

Envelope Surfaces

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ABSTRACT

We construct a class of envelope surfaces in \mathbb{R}^d , more precisely envelopes of balls. An envelope surface is a closed C^1 (tangent continuous) manifold wrapping tightly around the union of a set of balls. Such a manifold is useful in modeling since the union of a finite set of balls can approximate any closed smooth manifold arbitrarily close.

The theory of envelope surfaces generalizes the theoretical framework of skin surfaces [5] developed by Edelsbrunner for molecular modeling. However, envelope surfaces are more flexible: where a skin surface is controlled by a single parameter, envelope surfaces can be adapted locally.

We show that a special subset of envelope surfaces is piecewise quadratic and derive conditions under which the envelope surface is C^1 . These conditions can be verified automatically. We give examples of envelope surfaces to demonstrate their flexibility in surface design.

Categories and Subject Descriptors

F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems—*Geometrical Problems and Approximations*; I.3.5 [Computer graphics]: Computational Geometry and Object Modeling—*Curve, surface, solid, and object representations*

Keywords

Smooth surfaces, envelopes of spheres, Legendre-Fenchel transform, piecewise quadratic surfaces

General Terms

Algorithms, Theory

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1. INTRODUCTION

The construction of smooth surfaces has obtained considerable attention in computational geometry. In recent years research has focused on surface reconstruction from finite point samples or from a finite set of medial balls, on the design of smooth surfaces for molecular modeling starting from a finite set of balls, and on meshing such surfaces for further geometric processing.

In this paper we introduce envelope surfaces, a class of closed C^1 surfaces¹ in \mathbb{R}^d that wrap tightly around the union of a set of balls. Such objects are useful in surface design, since any closed smooth surface can be approximated to within arbitrarily small Hausdorff distance by the union of a finite set of balls. The ideas originate from the theory of skin surfaces [5] developed by Edelsbrunner for molecular modeling. In fact, skin surfaces are a special type of envelope surfaces. However, envelope surfaces are more flexible: for a set of n input balls in \mathbb{R}^d the number of degrees of freedom of an envelope surface is $\Omega(n)$, whereas a skin surface associated with such a set of balls is controlled by a single parameter (called the shrink factor).

1.1 Weighted points and envelope surfaces

A weighted point in \mathbb{R}^d is a pair (p, w) , with $p \in \mathbb{R}^d$ and $w \in \mathbb{R}$. If $w \geq 0$, we associate this pair with the ball $B(p, \sqrt{w})$ with center p and radius \sqrt{w} .

Given a finite set \mathcal{P} of weighted points (input balls) in \mathbb{R}^d , we construct a continuous real-valued weight function W on a domain D in \mathbb{R}^d , containing the centers of these points, such that W interpolates the given weights at the centers of the corresponding balls in \mathcal{P} . In other words: $W(p) = w$, for $(p, w) \in \mathcal{P}$. The *envelope surface* associated with this weight function is the boundary of the union of the infinite family of balls

$$\{B(p, \sqrt{W(p)}) \mid p \in D \text{ and } W(p) \geq 0\}. \quad (1)$$

In this paper D will be a compact domain, usually the convex hull of the set of centers of the balls in \mathcal{P} . Compactness of D implies that the envelope surface is bounded.

1.2 Results

Our contribution consists of three parts.

1. We derive necessary and sufficient conditions for smoothness (tangent continuity) of the envelope surface of the fam-

¹In this paper a surface is a hypersurface (codimension one submanifold) in \mathbb{R}^d .

ily (1). This envelope surface is the zero-set of the function $H : \mathbb{R}^d \rightarrow \mathbb{R}$, defined by

$$H(x) = \min_{p \in D} (\|x - p\|^2 - W(p)). \quad (2)$$

We prove that this function is C^1 if the *associated weight function* $W_1 : D \rightarrow \mathbb{R}$, defined by

$$W_1(p) = \|p\|^2 - W(p), \quad (3)$$

is strictly convex. Furthermore, we show that the zero-set of H is a regular hypersurface of \mathbb{R}^d if it does not contain any zeros of the weight function. In general, the envelope surface is C^k if W is C^k .

2. Subsequently, we present a flexible method for the design of tangent continuous (C^1) envelope surfaces from a finite set of input balls. The domain D of the interpolating weight is the convex hull of the set of centers of these balls, endowed with a triangulation T of the set of centers. We show how to design a tangent continuous piecewise quadratic envelope surface by constructing a continuous weight functions that is piecewise quadratic with respect to T and has a strictly convex associated weight function. Furthermore, we construct a polyhedral subdivision of \mathbb{R}^d such that each quadratic patch is the intersection of the envelope surface with one of the polyhedral cells.

The piecewise quadratic weight function is *controlled locally* by associating a control parameter to each edge of the triangulation T of D . Therefore, the number of degrees of freedom in the design of an envelope surface is considerably larger than in the case of skin surfaces, where the shrink factor is the single control parameter.

3. Finally, we show that our envelope surfaces extend the class of skin surfaces [5] and that our scheme generalizes the approximation method based on skin surfaces presented in [9]. The polyhedral subdivision defining the decomposition of the surface into quadratic patches is a generalization of the mixed complex associated with skin surfaces.

1.3 Previous work

A special class of envelope surfaces is formed by skin surfaces, introduced by Edelsbrunner in [5]. These are mainly used for modeling large molecules in biological computing. Each atom in the molecule is represented by a sphere and atoms that lie close to each other are connected by smooth patches. In our earlier work [9] we approximate a smooth surface by a skin surface. An algorithm that approximates a polyhedron by a skin surface can be found in [4]. Two examples showing the increased flexibility of envelope surfaces compared to skin surface are given in Figure 6 and 5. Several algorithms exist for meshing skin surfaces [2, 3, 10]. We expect that our algorithm [10] can be adapted to mesh envelope surfaces introduced in this paper.

Another surface representation that defines a surface by a set of interior balls is the Medial Axis Transform (MAT). An algorithm that constructs an approximating surface using the MAT is presented in [11]. The balls of the MAT are centered on the medial axis, which is a skeletal structure of the surface, whereas the balls defining an envelope surfaces lie on a convex three-dimensional domain.

Bloppy objects or metaballs [1] also control a surface based on a set of balls. A disadvantage of these surfaces over skin surfaces is that for metaballs there is no known combinatorial structure that decomposes the surface into manageable pieces. Moreover, the input balls for bloppy objects are en-

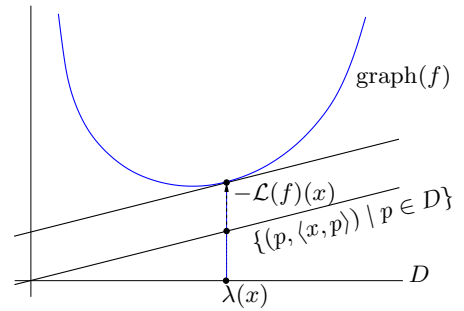


Figure 1: The function $\mathcal{L}(f)$ is the Legendre-Fenchel transform (conjugate) of the convex function f .

tirely contained inside the surface, and are not maximal. In fact, for bloppy objects two balls centered close to each other are approximately equivalent to a single ball with the sum of the radii of the two balls. This is not desirable in our setting where we want to increase the ‘sampling density’ of the balls to obtain a better approximation.

1.4 Outline

In Section 2 we introduce the Legendre-Fenchel transform, our main theoretical tool. This tool is applied in Section 3 to derive necessary and sufficient conditions for tangent continuity of envelope surfaces. Section 4 introduces piecewise quadratic envelope surfaces, associated with a weight function that is piecewise quadratic with respect to a triangulation of the set of centers of a finite set of input balls. Here we also introduce the polyhedral complex, defining the decomposition of the envelope surface into quadratic patches. Section 5 shows how skin surfaces fall in this framework, and Section 6 presents examples of our locally adaptive construction of piecewise quadratic envelope surface. Section 7 concludes the paper and discusses possibilities for future work.

2. LEGENDRE-FENCHEL TRANSFORM

We use the Legendre-Fenchel transform to show under which conditions the envelope surface is C^1 . Let $f : D \rightarrow \mathbb{R}$ be a continuous function defined on a compact subset D of \mathbb{R}^d .

DEFINITION 1. *The Legendre-Fenchel transform (or: conjugate) of f is the function $\mathcal{L}(f) : \mathbb{R}^d \rightarrow \mathbb{R}$, defined by*

$$\mathcal{L}(f)(x) = \max_{p \in D} (\langle x, p \rangle - f(p)). \quad (4)$$

See Figure 1. Note that the maximum exists, since f is continuous and D is compact. Let $f : D \rightarrow \mathbb{R}$ be a strictly convex function defined on a compact subset D of \mathbb{R}^d .

LEMMA 2. *For $x \in \mathbb{R}^d$ there is a unique point, denoted by $\lambda(x)$, at which the maximum in (4) is attained. The function $\lambda : \mathbb{R}^d \rightarrow D$ is continuous, and the set $\lambda^{-1}(p)$ is convex, for $p \in D$.*

PROOF. Uniqueness of $\lambda(x)$ follows from the strict convexity of f (See also Figure 1). The function $\lambda : \mathbb{R}^d \rightarrow D$ is continuous (see full version²).

To show that $\lambda^{-1}(p)$ is convex, let $x_0, x_1 \in \mathbb{R}^d$ such that $\lambda(x_0) = \lambda(x_1) = p_0$. From the strict convexity of f it follows

²A full version of this paper can be found at: <http://www.cs.rug.nl/~nico/EnvelopeSurfaces.ps.gz>

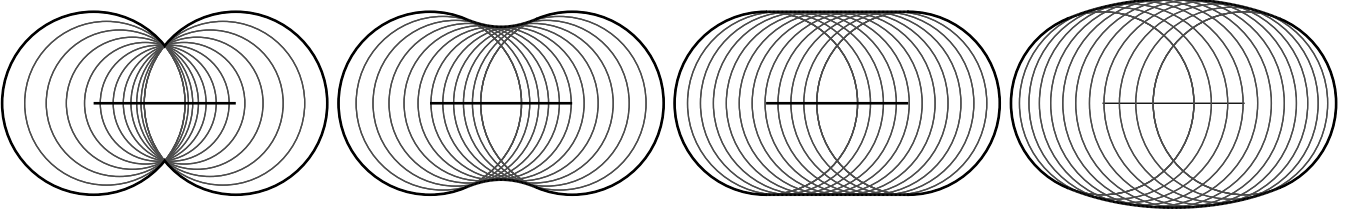


Figure 2: An example of an envelope surface. The weight function, defined on a line segment, is a quadratic univariate function with leading coefficient 1, 0.5, 0 and -0.5, respectively (from left to right).

that $\langle x, p \rangle - f(p) < \mathcal{L}(f)(x)$, for $p \neq \lambda(x)$. Let $x' = (1 - \gamma)x_0 + \gamma x_1$, with $0 \leq \gamma \leq 1$, be a point on the line segment $x_0 x_1$. Then, for $p \neq p_0$:

$$\begin{aligned} \langle x', p \rangle - f(p) &< (1 - \gamma)\mathcal{L}(f)(x_0) + \gamma\mathcal{L}(f)(x_1) \\ &= \langle x', x_0 \rangle - f(p_0). \end{aligned}$$

Therefore, $\lambda(x') = p_0$, so $\lambda^{-1}(p_0)$ is convex. \square

Note that the set $\lambda^{-1}(p)$ is connected since it is convex.

The one-sided directional derivative of f at $x_0 \in \mathbb{R}^d$ in the direction $v \in \mathbb{R}^d$, is

$$f'(x_0; v) = \lim_{h \downarrow 0} \frac{f(x_0 + hv) - f(x_0)}{h},$$

provided this limit exists.

A set A is convex if for any two points $x, x' \in A$ the line segment xx' lies in A and A is strictly convex if the open line segment (x, x') lies in the interior of A for any two points $x, x' \in A$. A function f is (strictly) convex if the set of points above the graph of f is (strictly) convex.

PROPOSITION 3. *If $f : D \rightarrow \mathbb{R}$ is a strictly convex continuous function defined on a compact subset D of \mathbb{R}^d , then the Legendre-Fenchel transform $\mathcal{L}(f)$ is a convex C^1 -function. Its gradient at $x \in \mathbb{R}^d$ is*

$$(\nabla \mathcal{L}(f))(x) = \lambda(x). \quad (5)$$

A point p_0 is equal to $\lambda(x_0)$ if and only if the directional derivatives are non-positive:

$$\frac{\partial \Phi}{\partial p}(x_0, p_0)(v) \leq 0, \text{ for all } v \in \mathbb{R}^d,$$

where the function $\Phi : \mathbb{R}^d \times D \rightarrow \mathbb{R}$ is defined by $\Phi(x, p) = \langle x, p \rangle - f(p)$, and $\frac{\partial \Phi}{\partial p}(x_0, p_0)(v)$ is the directional derivative of the function $p \mapsto \Phi(x_0, p)$ at p_0 in the direction v .

PROOF. Omitted from this version. \square

Remark. In the full version of the paper we also show that $\mathcal{L}(f)$ is C^k if f is a C^k -function, with $k \geq 2$. If the function $\Phi(x_0, \cdot)$ is C^k in a neighborhood of an interior point $p_0 \in D$, then there is a unique point $x_0 \in \mathbb{R}^d$ with $p_0 = \lambda(x_0)$, and $p = \lambda(x)$ is a solution of the equation $\frac{\partial \Phi}{\partial p}(x, p) = 0$ for x near x_0 .

3. ENVELOPE SURFACES

Let D be a compact convex subset of \mathbb{R}^d , and let $W : D \rightarrow \mathbb{R}$ be a continuous function. Consider the family of spheres $\{C_p \mid p \in D\}$, where C_p is the sphere with center p and weight $W(p)$. A point $x \in \mathbb{R}^d$ lies on the envelope S of

this family of spheres if it lies on at least one sphere, and on or outside all other spheres. In other words, x lies on the envelope if $H(x) = 0$, where $H : \mathbb{R}^d \rightarrow \mathbb{R}$ is defined by (2). Since D is compact and W is continuous, the minimum is attained in at least one point of D . The main result of this section states that, under certain conditions on the weight function W , the zero set of H is a C^1 -submanifold of \mathbb{R}^d .

PROPOSITION 4. *Assume that the associated weight function W_1 , defined by (3), is a strictly convex function. Then*

1. *For $x \in \mathbb{R}^d$ the minimum in (2) is attained in a unique point $\lambda(x)$, and the function $\lambda : \mathbb{R}^d \rightarrow D$ is continuous.*
2. *H is a C^1 -function, with derivative $\nabla H(x) = 2(x - \lambda(x))$. In particular, x is a critical point of H if and only if $x \in D$ and $x = \lambda(x)$.*
3. *The zero set of H is a C^1 -submanifold of \mathbb{R}^d if it does not contain any zeros of W .*

PROOF. 1. Rewrite H as follows

$$\begin{aligned} H(x) &= \|x\|^2 - \max_{p \in D} 2\langle p, x \rangle - \|p\|^2 + W(p) \quad (6) \\ &= \|x\|^2 - 2\mathcal{L}(1/2 W_1)(x). \end{aligned}$$

Here $\mathcal{L}(1/2 W_1) : \mathbb{R}^d \rightarrow \mathbb{R}$ is the Legendre-Fenchel transform (or: conjugate) of $1/2 W_1$. We refer to Section 2 for the definition, and some key properties relevant for our context. Since W_1 is strictly convex, for $x \in \mathbb{R}^n$ there is a unique point $\lambda(x)$ in D in which the maximum in (6) is attained. Obviously, the minimum in (2) is also attained at $\lambda(x)$. The function $\lambda : \mathbb{R}^d \rightarrow D$ is continuous, cf. Lemma 2.

2. According to Proposition 3 the $\mathcal{L}(1/2 W_1)$ is C^1 , since D is compact and W_1 is strictly convex. Therefore H is a C^1 -function, the derivative of which satisfies

$$H'(x) = 2\langle x, \cdot \rangle - 2\mathcal{L}(1/2 W_1)'(x) = 2\langle x - \lambda(x), \cdot \rangle.$$

The expression for the derivative of $\mathcal{L}(1/2 W_1)$ is given in Proposition 3.

3. From the second part we conclude that H is C^1 and we assume that $k \geq 2$. By the Implicit Function Theorem, it is sufficient to prove that H is regular at every point of its zero set. Let x be a point of $H^{-1}(0)$. Since $\lambda(x) \in D$, every point x in the complement of D is different from $\lambda(x)$. According to the second part of the proposition, H is regular at such points. Next consider $x \in D$. Since $W(x) \neq 0$, it follows from $\|x - \lambda(x)\|^2 - W(\lambda(x)) = H(x) = 0$ that $x \neq \lambda(x)$. Therefore, H is regular at x . \square

Remarks. 1. If W_1 is a C^k -function in a neighborhood of a point p_0 in the interior of D , with $k \geq 1$, then there is a

unique point x_0 with $p_0 = \lambda(x_0)$, and H is a C^k -function on a neighborhood of x_0 .

2. If W_1 is not strictly convex, then the zero set of H need not be a C^1 -surface. An example is given in the left figure of Figure 2. The weight function is defined on a line segment by $W(p) = \|p\|^2 + c$, with $c > 0$. The function W_1 is constant and therefore convex, but not strictly convex. In this case the zero set of H is not C^1 since it is the boundary of the union of the two input circles.

Piecewise smooth envelopes. Piecewise smooth envelope surfaces can be constructed from piecewise smooth weight functions. So let D be a compact convex polyhedron in \mathbb{R}^d , endowed with a triangulation T . The function $W : D \rightarrow \mathbb{R}$ is piecewise C^k with respect to T if

1. W is continuous on D , and
2. the restriction W_σ of W to a d -dimensional simplex σ of T can be extended to a C^k -function on a neighborhood of the closure of σ .

The envelope surface associated with W is a C^1 surface if the function W_1 defined in (3) is strictly convex, cf. Proposition 4. Each of the functions W_σ should therefore satisfy the following condition:

Strict Convexity Condition. For each simplex $\sigma \in T$, the function $W_{\sigma,1}$, defined by $W_{\sigma,1}(p) = \|p\|^2 - W_\sigma(p)$ on σ , is strictly convex.

Obviously, this condition does not guarantee strict convexity of W_1 on the transition between simplices, i.e., at lower dimensional simplices of T . Therefore we require that the family $\{W_\sigma\}$ satisfies the following condition:

Monotonous Transition Condition. If the d -dimensional simplices σ and σ' share a facet τ with normal v directed from σ to σ' , then for $p_0 \in \tau$:

$$W'_{\sigma',1}(p_0; v) \leq W'_{\sigma,1}(p_0; v).$$

LEMMA 5. *Let $W : D \rightarrow \mathbb{R}$ be a piecewise C^1 -function with respect to the triangulation T , satisfying the Strict Convexity Condition and the Monotonous Transition Condition. Then the function $W_1 : D \rightarrow \mathbb{R}$, defined by (3), is strictly convex.*

PROOF. We shall prove that W_1 is strictly convex along line segments $l \cap D$, where l is a line in \mathbb{R}^d . To this end, it is sufficient to prove that W_1 is convex along such lines. Indeed, the triangulation T partitions $l \cap D$ in a finite alternating sequence of points and open intervals, formed by the intersection of l with the simplices of T . Let $l \cap \tau$ be such an open interval, and suppose the simplex τ lies in the closure of the d -dimensional simplex σ . Then W_1 coincides with W_σ on this interval, so $W_1|_{l \cap \tau}$ is strictly convex. Therefore W_1 is strictly convex once we know that W_1 is convex along l (or, more precisely, that $W_1|_{l \cap D}$ is convex).

We first prove that W_1 is convex on any line disjoint from the $(d-2)$ -skeleton of T , and conclude, via a limit process, that W is convex on any line.

So consider a line l with direction v that only intersects simplices of dimension d and $d-1$. Since W satisfies the Strict Convexity Condition, it is sufficient to prove that $W'_{\sigma',1}(p_0; v) \leq W'_{\sigma,1}(p_0; v)$, for a point p_0 on a $(d-1)$ -simplex

τ shared by two d simplices σ and σ' , such that v is directed into σ . Note that v is not parallel to τ since l is disjoint from the $(d-2)$ -skeleton of T . Write $v = v_{\parallel} + v_{\perp}$, where v_{\parallel} is parallel to τ and v_{\perp} is perpendicular to τ . Since $W_{\sigma,1}$ and $W_{\sigma',1}$ are the restrictions of C^1 -functions defined on a neighborhood of p_0 in \mathbb{R}^d , the derivatives of these functions at p_0 exist and are linear, so: $W'_{\sigma,1}(p_0; v) = W'_{\sigma,1}(p_0; v_{\parallel}) + W'_{\sigma,1}(p_0; v_{\perp})$. and $W'_{\sigma',1}(p_0; v) = W'_{\sigma',1}(p_0; v_{\parallel}) + W'_{\sigma',1}(p_0; v_{\perp})$. Since the restrictions of the functions $W_{\sigma,1}$ and $W_{\sigma',1}$ to the facet τ are equal, the derivatives of these functions at p_0 in directions parallel to v are equal. In particular $W'_{\sigma',1}(p_0; v_{\parallel}) = W'_{\sigma,1}(p_0; v_{\parallel})$. According to the Monotonous Transition Condition $W'_{\sigma',1}(p_0; v_{\perp}) \leq W'_{\sigma,1}(p_0; v_{\perp})$. These identities and inequalities imply that

$$W'_{\sigma',1}(p_0; v) \leq W'_{\sigma,1}(p_0; v).$$

Hence W_1 is convex along l .

To prove that W_1 is convex along any line l , let $\{l_n\}$, be a sequence of lines converging to l , such that l_n is disjoint from the $(d-2)$ -skeleton of T . The restriction of W_1 to l is a convex function, since it is the limit of the sequence of convex functions $W_1|_{l_n}$. \square

If the function W is piecewise C^k with respect to the triangulation T of D and W_1 is strictly convex, then the function H is piecewise C^k on the pull-back $\lambda^*(T)$ of T under $\lambda : \mathbb{R}^d \rightarrow D$. The pull-back $\lambda^*(T)$ is the subdivision of \mathbb{R}^d into the regions $\lambda^{-1}(\tau)$, where τ ranges over all simplices of T . In general this subdivision can have a weird structure, even though the regions $\lambda^{-1}(\tau)$ are *connected*. If W is *piecewise quadratic* with respect to T , the cells of the pull-back are polyhedra. This context is studied in detail in the next section.

4. PIECEWISE QUADRATIC WEIGHT FUNCTIONS

Let \mathcal{P} be a finite set of balls and let T be a triangulation of their centers. In accordance with the ideas presented in the previous section, we construct an envelope surface by defining a piecewise quadratic weight function on T . This weight function is differentiable except possibly at lower dimensional cells, where it is merely assumed to be continuous.

In this section, the weight function is piecewise quadratic. In this case we show that the envelope surface is piecewise quadratic and that there is a polyhedral complex partitioning the envelope surface into the quadratic pieces.

Quadratic functions. A quadratic function q on \mathbb{R}^d is of the form $q(x) = x^T Q x + a^T x + b$, for a symmetric $d \times d$ matrix Q , a d -vector a and a constant b . The matrix Q is called the defining matrix of q . Since Q is symmetric, it has real eigenvalues [8, Ch. 7]. We shall sloppily speak of the eigenvectors and values of the quadratic function q .

Quadratic weight function. Each quadratic function is defined on a simplex. We first analyze the envelope surface corresponding to a single quadratic weight function. In order to avoid boundary conditions, we extend the domain D of the weight function to an n -dimensional affine subspace of \mathbb{R}^d containing the domain. Hence we extend the line segment of centers in Figure 2 to the line containing the segment and avoid spherical patches where the outer circles touch the envelope surface. These weight functions are

of special interest since the corresponding envelope surfaces are quadrics. Note that in this setting, D is not compact as assumed in the text above. Therefore, the minimum in Equation (2) might not be attained. Write

$$\begin{aligned} H(x) &= \inf_{p \in D} (\|x - p\|^2 - W(p)) \\ &= \|x\|^2 - \inf_{p \in D} (2\langle p, x \rangle - W_1(p)), \end{aligned}$$

where W_1 is defined in (3). If W_1 is strictly convex, then $2\langle p, x \rangle - W_1(p)$ has a positive definite quadratic part and the infimum is attained. So,

$$H(x) = \min_{p \in D} (\|x - p\|^2 - W(p)).$$

LEMMA 6. *If W is a quadratic function, defined on an affine subspace D of \mathbb{R}^d and W_1 is a strictly convex function, then*

1. *The map λ defined in Lemma 2 is linear,*
2. *The set $\lambda^{-1}(p)$ is orthogonal and complementary to D .*
3. *The function H is a quadratic function that is rotationally symmetric in D ,*
4. *The eigenvectors of W and H in the direction of D coincide and an eigenvalue d_i of W corresponds to an eigenvalue $d_i/(d_i - 1)$ of H . The eigenvalues of H corresponding to eigenvectors in the direction orthogonal to D are equal to 1. The unique critical point p of W is also the unique critical point of H , and $W(p) = -H(p)$.*

PROOF. Omitted from this version \square

COROLLARY 7. *The type of the quadric $H^{-1}(0)$ only depends on the eigenvalues of the quadratic weight function.*

The eigenvalues of H determine the type of quadric. For example, let W be defined on a one-dimensional affine space with leading coefficient c . Then, according to Lemma 6, part 4, the envelope surface is a hyperboloid if $0 < c < 1$, a cylinder for $c = 0$ and an ellipsoid for $c < 0$. See also Figure 2.

Decomposition of the surface. For $x \in S$ the ball B_x with center $\lambda(x)$ and radius-squared $W(\lambda(x))$ is called the *defining ball* of x . It is tangent to S at x . For a simplex σ of T let S_σ be the set of points of the envelope surface S such that the centers of its defining balls lie on the relative interior of σ . We shall prove that $\{S_\sigma \mid \sigma \in T\}$ is a decomposition of S into quadratic patches. Even stronger, we shall construct a polyhedral decomposition $\{\mu_\sigma \mid \sigma \in T\}$ of \mathbb{R}^d such that S_σ is the intersection of S with the polyhedral cell μ_σ .

More precisely, for a simplex σ of T let $\mu_\sigma = \{x \in \mathbb{R}^d \mid \lambda(x) \in \text{rel-int}(\sigma)\}$. Since λ is continuous, it maps the closure $\overline{\mu_\sigma}$ of μ_σ onto σ .

Example. Let $W(p) = 1/2\|p\|^2$ be defined on the triangle $\langle p_0, p_1, p_2 \rangle$ with $p_0 = (-1, -1)$, $p_1 = (-1, 2)$ and $p_2 = (2, -1)$; see Figure 3(a). We compute the polyhedral cell $\mu_{\langle p_0 \rangle}$, which is formed by all points x where $H(x) = \Phi(x, p_0)$, since $\text{rel-int}\langle p_0 \rangle = p_0$. Hence, $\mu_{\langle p_0 \rangle} = \{x = (\xi_0, \xi_1) \mid \xi_0, \xi_1 \leq -0.5\}$. Since λ changes continuously with x , it follows that $\mu_{\langle p_0, p_1 \rangle}$ is bounded by polyhedral cells of incident simplices of $\langle p_0, p_1 \rangle$. Let $x = (\xi_0, \xi_1)$ and $p' \in \text{rel-int}\langle p_0, p_1 \rangle$ be the point minimizing $\Phi(x, p)$ over all $p \in \text{rel-int}\langle p_0, p_1 \rangle$.

Then $x \in \mu_{\langle p_0, p_1 \rangle}$ if $\Phi(x, p') = H(x)$. For $\xi_1 \leq -1/2$ we have $\Phi(x, p') > \Phi(x, p_0)$, hence $\Phi(x, p') > H(x)$. Similarly for $\xi_1 \geq 1$ we have $\Phi(x, p') > \Phi(x, p_1)$ and for $\xi_0 > -1/2$, $\Phi(x, p') > \Phi(x, p'')$, with p'' the point minimizing $\Phi(x, p)$ over all $p \in \text{rel-int}\langle p_0, p_1, p_2 \rangle$. Hence, $\mu_{\langle p_0, p_1 \rangle} = (-\infty, -1/2] \times (-1/2, 1)$.

The remaining cells of the polyhedral complex are determined similarly.

Remark. The set $\overline{\mu_\sigma}$ is different from the set $\{x \in \mathbb{R}^d \mid \lambda(x) \in \sigma\}$. The example in Figure 3(b),(c) illustrates this. The cell $\overline{\mu_\sigma}$ with $\sigma = \langle p_0, p_1, p_3 \rangle$ does not contain the cells μ_τ for $\tau = \langle p_0, p_3 \rangle$ or $\tau = \langle p_0 \rangle$. However, the set $\mu(\sigma) = \cup_{p \in \sigma} \mu(p)$ does contain these cells.

In the following proposition, we show that the cells $\{\overline{\mu_\sigma} \mid \sigma \in T\}$ form a polyhedral complex that decomposes the envelope surface in pieces of quadrics.

PROPOSITION 8. *The cells $\overline{\mu_\sigma}$, with $\sigma \in T$, and their non-empty intersections form a polyhedral complex decomposing \mathbb{R}^d . Moreover, the cells have the property that*

1. *for $\sigma, \sigma' \in T$, with $\sigma' \leq \sigma$, the cells $\overline{\mu_\sigma}$ and $\overline{\mu_{\sigma'}}$ intersect*
2. *the intersection of $\overline{\mu_\sigma}$ with S is contained in the quadratic envelope surface obtained from the extension of W_σ to $\text{aff}(\sigma)$*

PROOF. We use Lemma 9 to show that the cells $\overline{\mu_\sigma}$ and their non-empty intersections form a polyhedral complex. Before we can apply this lemma, we have to show that the cells $\overline{\mu_\sigma}$ are convex and partition \mathbb{R}^d .

We first show that the cells μ_τ with $\tau \in T$ form a partition of \mathbb{R}^d and are pairwise disjoint. To this end, let $x \in \mathbb{R}^d$ and $\tau \in T$ be the simplex containing the point $\lambda(x) \in D$ in its relative interior. There may be more simplices containing $\lambda(x)$, but only one simplex contains $\lambda(x)$ in its relative interior. Hence, the cells are pairwise disjoint and their union is \mathbb{R}^d .

To prove that the cell μ_σ is convex, let $x_0, x_1 \in \mu_\sigma$ and $x' = (1 - \gamma)x_0 + \gamma x_1$ for $\gamma \in [0, 1]$. The cell μ_σ is convex if $x' \in \mu_\sigma$. Let $p' = (1 - \gamma)\lambda(x_0) + \gamma\lambda(x_1)$. We shall prove that $\lambda(x') = p'$, then $x' \in \mu_\sigma$ since $p' \in \text{rel-int}(\sigma)$. To this end, let $\Phi : \mathbb{R}^d \times D \rightarrow \mathbb{R}$ be defined by $\Phi(x, p) = \langle x, p \rangle - W_1(p)$. By Proposition 3, the points p' and $\lambda(x')$ are equal iff

$$\frac{\partial \Phi}{\partial p}(x', p')(v) \leq 0, \text{ for all } v \in \mathbb{R}^d.$$

By the same proposition, we have $\frac{\partial \Phi}{\partial p}(x_0, \lambda(x_0))(v) \leq 0$ and $\frac{\partial \Phi}{\partial p}(x_1, \lambda(x_1))(v) \leq 0$. We expand the directional derivative to:

$$\frac{\partial \Phi}{\partial p}(x, p)(v) = \langle x, v \rangle - W_1'(p; v). \quad (7)$$

Since $x_0, x_1 \in \mu_\sigma$, the points $\lambda(x_0)$ and $\lambda(x_1)$ lie in the relative interior of σ . Let σ' be the simplex containing the points $\lambda(x_0) + \epsilon v$, for small positive ϵ . Then σ' also contains the points $\lambda(x_1) + \epsilon v$ for small positive ϵ . Hence, $W(\lambda(x_i) + \epsilon v) = W_{\sigma'}(\lambda(x_i) + \epsilon v)$, for $i = 0, 1$. If v is parallel to $\text{aff}(\sigma)$, then the simplices σ and σ' are equal, otherwise $\sigma \leq \sigma'$. Restricted to σ' , W is a quadratic function. In view of (7), the directional derivative of Φ is linear in x and p . But then, $\frac{\partial \Phi}{\partial p}(x', p')(v) = \frac{\partial \Phi}{\partial p}((1 - \gamma)x_0 + \gamma x_1, (1 - \gamma)\lambda(x_0) + \gamma\lambda(x_1))(v) = (1 - \gamma)\frac{\partial \Phi}{\partial p}(x_0, \lambda(x_0))(v) + \gamma\frac{\partial \Phi}{\partial p}(x_1, \lambda(x_1))(v) \leq 0$. Hence, $p' = \lambda(x')$, $x' \in \mu_\sigma$ and μ_σ is convex.

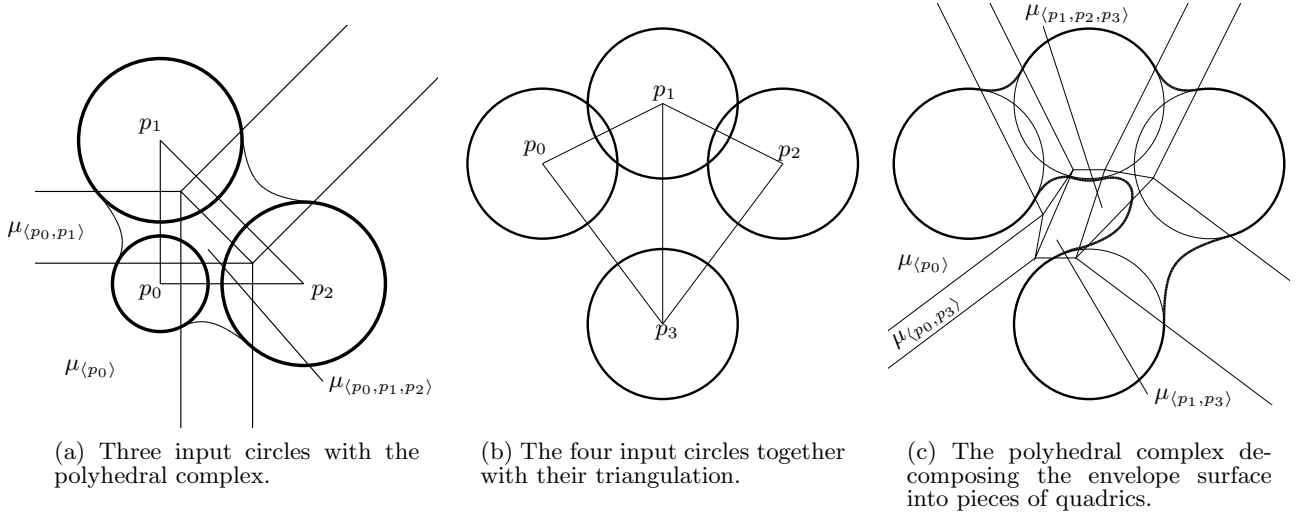


Figure 3: The decomposition of the envelope surface into quadrics.

In view of Lemma 9, the cells $\overline{\mu_\sigma}$ with a non-empty interior form a polyhedral complex. A cell with empty interior is a common face of two cells with non-empty interiors by the continuity of λ .

1. Let $x \in \overline{\mu_\sigma} \setminus \mu_\sigma$ and $\tau \in T$ the simplex such that $x \in \mu_\tau$. Note that $\sigma \neq \tau$. From the continuity of λ we derive that $\lambda(x)$ is a point on the boundary of σ . Since $\lambda(x)$ lies in the relative interior of τ , we have $\tau \leq \sigma$.

2. Let x be a point in $\overline{\mu_\sigma}$. Using the continuity of λ , the point $\lambda(x)$ is contained in σ . Hence $H(x) = H_\sigma(x)$, with $H_\sigma(x) = \min_{p \in \sigma} (\|x - p\|^2 - W_\sigma(p))$. Since $W_{\sigma,1}$ is strictly convex, H has a unique minimum $\lambda(x) \in \sigma$ for all $p \in \text{aff}(\sigma)$ and $H_\sigma(x) = \min_{p \in \text{aff}(\sigma)} (\|x - p\|^2 - W_\sigma(p))$. Lemma 6 states that H_σ is a quadratic function. The surface S is defined as the zero set of H , hence

$$\overline{\mu_\sigma} \cap S = \overline{\mu_\sigma} \cap H^{-1}(0) = \overline{\mu_\sigma} \cap H_\sigma^{-1}(0). \quad \square$$

The following general lemma is used in Proposition 8 to show that the cells μ_σ , with $\sigma \in T$, form a polyhedral complex.

LEMMA 9. *Let \mathcal{C} be a finite set of closed convex sets, with non-empty interior partitioning \mathbb{R}^d . The sets in \mathcal{C} and their non-empty intersections form a polyhedral complex if they only intersect in their boundaries.*

PROOF. First, we show that the sets are convex polyhedra. Since the sets are convex and their interiors are disjoint, there exists a, not necessarily unique, hyperplane separating the interiors of any two sets in \mathcal{C} . For two sets $c, c' \in \mathcal{C}$, with $c \neq c'$, denote with $H(c, c')$ the closed half-space containing c and bounded by a hyperplane separating c and c' .

For fixed $c \in \mathcal{C}$, let x be a point in the interior of the intersection of all halfspaces $H(c, c')$, for $c' \in \mathcal{C}$ and $c' \neq c$. Since x does not lie in any set $c' \in \mathcal{C}$ for $c' \neq c$, the point x is contained in the set c . Hence,

$$c = \bigcap_{c' \in \mathcal{C}, c' \neq c} H_{c, c'},$$

which is a polyhedron by definition.

The intersection of two convex polyhedra is empty or another convex polyhedron. Since the sets in \mathcal{C} only intersect in their boundary, the intersection is a face of both. \square

Mixed complex. The polyhedral complex for envelope surfaces is a generalization of the mixed complex for skin surfaces. For a skin surface with shrink factor s , the triangulation T is the weighted Delaunay triangulation and the graph of the weight function on a simplex σ is a paraboloid with leading term equal to s . The apex of the paraboloid lies on the intersection point of the affine hull of the Delaunay simplex δ_σ and the dual Voronoi cell ν_σ , called is the focus $f(\sigma)$ of σ .

A calculation in the spirit of Lemma 6 shows that $\{f(\sigma) + (1-s)(x - f(\sigma)) | x \in \delta_\sigma\}$ is the image of $\overline{\mu_\sigma}$ under orthogonal projection onto $\text{aff}(\delta_\sigma)$. From a similar analysis of the directional derivatives in directions orthogonal to δ_σ , it follows that $\{f(\sigma) + s(x - f(\sigma)) | x \in \nu_\sigma\}$ is the image of $\overline{\mu_\sigma}$ onto $\text{aff}(\nu_\sigma)$. Combining these two results, we obtain the following identity of the polyhedral cell: $\overline{\mu_\sigma} = (1-s) \cdot \delta_\sigma \oplus s \cdot \nu_\sigma$, which matches the definition of mixed cell.

Parameterization of the weight function. For ease of manipulating the piecewise quadratic weight functions, we define the weight function on a simplex by values at its vertices and at midpoints of its edges. The space of quadratic functions in \mathbb{R}^d has dimension $\frac{1}{2}d(d+1)$. Therefore, if the vertices are affinely independent, the quadratic function is uniquely determined by specifying its value on every vertex and at midpoints of the edges.

To show that the weight function thus obtained is continuous, let $\sigma, \sigma' \in T$ be two full dimensional simplices and let τ be their common n -dimensional face. Restricted to τ , W^* is quadratic and interpolates the weights at the vertices and at midpoints of the edges of τ . These weights determine the quadratic weight function uniquely, and $W^*|_\tau = W_\sigma|_\tau = W_{\sigma'}|_\tau$.

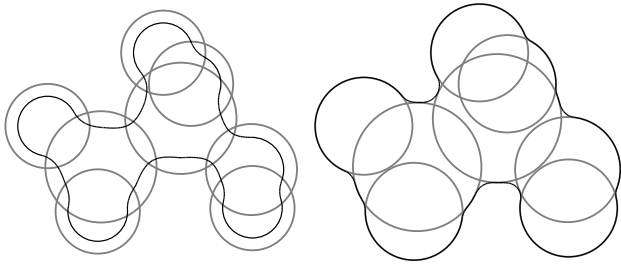


Figure 4: Two existing interpolation schemes. On the left the skin curve [5] ($s = 1/2$) and on the right the extended skin curve presented in [9].

5. EXISTING SCHEMES

The new interpolation scheme, to be presented in Section 6, generalizes two existing methods. In this section, we describe these methods in the context of the previous sections. An example in 2D of both methods is shown in Figure 4.

We first show how skin surfaces fit in our approach. The radii of the input balls are shrunk in the construction of a skin surface by the same shrink factor (see the full version).

In [9], we adapted the construction of skin surfaces such that the weight function interpolates the weights of the initial balls to the effect that the input balls are contained in the resulting skin surface. We call the skin surface defined in this way the extended skin surface.

Both methods define a one-parameter family of envelope surfaces, depending only on the shrink factor s . For a shrink factor equal to one, the weight functions of both methods are equal and so are their envelope surfaces. We also use this weight function denoted by W^* and called the *initial weight function* as starting point for the new interpolation scheme.

Initial weight function. Let T be the (weighted) Delaunay triangulation of the input set \mathcal{P} . The initial weight function W^* is piecewise quadratic with respect to T . Each quadratic function W_σ^* on a d -simplex σ is uniquely defined by the following two conditions:

- W_σ^* interpolates the weights on vertices of σ
- the defining matrix is the identity matrix.

We show that the W^* satisfies the Monotonous Transition Condition.

LEMMA 10. *The weight function W^* satisfies the Monotonous Transition Condition.*

PROOF. The paraboloids defining W^* become hyperplanes in W_1^* , hence W_1^* is simplexwise affine. The Delaunay triangulation has the property that these hyperplanes define a convex function [6], hence W_1^* is convex.

Since W_1^* is convex, the function W_1^* satisfies the Monotonous Transition Condition (see the full version), and W^* satisfies the Monotonous Transition Condition as well. \square

Even though the Strict Convexity Condition is not satisfied, the initial weight function W^* is a good starting point for our construction since the envelope surface it defines is the boundary of the union of the input balls.

LEMMA 11. *The envelope surface defined by the initial weight function W^* is equal to the boundary of the union of the input balls.*

PROOF. The weight function interpolates the weights on centers of the input balls. Hence, the input balls are contained in the envelope surface. It remains to show that the approximating balls lie within the union of the input balls.

Let p' be a point in D and let $\sigma \in T$ be the simplex containing p' . Write p' as a convex combination of the vertices p_i of σ : $p' = \sum_i \gamma_i p_i$ with $\gamma_i \geq 0$ and $\sum_i \gamma_i = 1$. Since $W^*(x) = \|p\|^2 + W_1^*(p)$, so $\Phi(x, p) = \|x\|^2 - 2\langle p, x \rangle - W_1^*(p)$ is affine in p when p ranges over σ and $\Phi(x, \sum_i \gamma_i p_i) = \sum_i \gamma_i \Phi(x, p_i)$. Let $x \in \mathbb{R}^d$ be a point outside the union of the balls in \mathcal{P} . Then, $\Phi(x, p_i)$ is positive, for any weighted point $(p_i, p_i) \in \mathcal{P}$, since $W^*(p_i) = p_i$. But then $\Phi(x, p') > 0$ and, hence, x also lies outside the ball centered at p' . \square

Recall from the end of Section 4 that each quadric is uniquely determined by specifying the function values at vertices and at midpoints of the edges. We denote the initial weight at the vertex $p_i \in \mathcal{P}$ of T with W_i^* and the initial weight at the midpoint of the edge $p_i p_j$ with W_{ij}^* . The initial weight function W^* interpolates the weights of the input balls if the value W_i^* is set to the weight of the input ball centered at p_i . Moreover, the defining matrix of $W^*|_\sigma$ is the identity matrix if the quadratic weight function restricted to any edge has leading coefficient equal to one. If p_i, p_j are two vertices of σ , then the quadratic weight function on the edge p_i, p_j has leading coefficient equal to one if the weight W_{ij}^* at the midpoint of the edge is: $W_{ij}^* = (W_i^* + W_j^*)/2 - \|p_i - p_j\|^2/4$.

Skin surfaces. In the full version of the paper, we show that the weight function for a skin surface is obtained by multiplying the initial weight function with s : $W^s(x) = s W^*(x)$. In terms of the parameterization described above, the weight function is obtained by multiplying the weights at vertices and at midpoints of edges with s .

COROLLARY 12. *For $s < 0 < 1$, the skin surface $\text{skn}^s(\mathcal{X})$ is tangent continuous.*

PROOF. The eigenvalues of the quadratic weight function restricted to a simplex are all equal to s . Hence, W^s satisfies the Strict Convexity Condition. By Lemma 10, W^* satisfies the Monotonous Transition Condition and so does W^s . Hence, the envelope surface obtained from the weight function W^s is C^1 , which is also shown in [5]. \square

Since all weights are multiplied by s , the weights on the vertices and edges controlling the weight function are also multiplied by s . The weight of the input balls are scaled with a factor s , and therefore the input balls are not contained in the skin surface. See also Figure 4 (left).

Extended skin surfaces. The fact that the input balls are shrunk in the construction of skin surfaces makes them not directly suitable for the construction of envelope surfaces containing the input balls. In [9], we proposed a method that first multiplies the weights of the input balls with $1/s$, and then computes the skin surface with shrink factor s . Further, we construct a range of s -values such that the skin surface is homeomorphic to the boundary of the union of

the input balls. An example of this construction is shown in the right part of Figure 4.

Since the extended skin surface is a skin surface (of the grown set of input balls), it is a C^1 -manifold, as is shown in Corollary 12. The weight function defining the extended skin surface interpolates the weights on vertices. The weight at the midpoint of an edge $p_i p_j$ depends linearly on s and is given by $W_{ij} = \frac{W_i + W_j}{2} - s \frac{\|p_i - p_j\|^2}{4}$, where W_i and W_j denote the weight on vertex p_i and p_j respectively.

During the growth of the input balls the (weighted) Delaunay triangulation may change. If we only increase the weights on the edges and do not adapt the triangulation accordingly, the Monotonous Transition Condition is not satisfied and the envelope surface is not C^1 . Therefore, we have to adjust the the Delaunay triangulation, as is done in [9].

6. THE NEW INTERPOLATION SCHEME

The schemes described in the previous section are global in the sense that they define a one parameter family of weight functions. In this section we propose an adaptive interpolation scheme, adapting the weights on the edges independently. This results in a local control of the envelope surface, and hence in a class of envelope surfaces which are much more flexible than skin surfaces. However, the envelope surface is only C^1 if Strict Convexity Condition and the Monotonous Transition Condition are satisfied, cf. Lemma 5.

Checking the smoothness conditions. The Strict Convexity Condition is satisfied if the Hessian of the function $W_{\sigma,1} : D \rightarrow \mathbb{R}$, defined by $W_{\sigma,1}(p) = \|p\|^2 - W_\sigma(p)$, is positive definite. For a quadratic function, this is the case if the eigenvalues of the defining matrix are positive. Hence, the eigenvalues of the defining matrix of W_σ have to be smaller than one. The Strict Convexity Condition with respect to a simplex σ only depends on W_σ and, therefore, only on the weights at the vertices and at midpoints of the edges of σ .

To see how to check the Monotonous Transition Condition, let $\sigma, \sigma' \in T$ be two full dimensional cells with a common facet τ and let v be the normal of τ directed from σ to σ' . Let the map w be defined on τ by $w(p) = W'_{\sigma'}(p; v) - W'_\sigma(p; v)$. Since the gradient of a quadratic function is a linear map, w is linear. Therefore, $w(p) \geq 0$ for all $p \in \tau$ iff $w(p) \geq 0$ for every vertex p of τ . Hence, the Monotonous Transition Condition is satisfied at every point of τ iff $w(p) \geq 0$ for every vertex p of τ . The Monotonous Transition Condition of a face τ incident to σ and σ' depends on W_σ and $W_{\sigma'}$ and therefore on the weights at the vertices and at midpoints of the edges of σ and σ' .

Changing the weights. Conceptually, the algorithm is similar to the extended skin surface algorithm. First it constructs the initial weight function W^* and then it continuously increases the weights on midpoints of the edges. We define a local growth parameter t_{ij} for every edge $p_i p_j$ and parameterize the weight at the midpoint of an edge by:

$$W_{ij} = \frac{W_i + W_j}{2} + (t_{ij} - 1) \frac{\|p_i - p_j\|^2}{4}.$$

Note that the envelope surface is an extended skin surface with $s = 1 - t_{ij}$, if all growth parameters are equal. Initially all scalars t_{ij} are equal to zero. We continuously increase the local growth parameters as long as the Strict Convexity Condition and the Monotonous Transition Condition are satisfied. The envelope surface is therefore C^1 .

Since the conditions are locally determined, we can fix the local growth parameters that would invalidate one of the conditions and increase the other local growth parameters.

Another method fixes the local growth parameter on edges between disjoint balls in an early stage and increases the local growth parameter at midpoints of the edges between intersecting weighted points as much as possible. This gave us the interpolation on the fingers of the hand in Figure 5.

Examples. Envelope curves that have an increasing weight on the midpoint of an edge are shown in Figure 2. Two examples of the interpolation scheme of [9] versus the local interpolation scheme are given in Figure 6, 7 and 5. By construction, the extended skin surface interpolates the input balls with concave patches. The local interpolation scheme, on the other hand, allows for envelope surfaces with interpolating patches that are both convex and concave, viz. Figure 6. In Figure 5, we increased the weight on the edges in the direction of the fingers and not on edges between balls of different fingers. The result is an envelope surface that interpolates nicely in the direction of the fingers. The extended skin curve is either bumpy (b), or contains patches in between fingers (c).

The envelope surfaces in Figure 7 show the flexibility of these surfaces for only four input balls (top left). The envelope can be decomposed into quadric patches determined by simplices of the triangulation, see Section 4. The patches are color-coded by the dimension of this simplex. The top row of Figure 7 shows envelope surfaces for which the local growth parameter at midpoints of the edges is increased with the same amount. These envelope surfaces are also extended skin surfaces. In the bottom row, we increased the local growth parameter on midpoints of one, two and three edges, respectively. The spheres are connected due to this increase.

7. CONCLUSIONS AND FUTURE WORK

We introduce envelope surfaces, a class of smooth surfaces constructed from a finite set of balls. Envelope surfaces extend the class of skin surfaces, and increase the number of free design parameters considerably: a skin surface determined by n input balls in \mathbb{R}^d has a single degree of freedom (its shrink factor), whereas a piecewise quadratic envelope surface for such input has $\Omega(n)$ free parameters (although confined to a convex domain by smoothness constraints). An envelope surface is determined by a weight function, which interpolates the square of the radii of the input balls on the convex hull of their centers. We introduce two conditions the weight function should satisfy in order to guarantee smoothness (tangent continuity) of the envelope surface. These conditions can be verified automatically. If the weight function is piecewise quadratic, then the envelope surface is also piecewise quadratic.

Envelope surfaces form a rich area for further research.

1. We conjecture that level sets of the weight function W and those of the function H defining the envelope surface implicitly are isotopic for all isovalues. The conjecture is true for skin surfaces. Also, if W is differentiable then the critical points of W and H coincide and at a critical point p , $W(p) = -H(p)$, cf. Proposition 4.

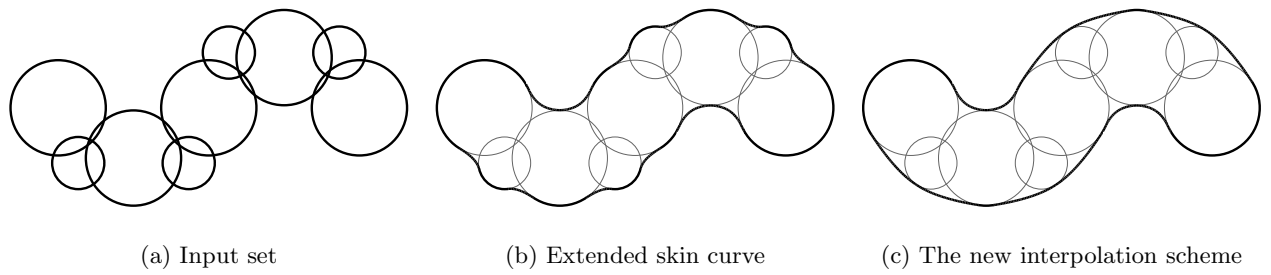


Figure 6: The extended skin surface introduced in [9] defines concave patches between the input balls. With envelope surfaces introduced in this paper it is possible to interpolate both with convex and with concave patches.

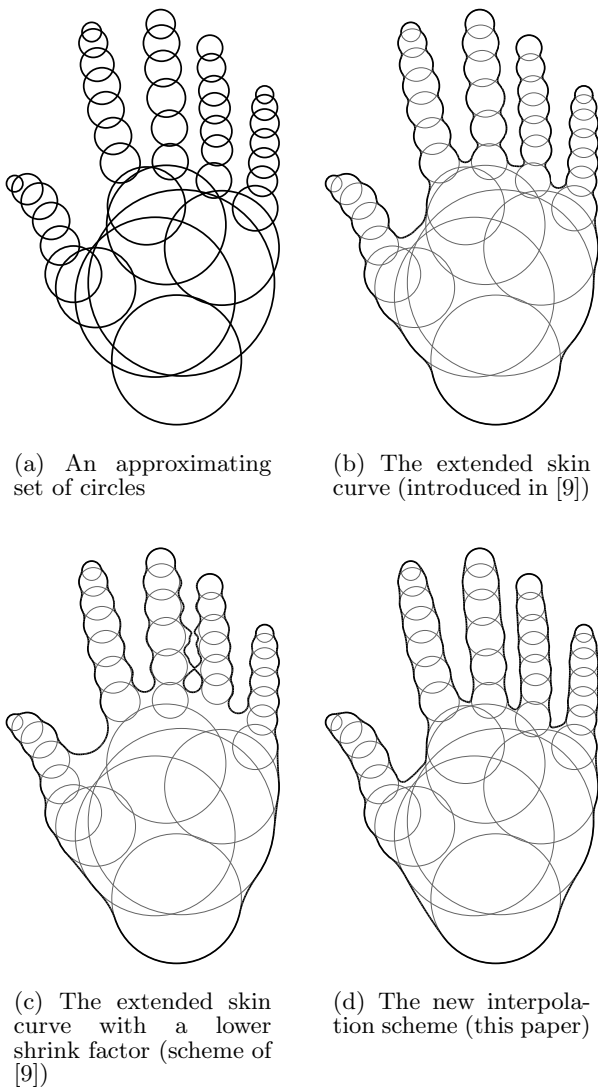


Figure 5: Different interpolation schemes

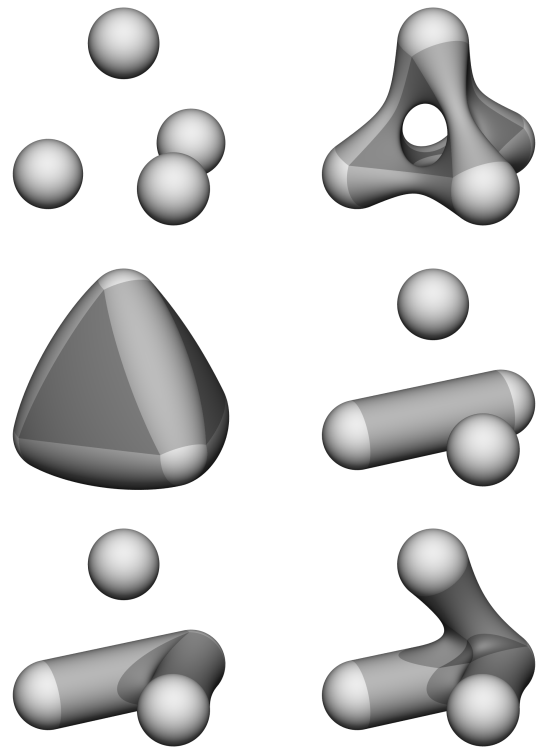


Figure 7: Envelope surfaces of four balls with different values for the local growth parameters.

2. Except for some prototypes used to give a proof of concept, we have not developed a modeler for these surfaces so far. With such a modeler it is possible to construct and modify an envelope surface interactively and test the applicability of envelope surfaces to their full extent. Several schemes are proposed in Section 6 to construct a weight function interactively. Validation of these schemes on realistic data is also possible with a modeler.
3. In Section 4 necessary and sufficient conditions are introduced for smoothness of envelope surfaces defined by piecewise smooth weight functions on a triangulation T of the set of centers of input balls. These conditions define a convex feasible region in the space of weights of the edges of T . We plan to investigate this feasible region, and to design strategies for flexible design of envelope surfaces by manipulating the edge-weights.
4. We plan to extend our meshing algorithm for skin surfaces [10] to the meshing of envelope surfaces.
5. Differential geometry provides an other approach to the study of envelopes of spheres. In particular, spheres in \mathbb{R}^d are represented by points of the Lorentz-sphere (also known as the de Sitter space) with respect to some semi-definite inner product in \mathbb{R}^{d+1} [7]. A systematic study of one-parameter families of spheres in 3D leads to the Dupin cyclides, a class of quartic envelopes of two-dimensional spheres. We plan to further investigate envelopes of families of spheres using this context.

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