An occidental way of dressing the table and an oriental one.

rather than commit to a specific domain of application. Furthermore, we do not mention bisimulation links that hold of the 'background' points. We consider these points always tied with a universal relation, and all verifying a common proposition letter that stands for background.

Example 7 *A spoon, a fork and a chop-stick.* We now turn to more visually concrete examples. First, we compare a spoon with a thick chop-stick (Figure 3). Geometrically we intend to compare a filled ellipse plus a touching straight line with a filled triangle plus a touching line. We start by noting that, as mentioned in the definition of bisimilarity, the notion can be lifted from points to spaces. Now we define the bisimulation. We relate the points in the spoon's ellipse with those in the chop-stick's triangle, bearing in mind that interior points should be mapped to interior points and frontier points to frontier points. Then we relate the points of the handle of the spoon with the spike coming out of the chop-stick: a simple relation between two straight lines.

A similar line of thought drives the comparison of a fork with a thin chop-stick (Figure 4). The latter is now different from the previous example, it is only a straight line. The fork, on the other hand, is a straight line plus four extra touching segments. Again the two pieces of cutlery are bisimilar. A possible way of relating the two spaces is indicated in the figure. The central straight line of the fork corresponds to the entire chop-stick. Then the other prongs of the fork are related to the upper part of the chop-stick.

It is time to put cutlery together with plates and start comparing dressed tables.

Example 8 *Bisimulation between two different visual scenes.* In Figure 5, we have two visual scenes. On the left, an occidental way of dressing the table for a soup and ice-cream. On the right, an oriental way of dressing the table for sushi. By a similar way of reasoning as the previous example, the two scenes are bisimilar: each chop-stick is in relation with one of the spoons (as we have seen in Figure 3) while the two plates are in relation (simply relate every point of the boundary of the round plate with the boundary of the rectangular one and vice versa, then relate every interior point of one plate with any interior point of the other and vice versa).

3.2.2 Invariance of modal formulas

Bisimulation is a standard notion of comparison, but which explicit spatial properties of points are preserved under it? Here is half of the answer.

Theorem 1 *Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two models, $x \in X$, and $x' \in X'$ two bisimilar points. Then, for any modal formula φ, $M, x \models \varphi$ iff $M', x' \models \varphi$. In words, modal formulas are invariant under bisimulations.*

Proof We prove this by induction on φ. The case of any proposition letter p is given by the first condition on $\leftrightarrows$. As for intersection, $M, x \models \varphi \wedge \psi$ is equivalent by the truth definition to $M, x \models \varphi$ and $M, x \models \psi$, which by induction hypothesis is equivalent to $M', x' \models \varphi$ and $M', x' \models \psi$, which by the truth definition amounts to $M', x' \models \varphi \wedge \psi$. The other boolean cases are similar. For the modal case, we do one direction. If $M, x \models \Box\varphi$, then by the truth definition we have that $\exists o \in O : x \in o \wedge \forall y \in o : M, y \models \varphi$. By the forth condition, we have that corresponding to o there must exist an $o' \in O'$ such that $\forall y' \in o' \; \exists y \in o \; y \leftrightarrows y'$. By the induction hypothesis applied to these y and y' with respect to φ, then $\forall y' \in o' : M', y' \models \varphi$. By the truth definition of the modal operator we have $M', x' \models \Box\varphi$. Using the back condition one can prove the other direction of the implication $M \models \Box\varphi$ iff $M' \models \Box\varphi$. QED

To clinch the fit, we need a converse result. From standard modal logic on Kripke models, we know that this is a somewhat delicate matter (cf. [Blackburn et al., 1999). The result holds in an *infinitary* modal language, but also for a finite one, provided we restrict to special classes of models, i.e. finite ones.

Theorem 2 *Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two finite models, $x \in X$, and $x' \in X'$ such that for every φ, $M, x \models \varphi$ iff $M', x' \models \varphi$. Then there exists a bisimulation between M and M' connecting x and x'. In words, finite modally equivalent models are bisimilar.*

Proof To get a bisimulation between the two finite models, we stipulate that $u \leftrightarrows u'$ if and only if u and u' satisfy the same modal formulas. The atomic preservation condition for a bisimulation is trivially true given that the modal φ. We now prove the forth condition. Suppose that $u \leftrightarrows u'$ with $u \in o$. We must find an open o' such that $u' \in o'$ and $\forall y' \in o' \exists y \in o : y \leftrightarrows y'$. Ab absurdum, suppose there is no such o'. Then for every o' containing x' $\exists y' \in o' : \forall y \in o : \exists \varphi_y : y \not\models \varphi_y$ and $y' \models \varphi_y$. In words, every open o' contains a point y' with no modally equivalent point in o. Taking the finite conjunction of all formulas φ_y, we get a formula $\Phi_{o'}$ such that $y' \models \Phi_{o'}$ and $\neg\Phi_{o'}$ is true everywhere in o. Forcing the notation, we can write $o \models \neg\Phi_{o'}$. This line of reasoning holds for any open o' containing x' as chosen. Therefore, there exists a collection of formulas $\neg\Phi_{o'}$ that are modeled by the open o, $o \models \bigwedge_{o'} \neg\Phi_{o'}$. Since $x \in o$, by the truth definition we have $x \models \Box \bigwedge_{o'} \neg\Phi_{o'}$. By the fact that x and x' satisfy the same modal formulas, it follows that $x' \models \Box \bigwedge_{o'} \neg\Phi_{o'}$. But then, there exists an open o^* (with $x' \in o^*$) such that $o^* \models \bigwedge_{o'} \neg\Phi_{o'}$. Since o^* is an open containing x', is one of the o', i.e. $o^* \models \neg\Phi_{o^*}$. But we had supposed that for all opens o' there was a point $y' \models \Phi_{o'}$, so in particular the y' of o^* satisfies Φ_{o^*}. We have thus reached a contradiction. Analogously, one proves the back condition.
QED

So far we have introduced a notion of bisimulation that works locally. It relates points of two models imposing conditions on opens containing the bisimilar points. This is enough to see the relation between bisimulations and topology, or to consider properties of certain portions of models. But for the purpose of visual patterns one may want more. The ability of looking at the models more globally, to consider spaces in their wholeness rather than points belonging to them is what we seek. In Section 4.1, we extend the language $\mathcal{L}$ to $\mathcal{L}(E|U)$ with modal existential E and universal U operators. This extension lets us consider regions rather than points and reflects in the bisimulations, making the latter a mean of global comparison.

Refining the reasoning behind Theorem 1 provides further useful results. In particular, consider 'existential' modal formulas constructed using only atomic formulas and their negations, $\wedge, \vee$ and $\Box$ (note that in the topological interpretation, unlike the Kripke one, the $\Box$ is an existential modal quantifier and the $\Diamond$ an universal one).

Corollary 3 *Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two models, with a simulation $\rightharpoondown$ from M to M', such that $x \rightharpoondown x'$. Then, for any existential modal formula φ, $M, x \models \varphi$ only if $M', x' \models \varphi$. In words, existential modal formulas are preserved under simulations.*

The proof is straightforward from that of Theorem 1. It is also easy to extend the result to the extended modal language $\mathcal{L}(E|U)$, provided that the simulation is total and surjective. Results like this can be used for a systematic analysis of well-known topological preservation phenomena. (The following result may be postponed until after reading Section 3.3.)

Corollary 4 *Consider $\langle X, O \rangle$ and $\langle X', O' \rangle$, and a continuous surjective map $f : X \rightarrow X'$. If the topological space $\langle X, O \rangle$ is connected, then the space $\langle X', O' \rangle$ is connected.*

Proof Our first observation is a modal definition for connectedness, in the extended modal language $\mathcal{L}(E|U)$. We say that a topological space $\langle X, O \rangle$ *validates* a modal formula φ if φ is true at every point under every valuation. Now we have that the following two statements are equivalent:

(i) $\langle X, O \rangle$ is connected

(ii) $\langle X, O \rangle \models U(\Diamond p \rightarrow \Box p) \rightarrow Up \vee U\neg p$

To see this, note that the antecedent of this extended modal formula holds if the denomination of p is both open and closed, while the consequent says that either $p = X$ or $p = \emptyset$.

Now, we return to the statement of the Corollary. We must show that $\langle X', O' \rangle$ is connected. Suppose that it is not. Then there exists a valuation ν' and a point x' such that $\langle X', O', \nu' \rangle, x' \models \neg$(ii). Next, we use the given continuous map f to define a simulation $\leftharpoondown$ from M' to M (note the reversal in direction here):

$$x \leftharpoondown x' \qquad \text{iff} \qquad x' = f(x)$$

In particular, the definition of continuous map gives the forward simulation clause. Moreover, the surjectiveness of f guarantees that $\leftharpoondown$ is surjective and total on M'. Next, we define a valuation ν on M by 'copying ν' along f':

$$\nu(p) = f^{-1}(\nu'(p))$$

The result is a simulation $\leftharpoondown$ from $\langle X', O', \nu' \rangle$ onto $\langle X, O, \nu \rangle$ such that $x \leftharpoondown x'$ for some point $x \in X$.

Finally, we note that the negated formulas $\neg$(ii) is logically equivalent (by some syntactic manipulation) to the $\mathcal{L}(E|U)$ formula without $\Diamond$

$$U(\Box\neg p \vee \Box p) \wedge E\neg p \vee Ep$$

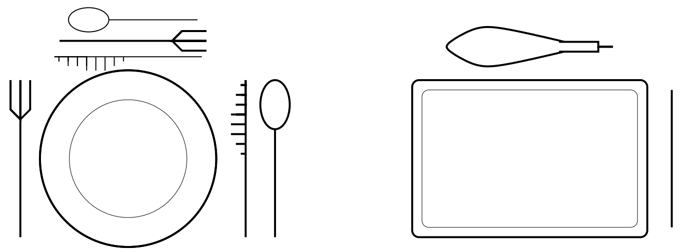

Figure 6: With added cutlery the scenes are not homeomorphic, but they are bisimilar.

By Corollary 3, this formula also holds for X in M, and hence $\langle X, O\rangle$ is not connected. A contradiction. QED

3.3 Bisimulations, homeomorphisms and continuous maps

Bisimulation is a form of 'topological sameness', but much weaker than the usual topological ones. To facilitate comparison, we consider *total* bisimulations, whose domain and image are the full topological spaces at hand.

We start with continuous maps between topological spaces. These naturally induce simulations (cf. the proof of Corollary 4). We now provide further details.

Lemma 5 (Continuity implies surjective simulation) Consider two topological spaces $\langle X, O\rangle$, $\langle X', O'\rangle$ and a continuous map $f : X \to X'$. Let ν' be any valuation on $\langle X', O'\rangle$, and set $\nu(p) = f^{-1}(\nu'(p))$. Then, the graph of f (i.e., the relation $\{(x, f(x)) : x \in X\}$) is a surjective simulation from $M' = \langle X', O', \nu'\rangle$ onto $M = \langle X, O, \nu\rangle$.

Proof This is immediate from the fact that continuous maps inversely preserve open sets. Note that, if $f(x) \in o$, every $x \in f^{-1}(o)$ has an image in o. QED

Theorem 6 *Homeomorphism implies total bisimulation.*

Proof This follows as with the previous lemma, noticing that homeomorphism implies the existence of two continuous maps that composed yield the identity mapping of each space. QED

Example 9 *Bisimulation is rougher than homeomorphism.* Consider Figure 5. The two visual scenes were bisimilar and, considering the scenes as subsets of a metric dense bidimensional space also homeomorphic. Let us slightly change cutlery both in shape and in number, as in Figure 6. We now have that the two visual scenes still bisimulate, but the underlying topological spaces are not homeomorphic anymore. The bisimulation is built, for instance, relating the two forks with one of the thin chop-sticks (as shown in Figure 4), the two knives

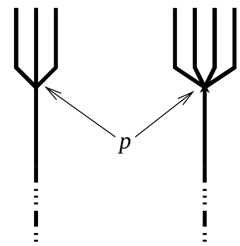

Figure 7: The two visual scenes are not homeomorphic (the point labeled p distinguishes them, leaving a different number of connected components), but they are bisimilar.

with the other chop-stick (following a similar reasoning of that used for the fork and the chop-stick in Figure 4), the two occidental spoons with the oriental one (the interiors are related, while the short oriental handle goes with the long occidental one), finally the plates are related as for the previous example. On the other hand, to show that they are not homeomorphic it is enough to note that the fork is not homeomorphic with anything, not even a chop-stick. In fact, the point where the prongs of the fork joins the handle, if removed from the fork leaves four connected components, while every point of the chop-stick, if removed can leave at most two connected components.

Some general remarks on the role of bisimulations in topology are in order.

Remark 10 (Any two topological spaces are bisimilar) Two topological spaces are always bisimilar, viz. by the universal relation relating all the points in one space with those of the other. This does not trivialize the notion of bisimulation though.

(i) The question of *interesting specific relations* that are bisimulations remains open. The proofs that we provided in this sections were based on non-trivial bisimulations and not on the universal relation. Often, we want bisimulation that relate as few elements as possible.

(ii) Topological models come with a valuation function, which allows us to 'name' regions with atomic propositions. For the purpose of visual patterns this is quite relevant. On a set table, we have several regions, marked by atomic propositions, `fork, knife, spoon` and `plate`. In that case, the universal relation need not be a bisimulation at all!

Remark 11 (Information transfer across models) Specific simulations and bisimulations can be used to transfer logical and topological information across topological models and spaces. A typical illustration is the preservation of existential modal formulas via continuous maps in Corollary 3. In a more abstract setting, these issues also return with so-called 'Chu morphisms' relating Chu spaces (cf. [van Benthem, 1998]). Existential modal formulas are then related to more general first-order 'flow formulas'.

We end this section by fitting another topological morphism into our list of simulations. The observation in itself is rather trivial, since two topological spaces are always bisimilar using the universal relation and a 'flat' valuation function. But the proof provides a non-trivial specific bisimulation, in the spirit of Remark 10.

Theorem 7 *Homeomorphism⇒Homotopy⇒Bisimulation.*

Proof The first implication we have from topology. For the second we have to prove that given two topological spaces of the same homotopy type there exist a bisimulation between them. If the two spaces X and X' are of the same homotopy type then there exist two continuous mappings $f : X \to X'$ and $g : X' \to X$ such that $g \circ f \simeq i_X$ and $f \circ g \simeq i_{X'}$. Given the homotopy, there exists two continuous mappings $H : X \times I \to X$ and $L : X' \times I \to X'$ (where $I = [0,1]$), such that $\forall x \in X\ H(x,0) = g \circ f \wedge H(x,1) = i_X$ and $\forall x' \in X'\ L(x',0) = f \circ g \wedge L(x',1) = i'_X$.

To construct the bisimulation we define the relation in the following way:

$$x \leftrightharpoons x' \text{ iff } x = H(g(x'),1)$$

That $x = H(g(x'),1)$ indeed defines a bisimulation is immediate from the continuity of $H(g(x'),1)$, which in turn guarantees the required inverse preservations of open sets. QED

3.4 Games that compare visual scenes

In the previous section we considered global ways of comparing topological models. In this section, we want more. It is not only important to know when two visual scenes are bisimilar, but also 'how different' they are when not equivalent.

We measure this difference twice. First, we consider topological comparison games, analyzing winning strategies for a 'difference player'. Then, we relate this to syntactic differences, between modal formulas having different truth values in the two models.

3.4.1 Definition and examples

To understand the fine-structure of topological difference, we introduce model comparison games. These are the analogue of *Ehrenfeucht-Fraïssé* games between first-order models. For a definition in the original context of first-order logic, we refer the reader to [Doets, 1996], for an implementation to [Agostini and Aiello, 1999].

Topological games are two player games, between Spoiler and Duplicator. The game is on two models, played for an a priori fixed number of rounds, and starting at two given points, one in each model. Intuitively Spoiler is trying to prove that the points are different, while Duplicator maintains they are not. A play consists of a sequence of interactions between Spoiler and Duplicator according to the following schedule (Figure 8 illustrates a one round game). Spoiler is granted the first move, in which he chooses one of the two models and an open set containing the current point of the model. Duplicator replies

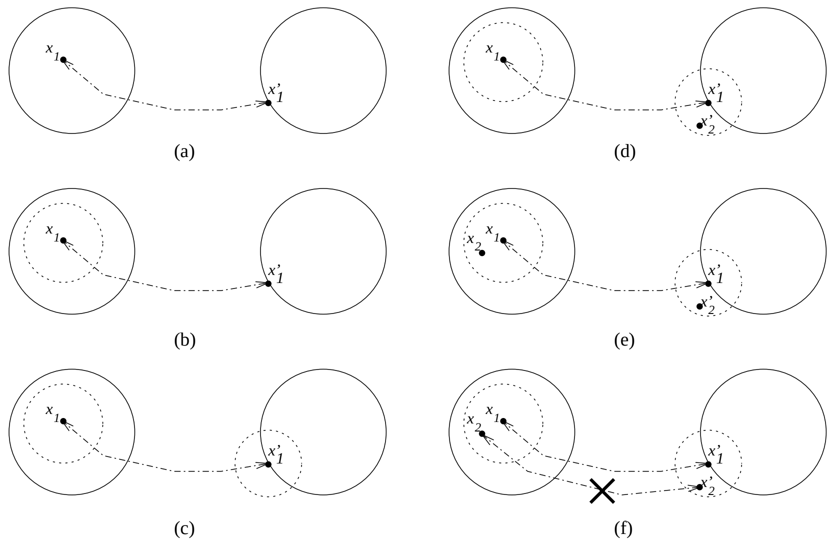

Figure 8: A simple one round play.

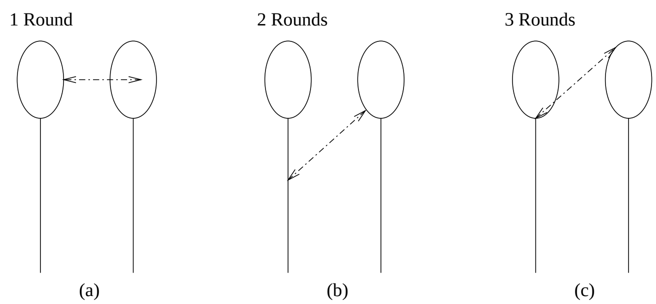

Figure 9: Games on two spoons with different starting points. On top, the number of rounds needed by Spoiler to win.

by selecting an open set in the other model containing the current point of that model. The round is not yet finished, since Spoiler must now pick a point in the open set just selected by Duplicator. This point becomes the new current point of that model. Duplicator has to reply selecting a point in the open chosen originally by Spoiler. This setup is iterated for the number of rounds fixed at the beginning. Spoiler can always choose in which model he wants to play the following round, thus he can switch models. By these sequences of rounds, the two players construct sequences of related points. If these points always agree pairwise in all atomic propositions, Duplicator has won, otherwise Spoiler has.

Example 12 (A one round game) In Figure 8, a one round game is depicted. (a) Two closed subsets p of two copies of $\mathbb{R}^2$ are shown. The starting points are x_1 and x'_1, inside and on the border of the closed sets, respectively. (b) Spoiler starts the game picking the left model and selecting an open (circu-

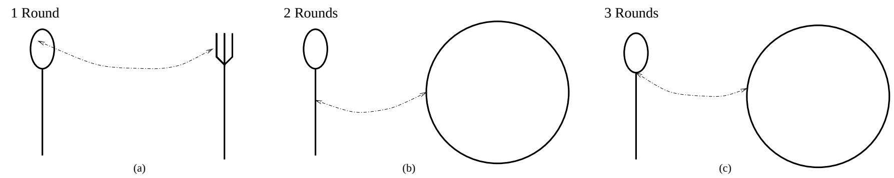

Figure 10: Games on different items for dressing a table. On top, the number of rounds needed by Spoiler to win.

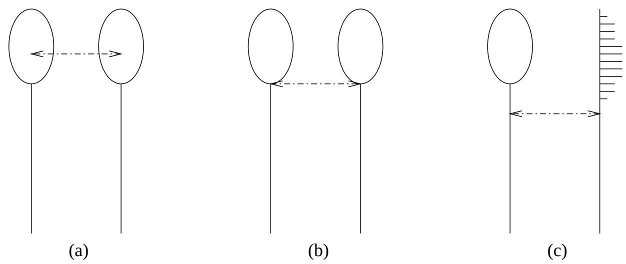

Figure 11: Example of games for which Duplicator has a winning strategy for any number of rounds.

lar) set surrounding the current point of the model x_1. (c) Duplicator does the same on the other model. He selects an open set surrounding x_1'. (d) To end the round, Spoiler chooses a point in Duplicator's open. A point x_2' outside the original set. (e) Duplicator replies with a point x_2 in Spoiler's open, which can only be inside the original set of the other model. (f) The two points are not in relation, since they do not satisfy the same propositional letters. In x_2 p is satisfied, while in x_2' $\neg p$ is. Spoiler won the game.

Example 13 (Playing on spoons) Next, consider Figure 9. In the three examples, the two same spoons are played upon. We view these as subsets of different copies of the space $\mathbb{R}^2$. The starting points are different though. (a) The leftmost game starts by comparing a point on the frontier of the spoon with an interior point of the other spoon. As we have seen in the previous example Spoiler can win this game in one round. (b) In the central game, a point on the handle is compared with a point on the boundary of the container of the spoon. Spoiler can again win the game, but needs two rounds this time. Here is a winning strategy for him. First, Spoiler chooses an open on the left spoon containing the starting point and taking care that no interior point is in his open. No matter how Duplicator chooses his open on the other spoon, it will contain an interior point. Spoiler will then choose such an interior point. Duplicator's response in the first round will be a frontier point of the other model (if he picks another point it will be outside the spoon and he will immediately lose). Now we are in the conditions of game (a), which Spoiler can win in the remaining round. (c) Finally, on the left the junction point between handle and container is related with a boundary point of the container. In this game, Spoiler will chose an open on the right model taking care to avoid all points on the handle of the spoon. Duplicator is forced to chose points on the handle. Spoiler then picks a point on the handle of the left spoon. Duplicator replies either with an interior point, or with a frontier point of the right spoon. Now we are in the conditions of game (b), and Spoiler can win in the remaining two rounds.

It is also instructive to check that other initial choices for Spoiler may very well lead to his losing the game! (E.g., let Spoiler start in the right-hand model in (b)). A strategy guarantees a win only for those who follow it ...

Example 14 (Set tables) Consider now Figure 10. We play this time different items for dressing a table, but the line of reasoning is the same. Again, it is the starting point that makes the game. We have a point on the prongs of a fork and in the interior of a spoon (a); a boundary point of a plate and a point of the handle of a spoon (b); the joint point of a spoon and a boundary point of a plate (c). We postpone the discussion of a determination of the numbers of rounds, and thus a winning strategy for Spoiler, to the Adequacy Theorem in the next section.

Example 15 (Infinite games) In this last example, see Figure 11, the starting points are such that Duplicator always has a winning strategy, independently from the number of rounds. We can see this intuitively. Whatever open Spoiler chooses around the starting point of one of the two models Duplicator can keep picking one with the 'same kind' of points. Spoons are the same if 'taken' from the same point, but also a knife and a spoon are if both 'taken' from the handle. Indeed, model comparison games may also go on for ever.

We now provide the full set of formal rules for topological games and the precise notion of a win for either player.

Definition 14 (Topological game) Consider two models $\langle X, O, \nu \rangle$, $\langle X', O', \nu' \rangle$, a natural number n and two starting points $x_1 \in X$ and $x'_1 \in X'$. A *topological game* of length n, with starting points x_1, x'_1—notation $TG(X, X', n, x_1, x'_1)$—consists of n rounds between two players: Spoiler and Duplicator. Each round consists of the following:

(i) Spoiler chooses a model X_s and an open o_s containing the current point x_s of that model

(ii) Duplicator chooses an open o_d in the other model X_d containing the current point x_d of that model

(iii) Spoiler picks a point $\bar{x}_d$ in Duplicator's open o_d in the X_d model

(iv) Duplicator replies by picking a point $\bar{x}_s$ in Spoiler's open o_s in X_s

The points $\bar{x}_s$ and $\bar{x}_d$ become the new current points of the X_s and X_d models, respectively. By this succession of actions, two sequences are built. After n rounds:

$$\{x_1, o_1, x_2, o_2, \ldots, o_{n-1}, x_n\}$$

$$\{x'_1, o'_1, x'_2, o'_2, \ldots, o'_{n-1}, x'_n\}$$

with $x_i \in o_i$, $x_{i+1} \in o_i$, and $o_i \in O$ (analogously for the second sequence). The opens in the game sequence do not play any role in determining which player wins, but it is visually convenient to keep track of them in the development of the game.

After n rounds, if x_i and x'_i (with $i \in [1, n]$) satisfy the same atoms, Duplicator *wins.* (Note that Spoiler already wins 'en route', if Duplicator fails to maintain the atomic match.) A *winning strategy* for Duplicator is a function from any sequence of moves by Spoiler to appropriate responses which always ends in a win for Duplicator. Dually for Spoiler.

An *infinite topological game* is one where no finite limit is set to the number of rounds. Duplicator wins if the matched points always satisfy the same atoms.

3.4.2 Strategies and modal formulas

We can now formulate the results that tie together topological games with distinctions expressible in our modal language. First, we define the notion of modal rank, then we dive into the main result of this section, the Adequacy Theorem.

Definition 15 The *modal rank* of a formula is the maximum number of nested modal operators that appear in it.

Example 16 The modal rank of formulas occurring in Figure 1: p, $\Box p$, $p \wedge \neg \Box p$, $\Diamond \Box p$, $p \wedge \neg \Diamond \Box p$, $\Diamond \Box p \wedge \Diamond (p \wedge \neg \Diamond \Box p)$ are 0, 1, 1, 2, 2, and 3, respectively.

Theorem 8 (Adequacy) Duplicator has a winning strategy in $TG(X, X', n, x, x')$ iff x and x' satisfy the same formulas of modal rank up to n.

Proof The left to right direction is proven by induction on the length n of the game $TG(X, X', n, x, x')$. If $n = 0$ and Duplicator has a winning strategy, this means that the points x, x' satisfy the same propositional letters, and hence the same boolean combinations of propositional letters, i.e. the same modal formulas of modal rank 0. The inductive step. Suppose that Duplicator has a winning strategy σ in $TG(X, X', n, x, x')$. We want to show that $X, x \models \varphi$ iff $X', x' \models \varphi$, when the modal rank of φ is n. Now, by simple syntactic inspection, φ must be a boolean combination of formulas of the form $\Box\psi$ where ψ has modal rank less or equal to $n - 1$. Thus, it suffices to prove that $X, x \models \Box\psi$ iff $X', x' \models \Box\psi$. Without loss of generality, let us consider the first model. Suppose that $X, x \models \Box\psi$. By the truth definition there exists an open o (with $x \in o$) such that $\forall u \in o : X, z \models \psi$. Now, assume that the n-round game starts with Spoiler choosing o in X. Using the strategy σ, Duplicator can pick an open o' such that $x' \in o'$ and $\forall u' \in o' : X, u' \models \psi$. Now Spoiler can pick any point u' in o'. Duplicator can use the information in σ to respond with a point $u \in o$, concluding the first round, so that the remaining strategy σ' is still winning for Duplicator in $TG(X, X', n - 1, u, u')$. By the inductive hypothesis, the fact that $X, u \models \psi$ (where ψ has modal rank $n - 1$) implies that $X', x' \models \psi$. Thus we have shown that all $u' \in o'$ satisfy ψ, and hence $X', x' \models \Box\psi$. The other direction is analogous.

The right to left direction is again proven by induction on n. If $n = 0$, then x and x' satisfy the same non-modal formulas. In particular, they satisfy the same atoms, which is winning for Duplicator, by the definition of topological game.

Now for the inductive step. Without loss of generality, let us assume that Spoiler picks an open set o containing x in X in the first round of $TG(X, X', n, x, x')$ game. Now, take the set $\{\mathrm{DES}_{n-1}(z) : z \in o\}$, where $\mathrm{DES}_{n-1}(z)$ denotes all the formulas up to modal rank $n-1$ satisfied at z. This set is not finite per se, but we can simply prove the following

Fact 17 (Logical Finiteness) There are only finitely many formulas of modal depth k up to logical equivalence.

Therefore, we can write one boolean formula to describe this open set o, namely $\bigvee \bigwedge \mathrm{DES}_{n-1}(z)$. Since this is true for all $z \in o$, by the truth definition we have that $X, x \models \Box \bigvee \bigwedge \mathrm{DES}_{n-1}(z)$ (a formula of modal rank n). By hypothesis, x and x' satisfy the same modal formulas of modal rank n, so $X', x' \models \Box \bigvee \bigwedge \mathrm{DES}_{n-1}(z)$. This last fact, together with the truth definition implies that there exists an open o' such that $\forall z' \in o' : X', z' \models \bigvee \bigwedge \mathrm{DES}_{n-1}(z)$. This is the open that Duplicator must choose to reply to Spoiler's move. Now Spoiler can pick any point u' in o'. Such a point satisfies at least one disjunct $\bigwedge \mathrm{DES}_{n-1}(z)$, and we let Duplicator respond with $z \in o$. As a result of this first round, z, u' satisfy the same modal formulas up to modal depth $n-1$. Hence by the inductive hypothesis, Duplicator has a winning strategy for $TG(X, X', n-1, z, u')$. Putting this together with our first instruction, we have a winning strategy for Duplicator in the n-round game. QED

Remark 18 (Shrinking opens) From the point of view of winning strategies, it is in Spoiler's interest to pick opens as small as possible, making it difficult for Duplicator to respond in the final move of a round. But it is also Duplicator's interest to pick a small open! Having a small open makes him less vulnerable to Spoiler's selection. This informal reasoning shows the following. If Duplicator has a winning strategy, then he has one for which every open he chooses during the game is contained in the previously chosen. The same is true for Spoiler. Thus, in implementing a perfect player, one can restrict the search of opens to a decreasing sequence, and not in the entire space.

Remark 19 (Infinite games) As we stated already, the definition of topological game easily extends to infinite games. Just let $n \to \infty$ and the sequences $x_i \in o_i$, $x'_i \in o'_i$ be infinite. If Duplicator has a winning strategy in the infinite game $TG(X, X', \infty, x, x')$, we retrieve an earlier notion: x, x' are bisimilar. Note that Duplicator's winning strategy in the infinite game on the points x and x' guarantees only a bisimulation between the two points, not between the whole spaces. We will need the 'global' existential and universal modalities of Section 4.1 to achieve the latter.

An analogue of the Adequacy Theorem runs as follows.

Fact 20 The following three assertions are equivalent:

(i) Duplicator has a winning strategy in the infinite game $TG(X, X', \infty, x, x')$,

(ii) x, x' satisfy the same formulas in the infinitary modal language,

(iii) x, x' are bisimilar.

The infinitary language mentioned above is the extension of $\mathcal{L}$ allowing arbitrary (i.e., infinite) conjunctions and disjunctions. In [Barwise and Moss, 1996], important results are obtained for infinitary modal languages, which go through to the case of infinitary $\mathcal{L}$ with the topological interpretation. For example, the construction of complete infinitary modal descriptions for bisimulation invariance classes carries over to our case.

Remark 21 (Matching strategies with formulas) The proof of the Adequacy Theorem yields more constructive information than we have stated so far. In particular, there exists a *direct correspondence* between

1. winning strategies σ for Spoiler in $TG(X, X', n, x, x')$
2. modal formulas φ of depth n such that $X, x \models \varphi$, $X', x' \models \varphi$

For details, we refer to [van Benthem, 1999]. Here we only give a concrete illustration. The strategies used by Spoiler in Figures 9, 10, 11 are immediately clear and linked to formulas true or not on the models. Let us take again the case of the spoons called p (Figure 9). In the left game, $\Box p$ is true of the starting point of the right spoon, and its negation $\Diamond\neg p$ is true of the starting point of the other spoon. The modal depth of these formulas is one and therefore Spoiler can win in one turn. In the central case, a distinguishing formula is $\neg\Diamond\Box p$, which holds on the starting point on the left spoon, but not of the starting point of the right one. The modal depth in this case is 2, which is the number of rounds that Spoiler needed to win the game. Finally, the formula of modal rank 3 that is true of the point on the left spoon of the leftmost game is: $\Diamond(p \wedge \neg\Diamond\Box p)$. The negation of this formulas is true on the other starting point, thus justifying Spoiler's winning strategy in 3 turns.

4 Extended Modal Logic of Space

Extended modal languages can be of many sorts: more of the first-order language of our models, higher-order extensions of that language, or multi-modal extensions. In this section, we keep the basic topological language introduced previously and look at multi-modal extensions, increasing expressive power, but still keeping the logic tractable.

4.1 Universal modality

The biggest limitation of the language $\mathcal{L}$ is its locality. It only talks about points and their neighbourhoods. Now we want to move from points to regions, allowing us to express spatial relations and constraints on these. To this end we consider an extension with a universal modality U (and its dual existential E), which can express global relations between regions, such as overlap, touch, and external connection. Here is the semantic definition of these two dual modalities in terms of topological models.

Definition 16 (Topological semantics of $\mathcal{L}(E|U)$) A topological model M is defined as for $\mathcal{L}$, with this additional definition for the two modal operators E, U at some point $x \in X$:

$$M, x \models E\varphi \quad \text{iff} \quad \exists y \in X,\ M, y \models \varphi$$
$$M, x \models U\varphi \quad \text{iff} \quad \forall y \in X : M, y \models \varphi.$$

Note that on the left-hand side of the definitions the point x has become irrelevant and could be omitted. E.g., in the universal definition, it is one specific open that should make the formula φ true, namely the whole topological space.

Various reasons motivate this extension of the simple language $\mathcal{L}$.

- The interest in total bisimulations between topological models (and therefore morphisms between topological spaces) rather than local bisimulations on a restricted number of points. We have seen an example of the power of global reasoning when we considered information transfer across topological spaces. (Cf. Remark 11 on preservation of connectedness under continuous maps.)
- With our initial visual patterns, one compares full regions, rather than single points in space. One wants to pick up a spoon, not 'grab' a point on the handle.
- Classical concepts introduced in the mereotopology literature, such as connection, part and external connection, can now be defined. Though there is no universal agreement on these notions, [Cohn and Varzi, 1998] identify the various approaches and what distinguishes them. E.g., the definition of connection between regions may require points to be shared by the sets, their closure or at least the closure of one of them. It is interesting to note that within the language $\mathcal{L}(E|U)$ all these different kinds could be defined. Here are some examples.

`C(x, y)`	connected	$E(x \wedge y)$
`P(x, y)`	part	$U(x \rightarrow y)$
`EC(x, y)`	external connection	$E(\Diamond x \wedge \Diamond y) \wedge \neg E(\Box x \wedge \Box y)$

The behavior of the U and E operators is that of the modal logic S5. Here are some typical valid principles for our new operations.

$$E\varphi \leftrightarrow \neg U \neg \varphi \qquad \text{(Dual.)}$$
$$U(\varphi \rightarrow \psi) \rightarrow (U\varphi \rightarrow U\psi) \qquad \text{(K)}$$
$$U\varphi \rightarrow \varphi \qquad \text{(T)}$$
$$U\varphi \rightarrow UU\varphi \qquad \text{(4)}$$
$$\varphi \rightarrow UE\varphi \qquad \text{(B)}$$

Of course, we also have 'connection principles' such as

$$\Diamond\varphi \rightarrow E\varphi$$

The above naturally suggests a normal form: nesting of universal and existential modalities is redundant. More precisely, we can prove the following proposition.

Proposition 9 *Every formula of $\mathcal{L}(E|U)$ is equivalent to one without nested occurrences of E, U.*

Proof Here is one way of seeing this. The following principle is valid in the semantics of $\mathcal{L}(E|U)$. Let $\varphi[E\psi]$ be any formula containing a subformula $E\psi$. Then we have

$$\varphi[E\psi] \leftrightarrow (E\psi \wedge \varphi[\top]) \vee (\neg E\psi \wedge \varphi[\bot])$$

The reason is that subformulas $E\psi$ are globally true or false, across modalities $\Box, \Diamond, E, U$. This observation also produces an effective algorithm for finding the normal form. E.g.

$$\begin{gathered}
\Box(Ep \wedge \neg\Box Eq) \leftrightarrow \\
(Ep \wedge \Box(\top \wedge \neg\Box Eq)) \vee (\neg Ep \wedge \Box(\bot \wedge \neg\Box Eq)) \leftrightarrow \\
(Ep \wedge \Box\neg\Box Eq) \vee (\neg Ep \wedge \Box\bot) \leftrightarrow \\
(Ep \wedge ((Eq \wedge \Box\neg\Box\top) \vee (\neg Eq \wedge \Box\neg\Box\bot)) \vee (\neg Ep \wedge \bot) \leftrightarrow \\
(Ep \wedge Eq \wedge \Box\neg\top) \vee (Ep \wedge \neg Eq \wedge \Box\neg\bot)) \vee \bot \leftrightarrow \\
(Ep \wedge Eq \wedge \bot) \vee (Ep \wedge \neg Eq) \leftrightarrow \\
Ep \wedge \neg Eq
\end{gathered}$$

QED

Another way of seeing this is by proving some more familiar reduction principles (either in the semantics, or from the given axioms), such as

$$\Diamond E\varphi \leftrightarrow E\varphi, \qquad \Box E\varphi \leftrightarrow E\varphi$$

Note that we do not get, e.g., $E\Box\varphi \leftrightarrow \Box\varphi$ or $E\Box\varphi \leftrightarrow E\varphi$. The normal forms that we obtain may be described as follows,

$$\bigvee\bigwedge [U|E]\varphi$$

where $[U|E]$ is U or E or nothing, and φ is a formula of our original modal language $\mathcal{L}$.

4.2 Bisimulation and games: moving in space randomly

The concepts of bisimulation and topological game introduced for the modal basic language $\mathcal{L}$ easily extend to $\mathcal{L}(E|U)$ (this is part of their *general* attraction). This time, bisimulations must talk about points 'at any distance', somewhat like teleporting randomly. Interestingly (but not surprisingly) the definition carries the same meaning of the definition of total topological bisimulations for the basic language $\mathcal{L}$. Forcing the bisimulation to be defined for all points in the model is like introducing the universal—existential modalities.

Definition 17 (Topological bisimulation for $\mathcal{L}(E|U)$) Given two topological models $\langle X, O, \nu\rangle$, $\langle X', O', \nu'\rangle$, a *total topological bisimulation* is a non-empty relation $\leftrightarrows \subseteq X \times X'$ such that:

(i) the three conditions of Definition 13 hold, as well as

(ii) (a) *(Forth condition):* $\forall x \in X\ \exists x' \in X' : x \leftrightarrows x'$

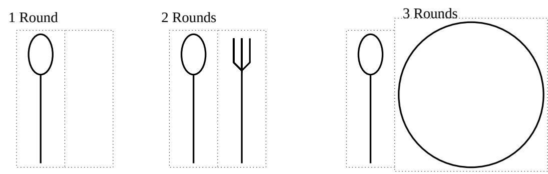

Figure 12: Plays of topological games for the universal language $\mathcal{L}(E|U)$. Above the two models is the number of rounds needed by Spoiler to win.

(b) *(Back condition):* $\forall x' \in X' \; \exists x \in X : x \leftrightharpoons x'$

Here are the analogue for the preservation results: Theorems 1, 2.

Theorem 10 *Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two models, $x \in X$, and $x' \in X'$ bisimilar points. Then, for any modal formula φ in $\mathcal{L}(E|U)$, $M, x \models \varphi$ iff $M', x' \models \varphi$. In words, extended modal formulas are invariant under total bisimulations.*

Theorem 11 *Let $M = \langle X, O, \nu \rangle$, $M' = \langle X', O', \nu' \rangle$ be two finite models, $x \in X$, and $x' \in X'$ such that for every φ in $\mathcal{L}(E|U)$, $M, x \models \varphi$ iff $M', x' \models \varphi$. Then there exists a total bisimulation between M and M' connecting x and x'. In words again, finite modally equivalent models are totally bisimilar.*

For the same reasons, and in the same style as we had for the language $\mathcal{L}$, we now define topological comparison games for the universal language $\mathcal{L}(E|U)$. The game is defined in the same way, only a new type of round is introduced to take into account the new modalities U and E. Accordingly the Adequacy Theorem will be modified later. There are two types of rounds. Those of Definition 14 we call *local* rounds and the new ones, *global* rounds. A global round consists of Spoiler choosing a model and a point (not necessarily related to any previously played point) and Duplicator replying with a point in the other model. If there is no initial match, the first round of a play is always a global round, while during the game Spoiler is free to choose which type of round he wants to engage in.

Example 22 (Comparing cutlery) As we did in Section 3.4, we can play on 'table items', i.e. regions in topological spaces. Differently from the local games, one may notice that there is no starting points in the two models. Spoiler can decide where to play, by means of a global move. By this added freedom, Spoiler can win games in which the players compare spoons and forks, spoons and plates or even spoons with an empty table cloth.

Similar to before, we have a way of tying Spoiler's winning strategies with formulas (of $\mathcal{L}(E|U)$) true in the models. Note that the formulas can be true in the entire model, not in only two particular starting points, as before. This reflects our earlier observation that E or U formulas are really true across a model.

Referring to Figure 12, we can write down a distinguishing formula of the appropriate multi-modal rank that is true in one model but not in the other. In the case of the 1 round game, Spoiler can win in one round since on the right model the formula Ep is true, while its negation is true in the other model. Think of it as the empty table which should be set, so there is no region p yet: $U\neg p$.

By a similar reasoning we can write the formula $E\Box p$ (the interior of p is non empty) for the 2 round game. This formula is only true in the left model. For the 3 round game, a distinguishing formula is $U(p \leftrightarrow \Diamond\Box p)$. This formula encodes closed regularity of regions, i.e. coincidence with the closure of its interior. This formula is true for the plate on the right but not for the spoon on the left. The negation of the regularity formula can be written as $E(p\wedge\Box\Diamond\neg p)\vee E(\neg p\wedge\Diamond\Box p))$. The first half of this accounts for external lower-dimensional spikes in the region p, the second for lower dimensional cracks. In the case of the spoon, the handle is a lower dimensional spike.

Definition 18 (Topological game) Given two models $\langle X, O, \nu\rangle$, $\langle X', O', \nu'\rangle$, a natural number n. A *topological game* of length n—notation $TG(X, X', n)$—consists of a interaction between two players: Spoiler and Duplicator. The two players move alternatively. Spoiler is granted the first move. There are two types of rounds, local and global.

- **local**

  as in Definition 14

- **global**

  (i) Spoiler chooses a model X_s and picks a point $\bar{x}_s$ anywhere in X_s

  (ii) Duplicator chooses a point $\bar{x}_d$ anywhere in the other model X_d

A play always begins with a global round. During the game Spoiler can decide whether to engage in a local or a global round, forcing Duplicator to follow his choice. (We can easily modify the game, however, to also allow initial starting points x, x'.)

By this succession of actions, two sequences are built. The form of this sequence after n rounds is:

$$\{x_1, x_2, x_3, \ldots, x_n\}$$

$$\{x'_1, x'_2, x'_3, \ldots, x'_n\}$$

After n rounds, if x_i and x'_i (with $i \in [1, n]$) satisfy the same propositional atoms, Duplicator *wins*, if not, Spoiler wins. A *winning strategy* for Duplicator is a function from any sequence of moves by Spoiler to appropriate responses which always ends in a win for him. Spoiler's winning strategies are defined dually.

As before a notion of adequacy for this game can be defined and proved. To this end, an extended concept of modal rank is needed.

Definition 19 (Multi-modal rank) The *multi-modal rank* of a $\mathcal{L}(E|U)$ formula is the maximum number of nested modal operators appearing in it (i.e. $\Box$, $\Diamond$, U and E modalities).

Theorem 12 (Adequacy) Duplicator has a winning strategy for n rounds in $TG(X, X', n)$ iff X and X' satisfy the same formulas of multi-modal rank at most n.

Remark 23 (Infinite games) The definition can be easily extended to infinite games. Just let $n \to \infty$ and hence the sequences x_n, x'_n be infinite. The Adequacy Theorem is still valid. Duplicator has a winning strategy in the infinite round game iff the models are bisimilar in our extended sense.

Remark 24 (Strategies and normal forms) From the practical point of view of playing topological games, Spoiler should bear in mind that identifying formulas that differentiate the models is not enough. Spoiler may consume too many turns if he is using a long formula (in terms of multi-modal depth) which has a shorter logical equivalent. Similarly Duplicator may have the illusion of a win, if he makes the same mistake. Once 'difference formulas' are identified in the models they should be reduced to logically equivalent ones with the lowest multi-modal depth. Normal forms are of great help for this purpose. E.g., here is the game-theoretic content of our earlier normal form for $\mathcal{L}(E|U)$. Having only one 'outermost' existential or universal modality means that Spoiler need engage only once in a global round. Furthermore, since such a modality is the first to appear, that is the first type of move Spoiler should play. This can also be seen directly in the game. If Spoiler engages in more than one global round, it is like jumping around the space, not having understood were the difference between the models resides.

One might try to extend this line of reasoning to the inner $\mathcal{L}$ part. After all, S4 validates reduction laws like $\Box\Box\varphi \leftrightarrow \Box\varphi$, or $\Box\Diamond\Box\Diamond\varphi \leftrightarrow \Box\Diamond\varphi$. Can this be used to simplify Spoiler's strategies? We have not been able to find a general principle here that would be of much use.

The use of normal forms can lead to a redefinition of the rules of the topological game. The new game would have always one starting global round and thereafter only local rounds.

4.3 General picture: first- and higher-order languages

So far we have considered a very simple modal language on topological spaces. We have extended it with universal and existential modalities, which is about the simplest extra one can think of. U, E modalities provide for a clean description of many topological properties, but other extensions are possible and interesting. Such extensions can go in two distinct directions. It is possible to *increase the logical power* of the language, keeping the type of spaces fixed. Or, it is possible to take spaces with *more geometrical structure*. In the next section we see the results of the latter move, moving from modal logics of topology to modal logics of affine geometry. Here though, we point out some interesting extensions of the logical power of the language over topological spaces.

We have already seen one such extension in logical power. In Fact 20, we made our modal language infinitary to get a 'perfect match' for topological bisimulations and games, $M, x \leftrightarrows M', x'$ iff $M, x \equiv_{\text{inf.ML}} M', x'$.

More complex, but rather natural are second-order extensions, involving quantification over sets of points as well. The naturality comes from two facts.

First, taking a closer look at the truth definition of $\Box\varphi$ in $\mathcal{L}$,

$$\Box\varphi \leftrightarrow \exists o \in O : x \in o\, \forall y \in o : \varphi(y)$$

one notices its second-order nature. The first quantification is over (open) sets, while the second is over points. Second, most interesting topological properties refer to open sets, again involving quantification over sets rather than points. The second-order language has the following vocabulary:

$\forall x$	quantification over points
$\forall A$	quantifications over sets of points
$x = y$	identity
$x \in A$	membership of points in sets
$O(A)$	predicate of openness of sets

This language really captures all fundamental topological notions. Here are two illustrations. In topology, homeomorphism is the basic 'equivalence' among spaces, it preserves the main properties. Accordingly, we expect homeomorphisms to preserve truth of formulas:

Fact 25 Formulas of the above second-order language are preserved under homeomorphisms.

Here is another piece of evidence.

Fact 26 All topological separation axioms T_i (with $0 \leq i \leq 4$) are expressible in the second-order language.

The topological taxonomy of spaces (see Definition 6) is built in terms of points, open sets, membership of points to opens, or disjointness of sets. All these items are expressible in the second-order language. For example, one can express the T_2 axiom (defining the Hausdorff spaces) in the following way:

$$\forall x, y : (x \neq y \,\rightarrow\, \exists A, B : O(A) \wedge O(B) \wedge \neg\exists z (z \in A \wedge z \in B) \wedge x \in A \wedge y \in B)$$

Similarly we can write the definition of the axiom for T_4 spaces:

$$\forall C, D : (O(\neg C) \wedge O(\neg D) \wedge \neg\exists z (z \in C \wedge z \in D)$$
$$\exists A, B : O(A) \wedge O(B) \wedge \neg\exists z (z \in A \wedge z \in B) \wedge \forall x \in C\; x \in A \wedge \forall x \in D\; x \in B)$$

Such a second-order language is extremely expressive, but of little practical use. Are there interesting fragments that lie in between it and our first very simple modal language? The extended modal language $\mathcal{L}(E|U)$ was a good example of this kind of 'guarded behavior', but not the only one. We mention two further examples of nice 'modal fragments'.

First, we borrow from temporal logics and define a sparse version of *Since* and *Until*.

$$\mathcal{U}\psi\varphi : \quad \exists A : O(A) \wedge x \in A \wedge \forall y \in A. \varphi(y) \wedge$$
$$\forall z (z \text{ is on the boundary of } A \rightarrow \psi(z))$$

The idea is of a fence ψ surrounding an open set φ. Temporal logic on $I\!R^1$, from which we took our inspiration, is a special case of this definition.

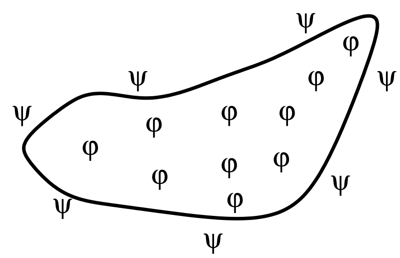

Figure 13: The region defined by $\mathcal{U}\psi\varphi$.

As a final example, we 'decompose' the modality of $\mathcal{L}$ accounting for its two quantifiers in a *two-level modal logic*. The idea is to have two sorts, points and sets, with different modalities on each sort (together with their duals). Intuitively the modality on the sets accounts for the second-order quantification appearing first in the semantic definition of $\Box$; while the modality on the points accounts for the first-order quantification. Formally,

$\Box\varphi$ is decomposed into $\langle s\rangle[p]\,\varphi$ with:

$x \models \langle s\rangle\varphi$ iff $\exists A : O(A) \wedge x \in A \wedge A \models \varphi$ and $A \models \langle p\rangle\,\psi$ iff $\exists x : x \in A \wedge x \models \psi$

Note that we now must have two different types of formulas: 'point-formulas' and 'set-formulas'. Obviously, this gives us some more expressive power than in $\mathcal{L}$. E.g. we can define $E\varphi$ as $\langle s\rangle\langle p\rangle\,\varphi$.

The two level language affords a nice new view on the S4-behavior of our original topological interpretation.

E.g., consider the behavior of the S4 axioms.

(1) $\Box\varphi \to \varphi$. This becomes $\langle s\rangle[p]\,\varphi \to \varphi$,

which, in a two-sorted modal logic, expresses the fact that the accessibility relation for s is contained in the *converse* of that for p. This is a natural connection between '$x \in A$' and '$A \ni x$'. Note that *reflexivity* vanishes!

(2) $\Box\varphi \to \Box\Box\varphi$. This becomes $\langle s\rangle[p]\,\varphi \to \langle s\rangle[p]\langle s\rangle[p]\,\varphi$,

Where, *transitivity* vanishes and we rather see the following phenomenon. $[p]\,\varphi \to [p]\langle s\rangle[p]\,\varphi$ holds as an instance of the Conversion axiom $\alpha \to [p]\langle s\rangle\alpha$, which expresses that R_p is contained in the converse relation $\breve{R}_s$. The rest is an application of the valid modal base rule "from $\gamma \to \sigma$ to $\langle s\rangle\gamma \to \langle s\rangle\sigma$."

(3) $\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$. This becomes $\langle s\rangle[p]\,\varphi \wedge \langle s\rangle[p]\,\psi \to \langle s\rangle[p]\,(\varphi \wedge \psi)$,

a principle which has no obvious meaning in a two-sorted modal language. We can analyze its meaning by standard *frame correspondence* techniques, however [Blackburn et al., 1999] to obtain:

$$\forall A, B : ((x \in A \wedge x \in B) \to \exists C : (x \in C \wedge \forall y \in C : y \in A \vee y \in B)).$$

5 Enriching Spatial Structure: Convexity and Betweenness

Our analysis so far explored topological spaces and modal languages expressing topological properties of visual patterns. We now take a look at more structured spaces, again both from the point of view of the models and of the visual languages. Our interest is in 'shapes' which takes us closer (asymptotically) to geometry. In line with our general philosophy, the extension steps remain modest.

More specifically, in what follows the crucial new aspect is *convexity* of regions, based on an underlying geometrical primitive relation of *betweenness*. Returning to our cutlery example, the objects in earlier table setting pictures naturally induce convex regions, either 'inside' or 'between' them. Thus, we get either convex regions, or finite unions of these, extending our earlier topological algebra. This is the view 'from below', adding expressive power. But also 'from above', we can look at what is involved in actual visual scenes. Not every wild subset of points in a space will occur as a reasonable visual region. In particular, some substantial literature deals with countable unions of convex polygons [Pratt and Lemon, 1997]. Thus, we will investigate two things: (a) the addition of the relevant structure, in a modal way, (b) possible restrictions to well-behaved regions.

There are various reasons for choosing finite unions of convex sets as basic elements of our spaces. First, for most computer applications these seems an ontology expressive enough to capture all the relevant details (think for example at Geographic Information Systems (GIS), or spatial databases). Secondly, qualitative spatial notions involved in spatial and visual reasoning can be often encoded in terms of such an ontology. Third, that one object is composed of many convex subregions seems to be similar to the way that we reason about objects. When thinking of a fork one thinks of its various prongs and its handle. The prongs that penetrate a one piece square-ish portion of a pizza. Furthermore, consider the fall of the plate and its braking. One ends up with several connected pieces all of which build up the plate. Last, but not least, object occlusion plays an important role in vision. Often objects in visual scenes appear partially occluded, thus we are presented with several disjoint portions of a region. Think of taking the fork from the table, keeping it by the handle. It is very likely that you see the upper part of the fork with its prongs and the bottom part of the handle. Two distinct pieces. Even more, if you spread out your fingers on the handle you are going to see several disjoint portions of the handle.

The preceding ontological choice provides inspiration for the next expressivity enhancement: *convexity modalities*.

5.1 Modal affine geometries

We now expand our topological models $\langle X, O, \nu\rangle$ to more general structures $\langle X, O, —, \nu\rangle$, where a ternary betweenness relation

$$x \in y—z$$

stands for: 'x lies on the segment with extreme points y and z' (y, z included). Natural examples are the usual spaces $\mathbb{R}^n$, but genuinely different 'affine' spaces

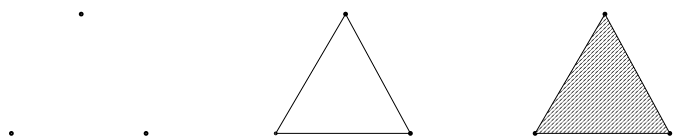

Figure 14: In a two dimensional space, the sequential application of the convexity operator to three non aligned points results in two different regions: a triangle (only the sides and corners of it) and the filled triangle.

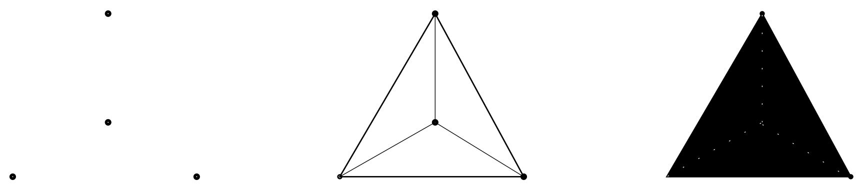

Figure 15: In a three dimensional space, the sequential application of the betweenness operator to four non coplanar points also yields two more regions: the wire-frame of a tetrahedron, and again its full interior.

are possible, too—such as Minkowski space (cf. [van Benthem, 1983]). In the case of $\mathbb{R}^n$, the segment is the usual notion of all points collinear with y and z and who's coordinates are in between those of y and z. But one can also identify an abstract notion of a segment in topological space, using the abstract relation among points.

5.1.1 Language and models

A first modal language arises simply as follows, using a new operator C,

$$M, x \models C\varphi \quad \text{iff} \quad \exists y, z : M, y \models \varphi \wedge M, z \models \varphi \wedge x \in y\text{—}z$$

As a result, $C\varphi$ defines the convex closure of the region φ. This simple extension enriches our expressive power considerably.

Example 27 [Distinguishing dimensions] In our modal topological languages, no difference will be apparent between $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ etc., by Tarski's Theorem. But using C, we do see a difference. For instance, the principle

$$CC\varphi \leftrightarrow C\varphi$$

holds in $\mathbb{R}$, not in $\mathbb{R}^2$ (cf. the triangle in Figure 14). What does hold in $\mathbb{R}^2$ is $CCC\varphi \leftrightarrow CC\varphi$. And this principle also holds in $\mathbb{R}^3$, witness the illustration of the tetrahedron in Figure 15.

Thus, we have an interesting topological-geometrical language now, with operators $\Box, E, C$, which can be studied with our earlier techniques. Nevertheless, from a technical point of view this language has a drawback: the new modality C is not *distributive*. We see this in Figure 16. We also illustrate a case that shows how the modality C is not distributive over intersections, Figure 17.

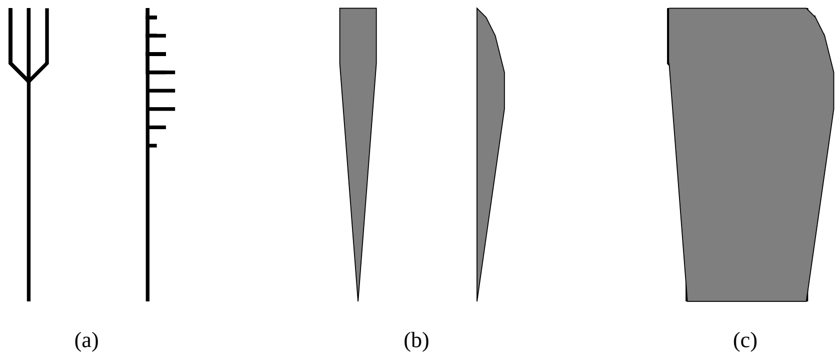

Figure 16: In (a) a fork and a knife `f,k` are displayed. If we take the convex closure of the two cutlery items and then the union, we get two distinct regions (b). On the other hand, if we consider the union of the the fork and the knife and then take the convex closure, we get a bigger region, as illustrated in (c). In formulas, $C(\mathtt{f}\vee\mathtt{k}) \not\leftrightarrow C\mathtt{f}\vee C\mathtt{k}$.

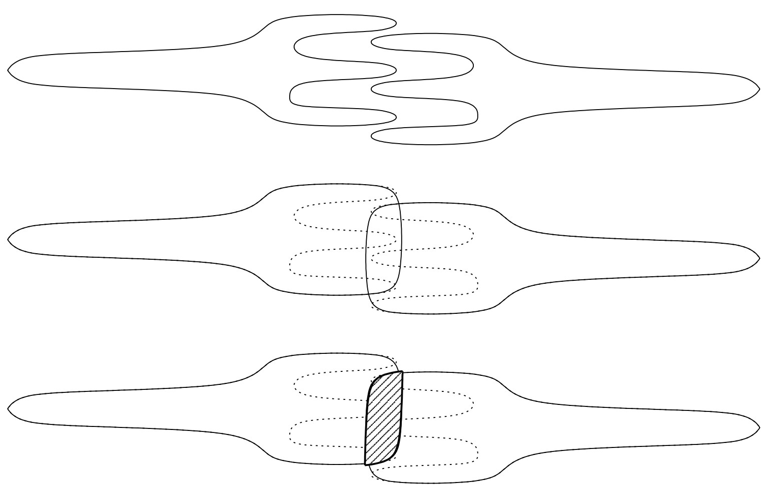

Figure 17: Consider two forks (top), `f1,f2`. If we take the convex closure of the two forks (center) and then intersect the result we get the striped region (bottom). On the other hand, if we consider the intersection of the two forks we get the empty region, the convex closure of which is the empty region. In formulas, $C(\mathtt{f1}\wedge\mathtt{f2}) \not\leftrightarrow C\mathtt{f1}\wedge C\mathtt{f2}$.

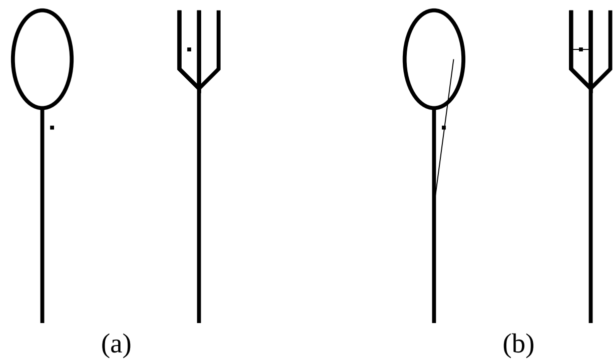

Figure 18: (a) Two cutlery items. A spoon with a point lying nearby and a fork with a point in between its prongs. (b) The point near the spoon lies in between an interior point and a boundary point of it. In the case of the fork the point lies in between two boundary points.

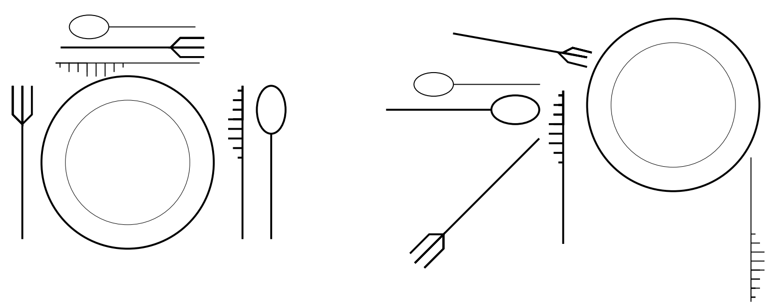

Figure 19: With the added betweenness operator we can distinguish between a properly dressed table and a 'messy' one.

The solution is to introduce a *binary* convexity modality $\bowtie$:

$$M, x \models \bowtie(\varphi, \psi) \quad \text{iff} \quad \exists y, z : M, y \models \varphi \wedge M, z \models \psi \wedge x \in y \text{—} z$$

The latter defines C as follow:

$$C\varphi = \bowtie(\varphi, \varphi).$$

But in addition, it satisfies the two distribution laws:

$$\bowtie(\varphi_1 \vee \varphi_2, \psi) \leftrightarrow \bowtie(\varphi_1, \psi) \vee \bowtie(\varphi_2, \psi)$$
$$\bowtie(\psi, \varphi_1 \vee \varphi_2) \leftrightarrow \bowtie(\psi, \varphi_1,) \vee \bowtie(\psi, \varphi_2,)$$

In what follows, we shall work with the latter language.

5.1.2 Simulations and games

The technology of bisimulations and games extends to the new language in a straightforward manner. We state a number of definitions and results, all obtainable by the earlier arguments.

Definition 20 (Affine Bisimulation) Given two topological models $\langle X, O, —, \nu\rangle$, $\langle X', O', —, \nu'\rangle$, an *affine bisimulation* is a non-empty relation $\leftrightarrows \subseteq X \times X'$ such that:

(i) the five conditions of Definition 17 hold

(ii) (a) *(forth condition):* $x \in y—z \Rightarrow \exists y'—z' : x' \in y'—z'$ and $y \leftrightarrows y'$ and $z \leftrightarrows z'$

(b) *(back condition):* $x' \in y'—z' \Rightarrow \exists y—z : x \in y—z$ and $y \leftrightarrows y'$ and $z \leftrightarrows z'$

where $x, y, z \in X$ and $x', y', z' \in X'$.

Thus, our affine bisimulations are total. (One can easily devise local versions, too.) The reader is invited to visualize the back and forth moves here: we can now also talk about 'geometrical contexts' for the points involved.

Definition 21 (Affine game) Given two models $\langle X, O, —, \nu\rangle, \langle X', O', —, \nu'\rangle$, an *affine game* of length n—notation $AG(X, X', n)$—has three types of rounds, local, global, and affine, for Spoiler and Duplicator:

- **local**
  as in Definition 14
- **global**
  as in Definition 18
- **affine**
  (i) Spoiler chooses a model X_s and a segment containing the current point x_s with end points y_s and z_s, such that $x_s \in y_s—z_s$
  (ii) Duplicator chooses a segment containing the current point x_d with end points y_d and z_d, $x_d \in y_d—z_d$ in the other model X_d
  (iii) Spoiler now picks either y_d or z_d which becomes the new current point of the model X_s
  (iv) Duplicator replies by picking either y_d or z_d which becomes the new current point of the model X_d

A play always begins with a global round. During the game Spoiler can decide whether to engage in a local, global or affine round, forcing Duplicator to follow his choice. The further description of game states and winning conventions is exactly like before.

We do not restate the relevant Adequacy Theorem, but it should be evident from earlier sections. Instead, we merely discuss examples.

Example 28 (Throwing cutlery) First, we note that we can now distinguish among cutlery at the 'frame' level. We do not need valuations to 'name' the regions differently. The regions describing a fork and a spoon are 'affinely' different. Consider Figure 18. The spoon is different from a fork from an affine point of view. Take the points highlighted in the picture. For the one near the spoon, $E(\bowtie(\Box\varphi, \Diamond\Box\neg\varphi) \wedge \neg\varphi)$ is true. The point lies outside the spoon but in between an interior point and a boundary point. This is not true of the point in between the prongs of the fork, or for any other point of/or near the fork. Look now at the tables of Figure 19. If topologically the two set tables are the same, then adding betweenness they are not. On the left-hand side, the plate lies in between the fork and the spoon, and so on. On the right instead, the cutlery has been 'thrown' and the plate is not between the fork and the spoon.

5.1.3 Some valid principles of affine reasoning

Without attempting to describe a complete logic, we identify some typical valid principles of reasoning with interior and convexity. First, as for general principles, we have the earlier mentioned distribution:

$$\bowtie(\varphi_1 \vee \varphi_2, \psi) \leftrightarrow \bowtie(\varphi_1, \psi) \vee \bowtie(\varphi_2, \psi)$$
$$\bowtie(\psi, \varphi_1 \vee \varphi_2) \leftrightarrow \bowtie(\psi, \varphi_1,) \vee \bowtie(\psi, \varphi_2,)$$

Other basic principles are:

$$\bowtie(\varphi, \psi) \rightarrow \bowtie(\psi, \varphi)$$
$$\varphi \rightarrow \bowtie(\varphi, \varphi)$$

At this general level, without further requirements on models $\langle X, O, —, \nu \rangle$, we do not get genuine interactions between $\bowtie$ and topological notions. But when we work on, say $\mathbb{R}$ or $\mathbb{R}^2$, this situation changes. Here are two principles valid in $\mathbb{R}$:

$$\bowtie(\Box\varphi, \Box\psi) \rightarrow \Box\bowtie(\varphi, \psi)$$
$$U(\gamma \rightarrow \bowtie(\varphi, \psi)) \rightarrow U(\Diamond\gamma \rightarrow \bowtie(\varphi, \psi))$$

Points in between interior points of two regions are contained in the interior of the 'betweenness area' of the original regions. Intuitively, on the left, one throws away boundary points, and then considers points in between, while on the right, one reverses the actions. The second principle states that if a region lies in the convex closure of two sets, then its closure also does.

Such principles reflect ways in which the topology has been generated from geometrical structure. Of course, on very specific structures, this connection may become very tight. We conclude with one example on $\mathbb{R}$.

Example 29 (Future and past) On the reals, we can take advantage of the ordering $<$, and define

$$M, x \models \bowtie(\varphi, \psi) \quad \text{iff} \quad \exists y, z : \; M, y \models \varphi \wedge M, z \models \psi \wedge z \leq x \leq y$$

Given this definition, we can use shorthand for other modalities that are well-known in the literature on temporal logic: future and past.

$$F\varphi := \bowtie(\text{true}, \varphi)$$
$$P\varphi := \bowtie(\varphi, \text{true})$$

(Note, both definitions include the present.) Conversely, on $\mathbb{R}$, these two unary modalities suffice for defining $\bowtie$:

$$\bowtie(\varphi, \psi) \leftrightarrow P\varphi \wedge F\psi$$

From here it is easy to get a complete axiomatization for our $\bowtie$-language, using the tense logic of future and past on $\mathbb{R}$ [Segerberg, 1970].

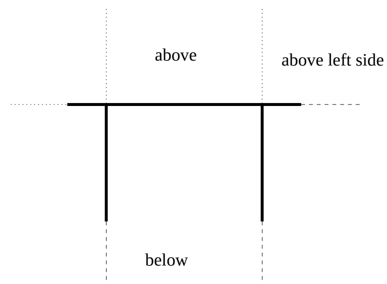

Figure 20: A table and the regions identified by means of versatile betweenness operators. The dotted lines correspond to the application of one of the operators to points belonging to the legs of the table and to its surface, while the dashed lines correspond to the application of the other versatile operator.

5.2 Towards full geometry

We have now extended our first simple topological modal language with convexity and betweenness operators in the hope of keeping the language tractable. But why not consider geometry in its full power from the very beginning? Its axiomatization has been around since ancient Euclid, so why not resort to it?

The reason is that we want maximally simple fragments which are expressive enough to describe visual patterns, preferably with a modal slant.

1. First, we consider *further betweenness modalities.* Betweenness was defined in terms of all the points that lie in between two given points. But here are two further positions. Given two points x and y, one can also consider all points z such that x lies in between y and z, or all points w such that y lies in between x and w. In this way, two points identify a direction and a weak notion of orientation. There are two obvious further modal operations corresponding to this. Together with $\bowtie$, they form a 'versatile' triple in the sense of [Venema, 1992]. Such triples are often easier to axiomatize together than in isolation.

   To exemplify this versatile version consider a table (see Figure 20), the one we have been dressing in the earlier sections. Using the versatile operators, the legs of the table and its main surface identify important directions in the visual scenes. Take for example all the points that lie in between the directions that stem from above the legs. We can identify this region within this versatile language, and indeed it is an interesting (at least linguistically) region: everything "above the table." Similarly, the other regions can be named.

2. Next we consider *mathematical morphology*, a theory at the crossroads of boolean algebra and geometry ([Serra, 1982]). The basic operations are

   $$A \oplus B = \{a + b | a \in A, b \in B\} \qquad \text{dilation}$$
   $$A \ominus B = \{a | a + b \in A, \forall b \in B\} \qquad \text{erosion}$$

   where the operation in the set definition is vector addition. These have a modal flavor. Think of modeling the implication and the sum in the following manner:

$$M, x \models A \oplus B \text{ iff } \exists y \exists z : \; x = y + z \wedge M, y \models A \wedge M, z \models B$$
$$M, x \models A \ominus B \text{ iff } \forall y \forall z : \; y = x + z \wedge M, z \models A \rightarrow M, y \models B$$

Modal operators like this occur in categorial and relevant logics (cf. [van Benthem, 91], [Kurtonina, 1995]). From such basic operations others are derived. Of particular interest in mathematical morphology are:

the structural *opening* of A by B	$(A \ominus B) \oplus B$
the structural *closing* of A by B	$(A \oplus B) \ominus B$

Dilations, erosions, openings and closings provide for weak characterizations of shape, distance and size. We leave their modal explanation for further study.

6 Conclusions

We have proposed the use of simulations and of games as a formal framework for the comparison of visual scenes. Such a framework not only provides for distinctions among different and similar scenes, but also enables a quality measurement of the difference in terms of winning strategies. Setting such a framework resulted in the analysis of a wide spectrum of modal languages describing spatial patterns, following a line that goes from topology to geometry. A pleasing side effect of the above analysis has been a new take on topology. Moreover,we have raised quite a few new issues for further investigation. In particular, we pursue questions of axiomatics in [Aiello and van Benthem, 1999]. All in all, there is much more to the good old 'topological interpretation' than meets the eye, once one starts following through its implications ...

A Appendix: Kripke Models and Neighbourhood Models

1. In this article we have used topological models for our modal languages. Historically this is the oldest semantics for modal logic, dating back to Tarski's work in the thirties. Today, the most widely used semantics for modal logics are *Kripke models.* A generalization of both is *neighbourhood semantics* (independently due to Montague and Scott in the seventies).

2. **Definition 22 (Kripke semantics)** A *Kripke model* M is a triple $\langle W, R, \nu \rangle$ where W is a set of states, $w \in W$, R is called the *accessibility* relation, and $\nu : P \rightarrow \mathcal{P}(W)$ is a valuation function. The key clause of the truth definition reads:

$$M, w \models \Box\varphi \quad \text{iff} \quad \forall y : \; xRy \rightarrow M, y \models \varphi$$

3. The minimal logic of all universally valid formulas is called K, whose axiomatization requires only one modal key principle:

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

This is much weaker than the modal logic S4 for topological semantics. Further axioms that correspond to specific conditions

(T)	reflexivity of the accessibility relation
(4)	transitivity of the accessibility relation
(5)	symmetry of the accessibility relation

In particular, a *tree-like* Kripke model corresponds to a topological model whose topology is generated by the R-closed subsets.

4. **Definition 23 (Neighbourhood Semantics)** A *neighbourhood model* M is a triple $\langle X, R, \nu\rangle$, where X is a set, $R \subseteq X \times \mathcal{P}(X)$, and ν is a valuation function. The key clause in the truth definition says:

$$M, x \models \Box\varphi \quad \text{iff} \quad \exists Y : Rx, Y \wedge \forall y \in Y \ M, y \models \varphi$$

The sets Y with Rx, Y are called *neighbourhoods* x.

5. The minimal logic is not K, but weaker. We do not even have

$$\Box\varphi \wedge \Box\psi \to \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

but only *monotonicity*

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$$

6. Kripke semantics is a special case of neighbourhood semantics: the neighbourhood Y consists only of singletons $\{y\}$, thus the relation R assumes the form $Rx, \{y\}$ or, more compactly, xRy.

7. Some frame correspondence principles for neighbourhood semantics are:

$$\begin{gathered} \Box\varphi \to \varphi \\ Rx, \emptyset \text{ or } Rx, \{x\} \end{gathered} \tag{T}$$

$$\begin{gathered} \Box\varphi \to \Box\Box\varphi \\ Rx, Y \wedge \forall y \in\text{:}\ Ry, Z_y \to Rx, \bigcup_{y \in Y} Z_y. \end{gathered} \tag{4}$$

8. Neighbourhood semantics is also a generalization of topological semantics. The relation R becomes the membership relation $\in$. The set of all neighbourhoods must fulfill the requirements of the open sets of a topological space. Rx, Y becomes $x \in Y$ where Y is an open set of the topological model. Thus, this is a good 'bottom level', where we can also study expressiveness and bisimulation (cf. [van Benthem, 1999]).

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