Approximation by Conic Splines

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Abstract. We show that the complexity of a parabolic or conic spline approximating a sufficiently smooth curve with non-vanishing curvature to within Hausdorff distance ε is $c_1\varepsilon^{-1/4} + O(1)$, if the spline consists of parabolic arcs, and $c_2\varepsilon^{-1/5} + O(1)$, if it is composed of general conic arcs of varying type. The constants c_1 and c_2 are expressed in the Euclidean and affine curvature of the curve. We also show that the Hausdorff distance between a curve and an optimal conic arc tangent at its endpoints is increasing with its arc length, provided the affine curvature along the arc is monotone. This property yields a simple bisection algorithm for the computation of an optimal parabolic or conic spline.

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1. Introduction

In the field of computer aided geometric design, one of the central topics is the approximation of complex geometric objects with simpler ones. An important part of this field concerns the approximation of plane curves and the asymptotic analysis of the rate of convergence of approximation schemes with respect to different metrics, the most commonly used being the Hausdorff metric.

Various error bounds and convergence rates have been obtained for several types of (low-degree) approximation primitives. For the approximation of plane convex curves by polygons with n edges, the order of convergence is $O(n^{-2})$ for several metrics, including the Hausdorff metric [13, 16, 17, 19]. When approximating by a tangent continuous conic spline, the order of convergence, for a strictly

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convex curve, is $O(n^{-5})$, where n is the number of elements of the conic spline, with respect to the Hausdorff distance metric [26]. For the approximation of a convex curve by a piecewise cubic curve, both curves being tangent and having the same Euclidean curvature at interpolation points (knots), the order of approximation is $O(h^6)$, where h is the maximum distance between adjacent knots [5]. As expected, the approximation order increases along with the degree of the approximating (piecewise-) polynomial curve.

As approximants, conic splines represent a good compromise between flexibility and modeling power. They have a great potential as intermediate representation for robust computation with curved objects. Some applications that come to mind are the implicitization of parametric curves (see works on approximate implicitization [8, 9]), the intersection of high-degree curves, the building of arrangements of algebraic curves (efficient solutions are known for sweeping arrangements of conic arcs [2]) and the computation of the Voronoi diagram of curved objects (the case of ellipses has been recently investigated [10, 11]).

While these applications necessitate a tight hold on the error of approximation, no previous work provides a sharp asymptotic error bound (i.e., the constant of the leading term in the asymptotic expansion) for the Hausdorff metric when the interpolant is curved.

In this paper, we study the optimal approximation of a sufficiently smooth curve with non-vanishing curvature by a tangent continuous interpolating conic spline, which is an optimal approximant with respect to Hausdorff distance. We present the first sharp asymptotic bound on the approximation error (and, consequently, a sharp bound on the complexity of the approximation) for both parabolic and conic interpolating splines. Our experiments corroborate this sharp bound: the complexity of the approximating splines we algorithmically construct exactly matches the complexity predicted by our complexity bound.

1.1. Related work

Fejes Tóth [13] considers the problem of approximating a convex C^2-curve C in the plane by an inscribed n-gon. Fejes Tóth proves that, with regard to the Hausdorff distance, the optimal n-gon P_n satisfies

$$\delta_H(C, P_n) = \frac{1}{8}\left(\int_0^l \kappa^{1/2}(s)\, ds\right)^2 \frac{1}{n^2} + O\Big(\frac{1}{n^4}\Big). \tag{1.1}$$

Here $\delta_H(A, B)$ is the Hausdorff distance between two sets A and B, l is the length of the curve, s its arc length parameter, and $\kappa(s)$ its curvature. An asymptotic expression for the *complexity* of the piecewise linear spline can easily be deduced: the number of elements is $c\,\varepsilon^{-1/2}\,(1+O(\varepsilon))$, where $c = \frac{1}{2\sqrt{2}}\int_{s=0}^{l} \kappa(s)^{1/2}\, ds$. Ludwig [17] extends this result by deriving the second term in the asymptotic expansion (1.1). If one considers the *symmetric difference metric* δ_S instead, one can prove that $\delta_S(C, P_n) = \frac{1}{12}\left(\int_0^l \kappa^{1/3}(s) ds\right)^3 \frac{1}{n^2} + O(\frac{1}{n^4})$ [19]. Again, this asymptotic expression can be refined, cf. [16].

Schaback [26] introduces a scheme that yields an interpolating conic spline with tangent continuity for a curve with non-vanishing curvature, and achieves an approximation order of $O(h^5)$, where h is the maximal distance of adjacent data points on the curve. A conic spline consists of pieces of conics, in principle of varying type. This result implies that approximating such a curve by a curvature continuous conic spline to within Hausdorff distance ε requires $O(\varepsilon^{-1/5})$ elements. However, the value of the constant implicit in this asymptotic expression of the complexity is not known. Ludwig [18] considers the problem of optimally approximating a convex C^4-curve with respect to the symmetric difference metric by a tangent continuous parabolic spline Q_n with n knots. She proves that $\delta_S(C, Q_n) = \frac{1}{240}\left(\int_0^\lambda \kappa^{1/5}(s)ds\right)^5 \frac{1}{n^4} + o(\frac{1}{n^4})$, where $\lambda = \int_0^l \kappa^{1/3}(s)ds$ is the *affine length* of the convex curve C.

These problems fall in the context of geometric Hermite interpolation, in which approximation problems for curves are treated independent of their specific parameterization. The seminal paper by De Boor, Höllig and Sabin [5] fits in this context. Floater [14] gives a method that, for any conic arc and any odd integer n, yields a geometric Hermite interpolant with $2n$ contacts, counted with multiplicity. This scheme gives a G^{n-1}-spline, and has approximation order $O(h^{2n})$, where h is the length of the conic arc. Ahn [1] gives a necessary and sufficient condition for the conic section to be the optimal approximation of the given planar curve with respect to the maximum norm used by Floater. This characterization does not however yield the best conic approximation obtained by the direct minimization of the Hausdorff distance. Degen [6] presents an overview of geometric Hermite interpolation, also emphasizing differential geometry aspects.

The problem of approximating a planar curve by a conic spline has also been studied from a more practical standpoint. Farin [12] presents a global method and discusses at length how curvature continuity can be achieved between conic segments. Pottmann [24] presents a local scheme, still achieving curvature continuity. Yang [28] constructs a curvature continuous conic spline by first fitting a tangent continuous conic spline to a point set and fairing the resulting curve. Li et al. [15] show how to divide the initial curve into simple segments which can be efficiently approximated with rational quadratic Bézier curves. These methods have many limitations, among which the dependence on the specific parameterization of the curve, the large number of conic segments produced or the lack of accuracy and absence of control of the error.

1.2. Results of this paper

Complexity of conic approximants. We show that the complexity – the number of elements – of an optimal *parabolic spline* approximating the curve to within Hausdorff distance ε is of the form $c_1\,\varepsilon^{-1/4} + O(1)$, where we express the value of the constant c_1 in terms of the Euclidean and affine curvatures (see Theorem 5.1, Section 5). An optimal *conic spline* approximates the curve to fifth order, so its complexity is of the form $c_2\,\varepsilon^{-1/5} + O(1)$. Also in this case the constant c_2 is expressed in the Euclidean and affine curvature. These bounds are obtained by first deriving an expression for the Hausdorff distance of a conic arc that is tangent to a (sufficiently short) curve at its endpoints, and minimizes the Hausdorff distance among all such bitangent conics. Applying well-known methods like those of [5] it follows that this Hausdorff distance is of fifth order in the length of the curve, and of fourth order if the conic is a parabola. However, we derive explicit constants in these asymptotic expansions in terms of the Euclidean and affine curvatures of the curve.

Algorithmic issues. For curves with monotone affine curvature, called *affine spirals*, we consider conic arcs tangent to the curve at its endpoints, and show that among such *bitangent conic arcs* there is a unique one minimizing the Hausdorff distance. This optimal bitangent conic arc C_{opt} intersects the curve at its endpoints and at one interior point, but nowhere else. If $\alpha : I \to \mathbb{R}^2$ is an affine spiral, its displacement function $d : I \to \mathbb{R}$ measures the signed distance between the affine spiral and the optimal bitangent conic along the normal lines of the spiral. The displacement function d has an *equioscillation property*: there are two parameter values $u_+, u_- \in I$ such that $d(u_+) = -d(u_-) = \delta_H(\alpha, C_{\text{opt}})$ and the points $\alpha(u_-)$ and $\alpha(u_+)$ are separated by the interior point of intersection of α and C_{opt}. Furthermore, the Hausdorff distance between a section of an affine spiral and its optimal approximating bitangent conic arc is a monotone function of the arc length of the spiral section. This useful property gives rise to a bisection based algorithm for the computation of an optimal interpolating tangent continuous conic spline. The scheme reproduces conics. We implemented such an algorithm, and compare its theoretical complexity with the actual number of elements in an optimal approximating parabolic or conic spline.

1.3. Paper overview

Section 2 reviews some notions from affine differential geometry. In particular, we introduce affine arc length and affine curvature, which are invariant under equi-affine transformations. Conic arcs are the only curves in the plane having constant affine curvature, which explains the relevance of these notions from affine differential geometry for our work. Section 3 introduces affine spirals, a class of curves which have a unique optimal bitangent conic. We show that the displacement function, which measures the distance of the curve to its offset curve along its normals, has an equioscillation property in the sense that it has extremes at exactly two points on the curve. Furthermore, the Hausdorff distance between an arc of an affine spiral and its optimal bitangent conic arc is increasing in the length of this arc. This useful property gives rise to a bisection algorithm for the computation of a conic spline approximating a smooth curve with a minimal number of elements. Section 6 presents the output of the algorithm for a collection of examples. The main result of Section 4 is a relation between the affine curvatures of a curve and a bitangent offset curve. We use this result in Section 5 to derive an expression for the complexity of optimal parabolic and conic splines approximating a regular curve. We do so by deriving a bound on the Hausdorff distance between an affine spiral arc and its optimal bitangent conic. We conclude with topics for future work in Section 7.

2. Preliminaries from differential geometry

Circular arcs and straight line segments are the only regular smooth curves in the plane with constant Euclidean curvature. Conic arcs are the only smooth curves in the plane with constant *affine curvature*. The latter property is crucial for our approach, so we briefly review some concepts and properties from affine differential geometry of planar curves. See also Blaschke [3].

2.1. Affine curvature

Recall that a regular curve $\alpha : J \to \mathbb{R}^2$ defined on a closed real interval J, i.e., a curve with non-vanishing tangent vector $T(u) := \alpha'(u)$, is parameterized according to Euclidean arc length if its tangent vector T has unit length. In this case, the derivative of the tangent vector is in the direction of the unit normal vector $N(u)$, and the Euclidean curvature $\kappa(u)$ measures the rate of change of T, i.e., $T'(u) = \kappa(u)\, N(u)$. Euclidean curvature is a differential invariant of regular curves under the group of rigid motions of the plane, i.e., a regular curve is uniquely determined by its Euclidean curvature, up to a rigid motion.

The larger group of *equi-affine transformations* of the plane, i.e., affine transformations with determinant one (in other words, area preserving linear transformations), also gives rise to a differential invariant, called the *affine curvature* of the curve. To introduce this invariant, let $I \subset \mathbb{R}$ be an interval, and let $\gamma : I \to \mathbb{R}^2$ be a smooth, regular plane curve. We shall denote differentiation with respect to the parameter u by a dot: $\dot\alpha = \frac{d\alpha}{du}$, $\ddot\alpha = \frac{d^2\alpha}{du^2}$, and so on. Then regularity means that $\dot\alpha(u) \neq 0$, for $u \in I$. Let the reparameterization $u(r)$ be such that $\gamma(r) = \alpha(u(r))$ satisfies

$$[\gamma'(r), \gamma''(r)] = 1. \tag{2.1}$$

Here $[v, w]$ denotes the determinant of the pair of vectors $\{v, w\}$, and derivatives with respect to r are denoted by dashes. The parameter r is called the *affine arc length* parameter. If $[\dot\alpha, \ddot\alpha] \neq 0$, in other words, if the curve α *has non-zero curvature*, then α can be parameterized by affine arc length, and (2.1) implies that

$$[\dot\alpha(u(r)), \ddot\alpha(u(r))]\, u'(r)^3 = 1. \tag{2.2}$$

Putting

$$\varphi(u) = [\dot\alpha(u), \ddot\alpha(u)]^{1/3}, \tag{2.3}$$

we rephrase (2.2) as

$$u'(r) = \frac{1}{\varphi(u(r))}. \tag{2.4}$$

From (2.1) it also follows that $[\gamma'(r), \gamma'''(r)] = 0$, so there is a smooth function k such that

$$\gamma'''(r) + k(r)\,\gamma'(r) = 0. \tag{2.5}$$

The quantity $k(r)$ is called the *affine curvature* of the curve γ at $\gamma(r)$. It is only defined at points of non-zero Euclidean curvature. A regular curve is uniquely determined by its affine curvature, up to an equi-affine transformation of the plane.

From (2.1) and (2.5) we conclude $k = [\gamma'', \gamma''']$. The affine curvature of α at $u \in I$ is equal to the affine curvature of γ at r, where $u = u(r)$.

2.2. Affine Frenet-Serret frame

The well known Frenet-Serret identity for the Euclidean frame, namely

$$\dot{\alpha} = T, \qquad \dot{T} = \kappa\, N, \qquad \dot{N} = -\kappa\, T, \tag{2.6}$$

where the dot indicates differentiation with respect to Euclidean arc length, have a counterpart in the affine context. More precisely, let α be a strictly convex curve parameterized by affine arc length. The *affine Frenet-Serret frame* $\{t(r), n(r)\}$ of α is a moving frame at $\alpha(r)$, defined by $t(r) = \alpha'(r)$, and $n(r) = t'(r)$, respectively. Here the dash indicates differentiation with respect to affine arc length. The vector t is called the affine tangent, and the vector n is called the affine normal of the curve. The affine frame satisfies

$$\alpha' = t, \qquad t' = n, \qquad n' = -k\, t. \tag{2.7}$$

Furthermore, we have the following identity relating the affine moving frame $\{t, n\}$ and the Frenet-Serret moving frame $\{T, N\}$.

Lemma 2.1. *1. The affine arc length parameter r is a function of the Euclidean arc length parameter s satisfying*

$$\frac{dr}{ds} = \kappa(s)^{1/3}. \tag{2.8}$$

2. The affine frame $\{t, n\}$ and the Frenet-Serret frame $\{T, N\}$ are related by

$$t = \kappa^{-1/3}\, T, \qquad n = -\tfrac{1}{3}\, \kappa^{-5/3}\, \dot{\kappa}\, T + \kappa^{1/3}\, N. \tag{2.9}$$

Here $\dot{\kappa}$ is the derivative of the Euclidean curvature with respect to Euclidean arc length.

Proof. 1. Let $\gamma(r)$ be the parametrization by affine arc length, and let $\alpha(s) = \gamma(r(s))$ be the parametrization by Euclidean arc length. Then $\dot{\alpha} = T$ and $\ddot{\alpha} = \kappa\, N$. Again we denote derivatives with respect to Euclidean arc length by a dot. Since $\gamma' = t$ and $t' = \gamma'' = n$, we have

$$T = \dot{\alpha} = \dot{r}t, \quad \text{and} \quad N = \kappa^{-1}\, \ddot{\alpha} = \kappa^{-1}(\ddot{r}t + (\dot{r})^2 n) \tag{2.10}$$

Since $[T, N] = 1$, and $[t, n] = 1$, we obtain $1 = \kappa^{-1}\dot{r}^3$. This proves the first claim.

2. The first part of the lemma implies $\ddot{r} = \frac{1}{3}\, \kappa^{-2/3}\, \dot{\kappa}$. Plugging this into the identity (2.10) yields the expression for the affine Frenet-Serret frame in terms of the Euclidean Frenet-Serret frame. $\square$

The affine Frenet-Serret identities (2.7) yield the following values for the derivatives—with respect to affine arc length—of α up to order five, which will be useful in the sequel:

$$\begin{gathered}\alpha' = t, \quad \alpha'' = n, \quad \alpha''' = -k\, t, \\ \alpha^{(4)} = -k'\, t - k\, n, \quad \alpha^{(5)} = (k^2 - k'')\, t - 2k'\, n.\end{gathered} \tag{2.11}$$

Combining these identities with the Taylor expansion of α at a given point yields the following affine local canonical form of the curve.

Lemma 2.2. *Let $\alpha : I \to \mathbb{R}^2$ be a regular curve with non-vanishing curvature, and with affine Frenet-Serret frame $\{t, n\}$. Then*

$$\begin{aligned}\alpha(r_0 + r) = \alpha(r_0) &+ \left(r - \tfrac{1}{3!}\, k_0\, r^3 - \tfrac{1}{4!}\, k_0'\, r^4 + O(r^6)\right) t_0 \\ &\qquad + \left(\tfrac{1}{2}\, r^2 - \tfrac{1}{4!}\, k_0\, r^4 - \tfrac{2}{5!}\, k_0'\, r^5 + O(r^6)\right) n_0,\end{aligned}$$

where t_0, n_0, k_0, and k_0' are the values of t, n, k and k' at r_0.

Furthermore, in its affine Frenet-Serret frame the curve α can be written locally as $x\, t_0 + y(x)\, n_0$, with

$$y(x) = \tfrac{1}{2}\, x^2 + \tfrac{1}{8}\, k_0\, x^4 + \tfrac{1}{40}\, k_0' x^5 + O(x^6).$$

The first identity follows directly from (2.11). As for the second, it follows from the first by a series expansion. Indeed, write

$$x = r - \tfrac{1}{3!}\, k_0\, r^3 - \tfrac{1}{4!}\, k_0'\, r^4 + O(r^6).$$

Computing the expansion of the inverse function gives

$$r = x + \tfrac{1}{3!}\, k_0\, x^3 + \tfrac{1}{4!}\, k_0'\, x^4 + O(x^6).$$

Plugging in $y = \frac{1}{2}\, r^2 - \frac{1}{4!}\, k_0\, r^4 - \frac{2}{5!}\, k_0'\, r^5 + O(r^6)$ gives the result.

2.3. Affine curvature of curves with arbitrary parameterization

The following proposition gives an expression for the affine curvature of a regular curve in terms of an arbitrary parameterization. See also [3, Chapter 1.6].

Proposition 2.3. *Let $\alpha : I \to \mathbb{R}^2$ be a regular C^4-curve with non-zero Euclidean curvature. Then the affine curvature k of α is given by*

$$k = \frac{1}{\varphi^5}\, [\ddot{\alpha}, \dddot{\alpha}] + \frac{\ddot{\varphi}\, \varphi - 3\dot{\varphi}^2}{\varphi^4}, \tag{2.12}$$

where $\varphi = [\dot{\alpha}, \ddot{\alpha}]^{1/3}$.

For a proof of this result we refer to Appendix A.

Remark. Proposition 2.3 gives the following expression for the affine curvature k in terms of the Euclidean curvature κ:

$$k = \frac{9\, \kappa^4 - 5\, (\dot{\kappa})^2 + 3\, \kappa\, \ddot{\kappa}}{9\, \kappa^{8/3}},$$

where $\dot{\kappa}$ and $\ddot{\kappa}$ are the derivatives of the Euclidean curvature with respect to arc length. This identity is obtained by observing that, for a curve parameterized by Euclidean arc length, the function φ is given by $\varphi = \kappa^{1/3}$. This follows from the Frenet-Serret identities (2.6) and the definition (2.3) of φ. Substituting this expression into (2.12) yields the identity for k in terms of κ.

2.4. Conics have constant affine curvature

Solving the differential equation (2.5) shows that a curve of constant affine curvature is a conic arc. More precisely, a curve with constant affine curvature is a hyperbolic, parabolic, or elliptic arc iff its affine curvature is negative, zero, or positive, respectively.

We now give expressions for the (constant) affine curvature of conics defined by an implicit quadratic equation.

Proposition 2.4 ([22], Theorem 6.4). *The affine curvature of the conic defined by the quadratic equation*

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

is given by $k = S\,T^{-2/3}$, where

$$S = \begin{vmatrix} a & b \\ b & c \end{vmatrix}, \qquad T = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}.$$

The next result relates the affine curvatures of a regular curve in the plane and its image under linear transformations.

Lemma 2.5. *Let α be the image of a regular planar curve β under a linear transformation $x \mapsto Ax$. The affine curvatures k_α and k_β of the curves α and β are related by $k_\alpha = (\det A)^{-2/3}\,k_\beta$.*

Proof. Assume that β is parameterized by affine arc length. Since $\alpha(u) = A\beta(u)$, it follows that the function φ, defined by (2.3), satisfies $\varphi = [A\dot\beta, A\ddot\beta]^{1/3} = (\det A)^{1/3}[\dot\beta, \ddot\beta]^{1/3} = (\det A)^{1/3}$. According to Proposition 2.3 the affine curvature of α is given by $k_\alpha = (\det A)^{-5/3}\,[A\ddot\beta, A\dddot\beta] = (\det A)^{-2/3}\,k_\beta$. □

2.5. Osculating conic at non-sextactic points

At a point of non-vanishing Euclidean curvature there is a unique conic, called the *osculating conic*, having fourth order contact with the curve at that point (or, in other words, having five coinciding points of intersection with the curve). The affine curvature of this conic is equal to the affine curvature of the curve at the point of contact. Moreover, the contact is of order five if the affine curvature has vanishing derivative at the point of contact. In that case the point of contact is a *sextactic point*. Again, see [3] for further details. At non-sextactic points the curve and its osculating conic cross (see also Figure 1):

Corollary 2.6. *At a non-sextactic point a curve crosses its osculating conic from right to left if its affine curvature is locally increasing at that point, and from left to right if the affine curvature is locally decreasing.*

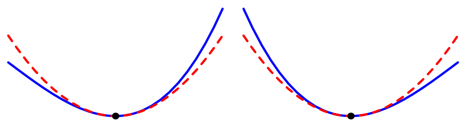


FIGURE 1. The curve and its osculating conic (dashed). The affine curvature is increasing in the left picture, and decreasing in the right picture.

2.6. The five-point conic

To derive error bounds for an optimal approximating conic we use the property that the approximating conic depends smoothly on the points of intersection with the curve. More precisely, let $\alpha : I \to \mathbb{R}^2$ be a regular curve without sextactic points, and let s_i, $1 \leqslant i \leqslant 5$, be points on I, not necessarily distinct. The unique conic passing through the points $\alpha(s_i)$ is denoted by C_s, with $s = (s_1, s_2, s_3, s_4, s_5)$. If one or more of the points coincide, the conic has contact with the curve of order corresponding to the multiplicity of the point. For instance, if $s_1 = s_2 \neq s_i$, $i \geqslant 3$, then C_s has first order contact with (is tangent to) the curve at $\alpha(s_1)$.

If $s_i \neq s_j$, for $i \neq j$, then the implicit quadratic equation of this conic can be obtained as follows. Let the *Veronese mapping* $\Psi : \mathbb{R}^2 \to \mathbb{R}^6$ be defined by $\Psi(x) = (x_1^2, x_1x_2, x_2^2, x_1, x_2, 1)$, $x = (x_1, x_2)$, then the equation of the conic C_s is $f(x, s) = 0$, with

$$f(x, s) = \det\bigl(\Psi(x), \Psi(\alpha(s_1)), \Psi(\alpha(s_2)), \Psi(\alpha(s_3)), \Psi(\alpha(s_4)), \Psi(\alpha(s_5))\bigr). \tag{2.13}$$

However, if $s_i = s_j$ for $i \neq j$, then $f(x, s) = 0$. We obtain a quadratic equation of the conic C_s by (formally) dividing $f(x, s)$ by $s_i - s_j$. More precisely:

Lemma 2.7. *If α is a C^m-curve, $m \geqslant 4$, then the conic C_s has a quadratic equation with coefficients that are C^{m-4}-functions of $s = (s_1, s_2, s_3, s_4, s_5) \in \mathbb{R}^5$.*

Proof. Put $\psi(s) = \Psi(\alpha(s))$. The *Newton development* of ψ in terms of the *divided differences* of ψ up to order four associated with the points $s_1, \ldots, s_5$ is given by $\psi(s_k) = \psi(s_1) + \sum_{i=2}^{k} \prod_{j=1}^{i-1}(s_k - s_j)\,[s_1, \ldots, s_k]\,\psi$, for $2 \leqslant k \leqslant 5$. See Appendix B. Plugging these identities into (2.13), we see that $f(x, s) = \prod_{1\leqslant j<k\leqslant 5} (s_k - s_j)\, F(x, s)$, with

$$F(x, s) = \det(\Psi(x), \psi(s_1), [s_1, s_2]\,\psi, \ldots, [s_1, \ldots, s_5]\,\psi).$$

Since ψ is C^m, with $m \geqslant 4$, it follows from Appendix B that F is a C^{m-4}-function, with $x \mapsto F(x, s)$ being a non-vanishing quadratic function. $\square$

In particular, if $\overline{\sigma} = (\sigma, \ldots, \sigma) \in \mathbb{R}^5$, then it follows from Appendix B, Lemma B.1, that

$$F(x, \overline{\sigma}) = \frac{1}{2!3!4!} \det\bigl(\Psi(x), \psi(\sigma), \psi'(\sigma), \psi''(\sigma), \psi'''(\sigma), \psi^{(4)}(\sigma)\bigr).$$

Since α contains no sextactic points, the equation $F(x, \overline{\sigma}) = 0$ defines a non-degenerate conic, the *osculating conic* at $\alpha(\sigma)$.

If σ, ϱ and u are three distinct points of I, then there is a unique conic $C_{\sigma,\varrho,u}$ which is tangent to α at $\alpha(\sigma)$ and $\alpha(\varrho)$, and passes through the point $\alpha(u)$. The equation of this conic is

$$\det\big(\Psi(x), \psi(\sigma), \psi'(\sigma), \psi(\varrho), \psi'(\varrho), \psi(u)\big) = 0.$$

As in the proof of Lemma 2.7 one proves that $C_{\sigma,\varrho,u}$ is a C^{m-4}-function of (σ, ϱ, u), and that it tends to the osculating conic at $\alpha(\sigma)$ as $u \to \sigma$ and $\varrho \to \sigma$.

3. Optimal conic approximation of affine spiral arcs

In this section we prove both the equioscillation property and the monotonicity property of the Hausdorff distance. Both properties are *global*, since the affine spiral is not necessarily short.

3.1. Intersections of conics and affine spirals

We start with a useful global property of affine spirals.

Proposition 3.1. 1. *A conic intersects an affine spiral in at most five points, counted with multiplicity.*
2. *The osculating conics of an affine spiral are disjoint, and do not intersect the spiral arc except at their point of contact.*

A proof of this theorem is given in [23, chapter 4]. The second part is an exercise in [3, chapter 1]. A modern proof is given in [27].

Now consider an affine spiral arc $\alpha : [u_0, u_1] \to \mathbb{R}^2$. Let C_u, $u_0 \leqslant u \leqslant u_1$, be the unique conic that is tangent to α at its endpoints, and intersects it at the point $\alpha(u)$. For $u = u_0$ and $u = u_1$ the conic has a triple intersection with the curve, or, in other words, it has contact of second order with α there.

Proposition 3.2. 1. *Two conics C_u and $C_{u'}$, $u \neq u'$, are tangent at $\alpha(u_0)$ and $\alpha(u_1)$, and have no other intersections.*
2. *Conic C_u intersects arc α at $\alpha(u_0)$, $\alpha(u)$, and $\alpha(u_1)$, but at no other point.*

Proof. 1. By Bezout's theorem, two conics intersect in at most four points, counted with multiplicity. Since conics C_u and $C_{u'}$ intersect with multiplicity two at each of the points $\alpha(u_0)$ and $\alpha(u_1)$, there are no other intersections.

2. This is a straightforward consequence of Proposition 3.1, part 1.

□

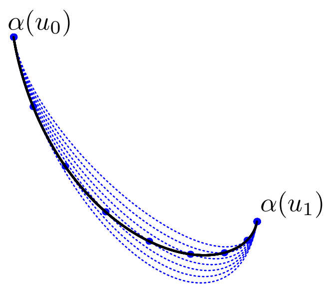


FIGURE 2. The curve and the family of conics C_u, $u_0 \leqslant u \leqslant u_1$, tangent at the endpoints $\alpha(u_0)$ and $\alpha(u_1)$ and passing through $\alpha(u)$.

3.2. Displacement function

A *bitangent conic* of a regular curve $\alpha : I \to \mathbb{R}^2$ is a conic arc which is tangent to α at its endpoints, such that each normal line of α intersects the conic arc in a unique point. Therefore, a bitangent conic has a parameterization $\beta : I \to \mathbb{R}^2$ of the form $\beta(u) = \alpha(u) + d(u)\,N(u)$, where $d : I \to \mathbb{R}$ is the *displacement function* of the conic arc. The Hausdorff distance between α and a bitangent conic C is equal to

$$\delta_H(\alpha, C) = \max_{u \in I} |d(u)|.$$

There is a one-parameter family of bitangent conics, so the goal is to determine an *optimal bitangent conic*, i.e., a conic in this family that minimizes the Hausdorff distance.

3.3. Equioscillation property

Denote the arc of the curve between $\alpha(u_0)$ and $\alpha(u)$ by α_u^-, and the arc between $\alpha(u)$ and $\alpha(u_1)$ by α_u^+. Similarly, C_u^- and C_u^+ denote the arcs of C_u between $\alpha(u)$ and $\alpha(u_0)$ and between $\alpha(u)$ and $\alpha(u_1)$, respectively.

Corollary 3.3 (Equioscillation property). *There is a unique conic C_{u_*} in the family $C_u, u_0 \leqslant u \leqslant u_1$, such that the Hausdorff distance d_{u_*} of α and C_{u_*} is minimal:*

$$d_{u_*} = \min_{u_0 \leqslant u \leqslant u_1} \delta_H(\alpha, C_u).$$

Furthermore,

$$d_{u_*} = \delta_H(\alpha_{u_*}^-, C_{u_*}^-) = \delta_H(\alpha_{u_*}^+, C_{u_*}^+).$$

Proof. Let $\delta^\pm(u) = \delta_H(\alpha_u^\pm, C_u^\pm)$. Then there are two cases: (i) $\delta^-(u)$ is increasing and $\delta^+(u)$ is decreasing as a function of u, and (ii) $\delta^-(u)$ is decreasing and $\delta^+(u)$ is increasing as a function of u. The situation depicted in Figure 2 corresponds to Case (i). This observation, which is a direct consequence of Proposition 3.2, part 2, implies that there is a unique u_* such that $\delta^-(u_*) = \delta^+(u_*)$. Obviously, d_{u_*} satisfies the two claimed identities. □

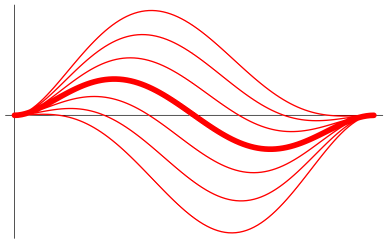


FIGURE 3. The graphs of the family of displacement functions. The bold graph corresponds to the displacement function of the optimal conic.

Let $d(s;u)$, $u_0 \leqslant s, u \leqslant u_1$, be the displacement function defined by the condition that the point $\alpha(s) + d(s;u)N(s)$, lies on the conic arc C_u. Here $N(s)$ is the unit normal of the curve at $\alpha(s)$. The graphs of the functions $s \mapsto d(s;u)$, $u_0 \leqslant s \leqslant u_1$, are disjoint, except at their endpoints. See Figure 3. We *conjecture* that the displacement function of an affine spiral is bimodal, i.e., its displacement function has the profile of any of the graphs depicted in Figure 3. More precisely, the function has one maximum, one minimum, and one interior zero, and there are no other interior extremal points.

3.4. Monotonicity of optimal Hausdorff distance

If one endpoint of the affine spiral moves along the curve α, the Hausdorff distance between the affine spiral and its optimal bitangent conic arc is *monotone* in the arc length of the affine spiral. This result shows that bisection methods can be used for the computation of an optimal approximating conic arc. We use this property for the implementation of the algorithm presented in Section 6.

Proposition 3.4 (Monotonicity of Hausdorff distance along spiral arcs). *Let $\alpha : I \to \mathbb{R}^2$ be an affine spiral arc, where I is an open interval containing 0. For $\varrho > 0$ let α_ϱ be the sub-arc between $\alpha(0)$ and $\alpha(\varrho)$, and let β_ϱ be the (unique) conic arc tangent to α_ϱ at its endpoints, and minimizing the Hausdorff distance between α_ϱ and the conic arcs tangent to α_ϱ at its endpoints. Then the Hausdorff distance between α_ϱ and β_ϱ is a monotonically increasing function of ϱ, for $\varrho \geqslant 0$.*

Proof. First we introduce some notation. The unique interior point of intersection of α_ϱ and β_ϱ occurs at $u = u(\varrho) \in I$. The sub-arcs of α_ϱ and β_ϱ between $\alpha(0)$ and $\alpha(u(\varrho))$ are denoted by α_ϱ^- and β_ϱ^-, respectively. The complementary sub-arcs of α_ϱ and β_ϱ are denoted by α_ϱ^+ and β_ϱ^+, respectively. According to the Equioscillation Property (Corollary 3.3) the Hausdorff distance between α_ϱ and β_ϱ is equal to the Hausdorff distances between $\alpha_\varrho^\pm$ and $\beta_\varrho^\pm$, and is attained as the distance between points $a_\pm(\varrho)$ on $\alpha_\varrho^\pm$ and $b_\pm(\varrho)$ on $\beta_\varrho^\pm$, i.e.,

$$\delta_H(\alpha_\varrho, \beta_\varrho) = \operatorname{dist}(a_-(\varrho), b_-(\varrho)) = \operatorname{dist}(a_+(\varrho), b_+(\varrho)).$$

The complete conic containing β_ϱ will be denoted by K_ϱ. We will repeatedly use the following consequence of Bezout's theorem:

Intersection Property: For $0 < \varrho_1 < \varrho_2$, the conics K_{ϱ_1} and K_{ϱ_2} have at most two points of intersection (possibly counted with multiplicity) different from $\alpha(0)$.

Let $\varrho_1, \varrho_2 \in I$, with $0 < \varrho_1 < \varrho_2$. The regions bounded by $\alpha^\pm_{\varrho_2}$ and $\beta^\pm_{\varrho_2}$ are denoted by $R^\pm$. Since K_{ϱ_1} is either compact or unbounded, and not disjoint from the boundary of R^+, it intersects this boundary in an even number of points (counted with multiplicity). Our strategy is to prove that $\beta^-_{\varrho_1}$ lies inside R^-, or that $\beta^+_{\varrho_1}$ lies inside R^+. In the former case, we see that

$$\delta_H(\alpha_{\varrho_1}, \beta_{\varrho_1}) = \mathrm{dist}(a_-(\varrho_1), b_-(\varrho_1)) < \mathrm{dist}(a_-(\varrho_2), b_-(\varrho_2)) = \delta_H(\alpha_{\varrho_2}, \beta_{\varrho_2}),$$

whereas in the latter case

$$\delta_H(\alpha_{\varrho_1}, \beta_{\varrho_1}) = \mathrm{dist}(a_+(\varrho_1), b_+(\varrho_1)) < \mathrm{dist}(a_+(\varrho_2), b_+(\varrho_2)) = \delta_H(\alpha_{\varrho_2}, \beta_{\varrho_2}).$$

We distinguish two cases, depending on the order of $u(\varrho_1)$ and $u(\varrho_2)$.

Case 1: $u(\varrho_1) > u(\varrho_2)$. Note that the conic K_{ϱ_1} is tangent to α at $\alpha(\varrho_1)$, a point contained in α_{ϱ_2}. Therefore, in this case K_{ϱ_1} intersects $\alpha^+_{\varrho_2}$ in an odd number of points, namely, once at the point $\alpha(u(\varrho_1))$ and twice at the point of tangency $\alpha(\varrho_1)$.

$\beta^+_{\varrho_2}$, the other part of boundary of R^+, in an odd number of points. By the Intersection Property, this odd number is equal to one. Since both endpoints of β_{ϱ_1} lie on the same side of β_{ϱ_2}, this point of intersection does not lie on β_{ϱ_1}. In other words, the interior of $\beta^+_{\varrho_1}$ lies inside the region R^+.

Case 2: $u(\varrho_1) < u(\varrho_2)$. In this case K_{ϱ_1} does not cross $\alpha^+_{\varrho_2}$, since it intersects $\alpha^+_{\varrho_2}$ in two coinciding points at the tangency $\alpha(\varrho_1)$, but at no other point. Therefore, K_{ϱ_1} intersects $\beta^+_{\varrho_2}$, the other part of the boundary of R^+, in at least two points (at least one entrance and at least one exit point). By the Intersection Property, apart from $\alpha(0)$, these are the only points in which K_{ϱ_1} and K_{ϱ_2} intersect. Therefore, $\beta^-_{\varrho_1}$ intersects neither $\beta^-_{\varrho_2}$ nor $\alpha^-_{\varrho_2}$ in an interior point. In other words, the interior of $\beta^-_{\varrho_1}$ lies inside the region R^-. $\square$

Remark. A similar monotonicity property holds for the Hausdorff distance between an affine spiral and a bitangent *parabolic arc.* The proof is omitted, since it is straightforward, and along the same lines as the proof of Proposition 3.4.

4. Affine curvature of offset curves

The main result of this section is a relation between the affine curvatures of a curve and a bitangent offset curve.

Let $\alpha : I \to \mathbb{R}^2$ be a regular curve parameterized by affine arc length, with affine arc length parameter $u \in I$. Here I is an open interval, containing 0. We consider offset curves tangent to α at $\alpha(0)$ and $\alpha(\varrho)$. The affine curvature of such a curve is related to the affine curvature k of α, as indicated in the first part of the following Lemma. In the second part, an analogous result relates these curvatures when there is an additional point of intersection at $\alpha(\sigma)$.

Lemma 4.1 (Affine curvature of offset curves). *Let α be a C^m-regular curve.*

1. Let $\beta : I\times I \to \mathbb{R}^2$ be a C^n-function, such that, $\beta(\cdot,\varrho)$ is a curve tangent to α at $\alpha(0)$ and $\alpha(\varrho)$, for $\varrho \in I$. If $m, n \geqslant 5$, there are C^l-functions $P, Q : I\times I \to \mathbb{R}$, with $l = \min(m-5, n-4)$, such that

$$\beta(u,\varrho) = \alpha(u) + d(u,\varrho)\big(\, P(u,\varrho)t(u) + Q(u,\varrho)n(u)\,\big), \tag{4.1}$$

where $d(u,\varrho) = u^2\,(u-\varrho)^2$. Here $t(u)$ and $n(u)$ are the affine tangent and the affine normal of α, respectively. Furthermore, the affine curvature $k_\beta(u,\varrho)$ of $\beta(\cdot,\varrho)$ at $0 \leqslant u \leqslant \varrho$ is given by

$$k_\beta(u,\varrho) = k(0) + 8\,Q(0,0) + O(\varrho). \tag{4.2}$$

2. Let $\beta : I\times I\times I \to \mathbb{R}^2$ be a C^n-function, such that, $\beta(\cdot,\sigma,\varrho)$ is a curve tangent to α at $\alpha(0)$ and $\alpha(\varrho)$ and intersecting α at $\alpha(\sigma)$, for $\sigma,\varrho \in I$ and $0 \leqslant \sigma \leqslant \varrho$. If $m, n \geqslant 6$, and, moreover, β also intersects α at $\alpha(\sigma)$, with $0 \leqslant \sigma \leqslant \varrho$, then there are C^l-functions $P, Q : I\times I \to \mathbb{R}$, with $l = \min(m-6, n-5)$, such that

$$\beta(u,\sigma,\varrho) = \alpha(u) + d(u,\sigma,\varrho)\big(\, P(u,\sigma,\varrho)t(u) + Q(u,\sigma,\varrho)n(u)\,\big), \tag{4.3}$$

where $d(u,\sigma,\varrho) = u^2\,(u-\varrho)^2\,(u-\sigma)$. Furthermore, the affine curvature $k_\beta(u,\sigma,\varrho)$ of $\beta(\cdot,\sigma,\varrho)$ at $0 \leqslant u \leqslant \varrho$ is given by

$$k_\beta(u,\sigma,\varrho) = k(0) + k'(0)\,u + 8\,(5u-\sigma-2\varrho)\,Q(0,0,0) + O(\varrho^2).$$

Proof. 1. If α is C^m, then the functions $(u,\varrho) \mapsto [\beta(u,\varrho)-\alpha(u), n(u)]$ and $(u,\varrho) \mapsto [\beta(u,\varrho)-\alpha(u), t(u)]$ are of class $C^{\min(m-1,n)}$. For fixed ϱ, these functions have double zeros at $u=0$ and $u=\varrho$. The Division Property, cf. Appendix B, Lemma B.2, guarantees the existence of $C^{\min(m-5,n-4)}$-functions P and Q satisfying $[\beta(u,\varrho)-\alpha(u), n(u)] = d(u,\varrho)\,P(u,\varrho)$ and $[\beta(u,\varrho)-\alpha(u), t(u)] = d(u,\varrho)\,Q(u,\varrho)$. In other words, P and Q satisfy identity (4.1).

According to Proposition 2.3 the affine curvature of the curve $\beta(\cdot,\varrho)$ is a C^{n-4}-function given by

$$k_\beta = \frac{1}{\varphi^5}\,[\beta_{uu},\beta_{uuu}] + \frac{1}{\varphi^4}\,(\varphi_{uu}\,\varphi - 3\varphi_u^2), \tag{4.4}$$

$\varphi = [\beta_u,\beta_{uu}]^{1/3}$. In (4.4), the functions k_β, φ, β, and their partial derivatives are evaluated at (u,ϱ). Since $n \geqslant 5$, and $0 \leqslant u \leqslant \varrho$, it follows that $k_\beta(u,\varrho) = k_\beta(u,0) + O(\varrho)$. So, to prove (4.2), it is sufficient to determine $\beta(u,0)$ and its derivatives up to order four. Writing $\beta_0(u) = \beta(u,0)$, we see that

$$\beta_0(u) = \alpha(u) + f(u)\,(P_0\,t(u) + Q_0\,n(u)) + O(u^5),$$

where $f(u) = u^4$, $P_0 = P(0,0)$ and $Q_0 = Q(0,0)$. In view of the affine Frenet-Serret identities (2.7) we get

$$\begin{aligned}\beta_0' &= (1 + f'\,P_0)\,t + f'\,Q_0\,n + O(u^4),\\ \beta_0'' &= f''\,P_0\,t + (1 + f''\,Q_0)\,n + O(u^3),\\ \beta_0''' &= (-k\, + f'''\,P_0)\,t + f'''\,Q_0\,n + O(u^2).\end{aligned} \tag{4.5}$$

Here the functions β_0, f, t, n and k, as well as their derivatives, are evaluated at u. Since $\varphi(u,0) = [\beta_0'(u), \beta_0''(u)]^{\frac{1}{3}}$, we use the first two identities of (4.5) to derive

$$\varphi(u,0) = 1 + \tfrac{1}{3}\,f''(u)\,Q_0 + O(u^3) = 1 + 4\,u^2\,Q_0 + O(u^3).$$

Similarly, using the second and third identity of (4.5) we get

$$[\beta_0''(u), \beta_0'''(u)] = k(u) + O(u) = k(0) + 8\,Q_0 + O(u).$$

Identity (4.2) is obtained by plugging these expressions into (4.4).

2. Now we turn to the case where the offset curve not only is tangent to α at its endpoints, but also has an additional point of intersection at $\alpha(\sigma)$. The existence of functions P and Q satisfying (4.3) is proven as in Part 1, using the Division Property. Again the affine curvature of β is given by (4.4), where this time the functions k_β, φ, β, and their partial derivatives are evaluated at (u, σ, ϱ).

In (4.3) we have $d(u, \sigma, \varrho) = u^5 - (2\varrho + \sigma)\,u^4 + O(\varrho^2 + \sigma^2)$, $P = P_0 + O(u)$, and $Q = Q_0 + O(u)$. Focusing on the essential terms only, we rewrite (4.3) as:

$$\beta = \alpha + (u^5 - (2\varrho + \sigma)\,u^4)\,(P_0\,t + Q_0\,n) + O(u^6) + O((\varrho + \sigma)u^5) + O(\varrho^2 + \sigma^2). \tag{4.6}$$

Here α, t and n are evaluated at u, and β at (u, σ, ϱ). For a smoother presentation, we introduce the following terminology. The class $O_i(u, \sigma, \varrho)$, $0 \leqslant i \leqslant 4$, consists of all C^{m-i}-functions of the form $O(u^{6-i}) + O((\varrho + \sigma)u^{5-i}) + O(\varrho^2 + \sigma^2)$. Using this notation we rewrite (4.6) as

$$\beta = \alpha + f\,(P_0\,t + Q_0\,n) + O_0(u, \sigma, \varrho).$$

where $f(u, \sigma, \varrho) = u^5 - (2\varrho + \sigma)u^4$.

If $g \in O_i(u, \sigma, \varrho)$, then $g_u \in O_{i+1}(u, \sigma, \varrho)$, for $1 \leqslant i \leqslant 4$. Therefore, we get, as in (4.5):

$$\begin{aligned}\beta_u &= (1 + f_u\,P_0)\,t + f_u\,Q_0\,n + O_1(u, \sigma, \varrho),\\ \beta_{uu} &= f_{uu}\,P_0\,t + (1 + f_{uu}\,Q_0)\,n + O_2(u, \sigma, \varrho),\\ \beta_{uuu} &= (-k\, + f_{uuu}\,P_0)\,t + f_{uuu}\,Q_0\,n + O_3(u, \sigma, \varrho).\end{aligned} \tag{4.7}$$

Since $\varphi = [\beta_u, \beta_{uu}]^{\frac{1}{3}}$, we use the first two identities of (4.7) to derive

$$\varphi = 1 + \tfrac{1}{3}\,f_{uu}\,Q_0 + O_2(u, \sigma, \varrho),$$

so $\varphi = 1 + O_3(u, \sigma, \varrho)$, $\varphi_u^2 = O_4(u, \sigma, \varrho)$, and $\varphi_{uu} = \frac{1}{3} Q_0 f_{uuuu} + O_4(u, \sigma, \varrho)$. Similarly, using the second and third identity of (4.5) we get

$$[\beta_{uu}, \beta_{uuu}] = k(u) + O_4(u, \sigma, \varrho).$$

It follows that

$$\begin{aligned} k_\beta(u, \sigma, \varrho) &= k(u) + \tfrac{1}{3} f_{uuuu} Q_0 + O_4(u, \sigma, \varrho) \\ &= k(0) + k'(0)\, u + 8\,(5u - \sigma - 2\varrho)\, Q_0 + O(\varrho^2). \end{aligned}$$

Note that in the last identity we used that $O_4(u, \sigma, \varrho) = O(u^2 + \sigma^2 + \varrho^2) = O(\varrho^2)$, since $0 \leqslant u, \sigma \leqslant \varrho$. This concludes the proof of the second part. □

If the offset curves are bitangent conics, the affine curvature of these conics can be expressed in the Euclidean and affine curvature of the curve α at the points of intersection. Furthermore, we can determine the displacement function up to terms of order five if the conic is a parabola, and up to terms of order six in the general case. These results will enable us to determine an asymptotic expression for the Hausdorff distance between a small arc and its optimal bitangent conic.

Corollary 4.2 (Bitangent conics). *Let α be a strictly convex regular C^m-curve.*

1. If $m \geqslant 8$, a parabolic arc tangent to α at $\alpha(0)$ and $\alpha(\varrho)$ has the form

$$\beta(u, \varrho) = \alpha(u) + u^2\,(\varrho - u)^2\, D(u, \varrho)\, N(u), \tag{4.8}$$

where D is a C^{m-8}-function with $D(0,0) = -\frac{1}{8}\, k(0)\, \kappa(0)^{1/3}$. Here $N(u)$ is the Euclidean normal of α, and κ is its Euclidean curvature.

2. If $m \geqslant 9$, a conic arc tangent to α at $\alpha(0)$ and $\alpha(\varrho)$ and intersecting at $\alpha(\sigma)$, with $0 \leqslant \sigma \leqslant \varrho$, has the form

$$\beta(u, \sigma, \varrho) = \alpha(u) + u^2\,(\varrho - u)^2\,(u - \sigma)\, D(u, \sigma, \varrho)\, N(u), \tag{4.9}$$

where D is a C^{m-9}-function with $D(0,0,0) = -\frac{1}{40}\, k'(0)\, \kappa(0)^{1/3}$. Moreover, its affine curvature is of the form

$$k_\beta(\sigma, \varrho) = \tfrac{1}{5}\,(2k(0) + k(\sigma) + 2k(\varrho)) + O(\varrho^2).$$

Proof. 1. Obviously, the family of parabolic arcs can be written in the form $\beta(u, \varrho) = \alpha(u) + d(u, \varrho)\, N(u)$, provided ϱ is sufficiently small. According to Lemma 2.7, β is a C^{m-4}-function, so $d = [T, \beta - \alpha]$ is a C^{m-4}-function with double zeros at $u = 0$ and $u = \varrho$. According to Lemma 4.1, the parabola has a parameterization of the form (4.1), where P and Q are C^{m-8}-functions. Therefore, $d(u, \varrho) = u^2\,(u - \varrho)^2\, Q(u, \varrho)\,[T(u), n(u)]$, so β is of the form (4.8) with $D = Q\,[T, n]$. In particular, D is a C^{m-8}-function. Comparing this expression with identity (4.1) in Lemma 4.1, we see that $D(u, \varrho) = Q(u, \varrho)\,[T(u), n(u)]$. From (2.9) we conclude that $D(0,0) = \kappa(0)^{1/3}\, Q(0,0)$. Since the affine curvature of a parabolic arc is identically zero, Part 1 of Lemma 4.1 yields $Q(0,0) = -\frac{1}{8}\, k(0)$, yielding the value for $D(0,0)$ stated in Part 1.

2. As in Part 1 we prove that β has a parameterization of the form (4.9), where D is a C^{m-9}-function. The affine curvature of a conic arc is constant, so Part 2 of Lemma 4.1 yields $Q(0,0,0) = -\frac{1}{40}\,k'(0)$. Since also in this case we have $D(0,0,0) = \kappa(0)^{1/3}\,Q(0,0,0)$, we conclude that $D(0,0,0)$ has the value stated in Part 2. Furthermore, (4.2) yields

$$k_\beta = k(0) + \tfrac{1}{5}\,(\sigma + 2\varrho)\,k'(0) + O(\varrho^2) = \tfrac{1}{5}\,(2k(0) + k(\sigma) + 2k(\varrho)) + O(\varrho^2).$$

This concludes the proof of the second part. □

Remarks. 1. The second part of Corollary 4.2 can be generalized in the sense that the affine curvature of a conic intersecting a strictly convex arc at five points is equal to the average of the affine curvatures of the curve at these five points, up to quadratic terms in the affine length of the arc. The proof is similar to the one given above.

2. We conjecture that the 'loss of differentiability' is less than stated in Corollary 4.2. More precisely, we expect that D is of class C^{m-4} for a bitangent parabolic arc, and of class C^{m-5} for a bitangent conic arc.

5. Complexity of conic splines

In this section our goal is to determine the Hausdorff distance of a conic arc of best approximation to an arc of α of Euclidean length $\sigma > 0$, that is tangent to α at its endpoints. If the conic is a parabola, these conditions uniquely determine the parabolic arc. If we approximate with a general conic, there is one degree of freedom left, which we use to minimize the Hausdorff distance between the the arc of α and the approximating conic arc β. As we have seen in Section 3, the optimal conic arc intersects the arc of α in an interior point.

The main result of this section gives an asymptotic bound on this Hausdorff distance.

Theorem 5.1 (Error in parabolic and conic spline approximation). *Let β be a conic arc tangent at its endpoints to an arc of a regular curve α of length σ, with non-vanishing Euclidean curvature.*

1. If α is a C^8-curve, and β is a parabolic arc, then the Hausdorff distance between these arcs has asymptotic expansion

$$\delta_H(\alpha,\beta) = \tfrac{1}{128}\,|k_0|\,\kappa_0^{5/3}\,\sigma^4 + O(\sigma^5), \tag{5.1}$$

where κ_0 and k_0 are the Euclidean and affine curvatures of α at one of its endpoints, respectively.

2. If α is a C^9-curve, and β is a conic arc, then the Hausdorff distance between these arcs is minimized if the affine curvature of β is equal to the average of the *affine curvatures of α at its endpoints, up to quadratic terms in the length of α. In this case this Hausdorff distance has asymptotic expansion*

$$\delta_H(\alpha,\beta) = \tfrac{1}{2000\sqrt{5}}\, |k_0'|\, \kappa_0^2\, \sigma^5 + O(\sigma^6), \tag{5.2}$$

where κ_0 is the Euclidean curvature of α at one of its endpoints, and k_0' is the derivative of the affine curvature of α at one of its endpoints.

Proof. 1. According to Corollary 4.2, the parabolic arc has a parameterization of the form (4.8). It follows from Appendix C, Lemma C.1, applied to the displacement function $d(u) = u^2\,(\varrho - u)^2\, D(u,\varrho)$, cf. (4.8), that

$$\delta_H(\alpha,\beta) = \tfrac{1}{16}\, |D(0,0)|\, \varrho^4 + O(\varrho^5). \tag{5.3}$$

From Lemma 2.1, part 1, we derive

$$\varrho = \kappa_0^{1/3}\, \sigma + O(\sigma^2). \tag{5.4}$$

Since $D(0,0) = -\frac{1}{8}\, k(0)\, \kappa(0)^{1/3}$, we conclude from (5.3) and (5.4) that the Hausdorff distance satisfies (5.1).

2. Again, according to Corollary 4.2, cf. (4.9), a best approximating conic arc has a parameterization of the form (4.9), with $D(0,0,0) = -\frac{1}{40}\, k'(0)\, \kappa(0)^{1/3}$. Applying Appendix C, Lemma C.1 to the displacement function $d(u) = u^2\,(u-\sigma)\,(\varrho - u)^2\, D(u,\sigma,\varrho)$, cf. (4.9), we see that

$$\delta_H(\alpha,\beta) = \tfrac{1}{50\sqrt{5}} |D(0,0,0)|\, \varrho^5 + O(\varrho^6), \tag{5.5}$$

where the optimal conic intersects the curve α for $\sigma = \sigma(\varrho) = \frac{1}{2}\,\varrho + O(\varrho^2)$. Identities (5.4) and (5.5) imply that the Hausdorff distance is given by (5.2). Finally, the affine curvature of this conic is

$$\tfrac{1}{5}(2k(0) + k(\tfrac{1}{2}\,\varrho + O(\varrho^2)) + 2k(\varrho)) + O(\varrho^2) = \tfrac{1}{2}(k(0) + k(\varrho)) + O(\varrho^2).$$

This concludes the proof of the main theorem of this section. □

Remark. It would be interesting to give a direct *geometric* proof of the fact that the best approximating conic has affine curvature equal to the average of the affine curvatures of α at its endpoints.

The preceding result gives an asymptotic expression for the minimal number of elements of an optimal parabolic or conic spline in terms of the maximal Hausdorff distance.

Corollary 5.2 (Complexity of parabolic and conic splines). *Let $\alpha : [0, L] \to \mathbb{R}^2$ be a regular curve with non-vanishing Euclidean curvature of length L, parameterized by Euclidean arc length, and let $\kappa(s)$ and $k(s)$ be its Euclidean and affine curvature at $\alpha(s)$, respectively.*

1. If α is a C^8-curve, then the minimal number of arcs in a tangent continuous parabolic *spline approximating α to within Hausdorff distance ε is*

$$N(\varepsilon) = c_1 \Big(\int_0^L |k(s)|^{1/4}\, \kappa(s)^{5/12} ds \Big)\, \varepsilon^{-1/4}(1 + O(\varepsilon^{1/4})), \tag{5.6}$$

where $c_1 = 128^{-1/4} \approx 0.297$.

2. If α is a C^9-curve, then the minimal number of arcs in a tangent continuous conic *spline approximating α to within Hausdorff distance ε is*

$$N(\varepsilon) = c_2 \Big(\int_0^L |k'(s)|^{1/5}\, \kappa(s)^{2/5} ds \Big)\, \varepsilon^{-1/5}(1 + O(\varepsilon^{1/5})), \tag{5.7}$$

where $c_2 = (2000\sqrt{5})^{-1/5} \approx 0.186$.

We only sketch the proof, and refer to the papers by McClure and Vitale [19] and Ludwig [17] for details about this proof technique in similar situations. Consider a small arc of α, centered at $\alpha(s)$. Let $\sigma(s)$ be its Euclidean arc length. Then the Hausdorff distance between this curve and a bitangent parabolic arc is $\frac{1}{128}\, |k_0|\, \kappa_0^{5/3}\, \sigma(s)^4 + O(\sigma(s)^5)$, cf. Theorem 5.1. Therefore,

$$\sigma(s) = \sqrt[4]{128}\, |k(s)|^{-1/4}\, \kappa(s)^{-5/12}\, \varepsilon^{1/4}(1 + O(\varepsilon^{1/4})).$$

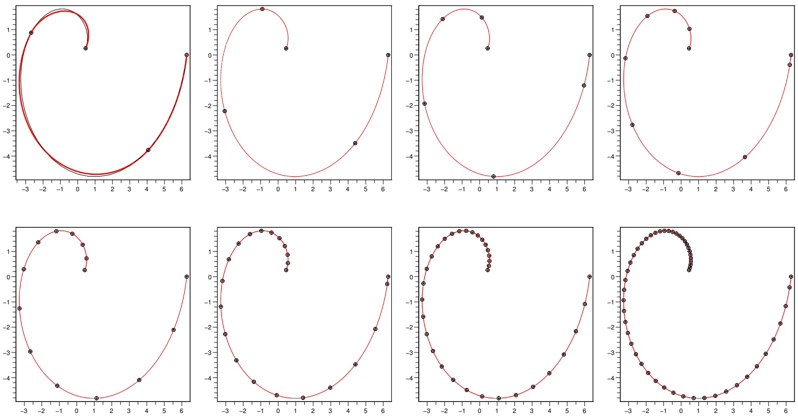
The first part follows from the observation that $N(\varepsilon) = \displaystyle\int_0^L \frac{1}{\sigma(s)}\, ds$. The proof of the second part is similar.

6. Implementation

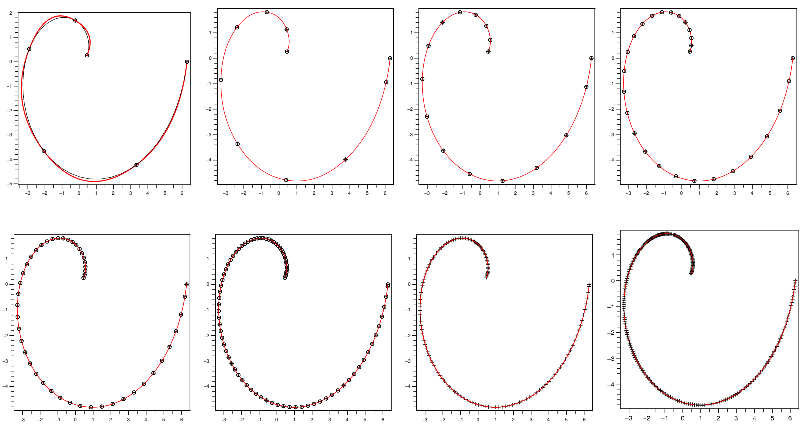
We implemented an algorithm in `C++` using the symbolic computing library `GiNaC`[1], for the computation of an optimal parabolic or conic spline, based on the monotonicity property. For computing the optimal parabolic spline, the curve is subdivided into affine spirals. Then for a given maximal Hausdorff distance ε, the algorithm iteratively computes optimal parabolic arcs starting at one endpoint. At each step of this iteration the next breakpoint is computed via a standard bisection procedure, starting from the most recently computed breakpoint. The bisection procedure yields a parabolic spline whose Hausdorff distance to the subtended arc is ε. An optimal conic spline is computed similarly. The bisection step is slightly more complicated, since the algorithm has to select the optimal conic arc from a one-parameter family. Here the equioscillation property gives the criterion for deciding whether the computed conic arc is optimal.

Below we present two examples of computations of optimal parabolic and conic splines. We compare the computed number of elements of these splines with the theoretical asymptotic complexity given in Corollary 5.2, thereby neglecting the higher order terms in (5.6) and (5.7).

[1] `http://www.ginac.de`



(a) Conic spline approximation



(b) Parabolic spline approximation

FIGURE 4. Approximation of the spiral for ε ranging between 10^{-1} to 10^{-8}.

6.1. A spiral curve

We present the results of our algorithm applied to the *spiral curve*, parameterized by $\alpha(t) = (t\cos(t), t\sin(t))$, with $\frac{1}{6}\pi \leqslant t \leqslant 2\pi$.

Figures 4(a) and 4(b) depict the result of the algorithm applied to the spiral for different values of the error bound ε, for the approximation by conic arcs and parabolic arcs respectively. For $\varepsilon \geqslant 10^{-2}$, there is no visual difference between the curve and its approximating conic.

ε	Parabolic Exp./ Th.	Conic Exp./ Th.
10^{-1}	5	3
10^{-2}	9	4
10^{-3}	15	6
10^{-4}	26	9
10^{-5}	46	13
10^{-6}	82	21
10^{-7}	145	32
10^{-8}	257	51

TABLE 1. The complexity (number of arcs) of the parabolic spline and the conic spline approximating the Spiral Curve. The theoretical complexity matches exactly with the experimental complexity, for various values of the maximal Hausdorff distance ε.

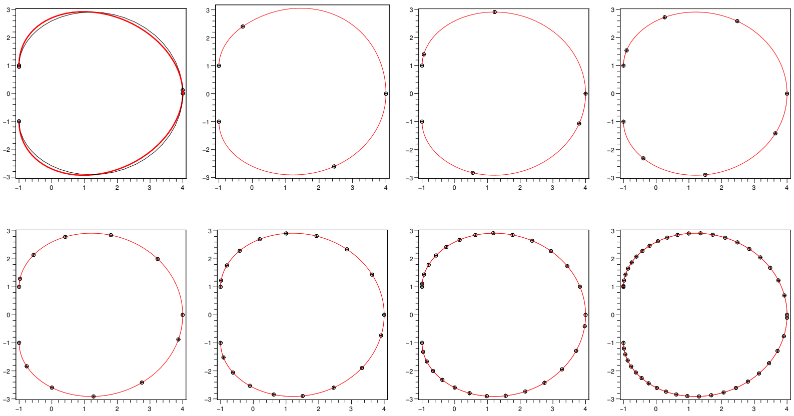
Table 1 gives the number of arcs computed by the algorithm, and the theoretical bounds on the number of arcs for varying values of ε, both for the parabolic and for the conic spline.

6.2. Cayley's sextic

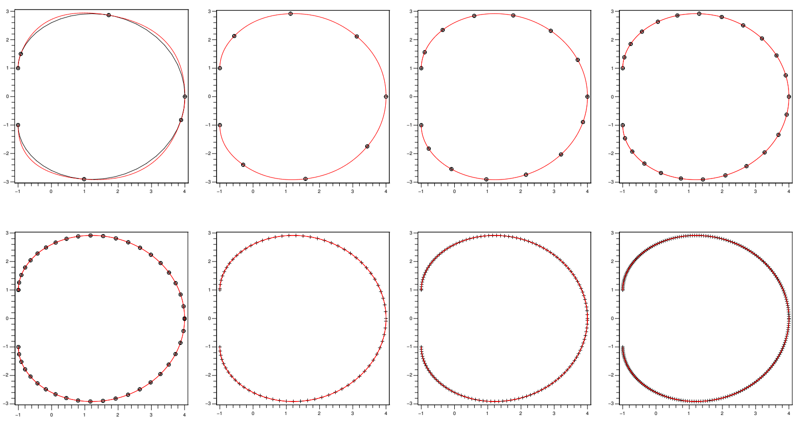
We present the results of our algorithm applied to the *Cayley's sextic*, the curve parameterized by $\alpha(t) = (4\cos(\frac{t}{3})^3\cos(t), 4\cos(\frac{t}{3})^3\sin(t))$, with $-\frac{3}{4}\pi \leqslant t \leqslant \frac{3}{4}\pi$. This curve has a sextactic point at $t = 0$. For all values of ε we divide the parameter interval into two parts $[-\frac{3}{4}\pi, 0]$ and $[0, \frac{3}{4}\pi]$ each containing the sextactic point as an endpoint, and then approximate with conic arcs using the Incremental Algorithm.

The pictures in Figure 5(a) give the *conic spline* approximation images for Cayley's sextic for different values of ε. The first picture in Figure 5(b) gives the original curve and its *parabolic spline* approximation for $\varepsilon = 10^{-1}$. The rest of the pictures in Figure 5(b) gives only the parabolic spline approximation for Cayley's sextic for different errors, since the original curve and the approximating parabolic spline are not visually distinguishable.

Table 2 gives the number of arcs computed by the algorithm, and the theoretical bounds on the number of arcs for varying values of ε, both for the parabolic and for the conic spline. The difference in the experimental and theoretical bound in the conic case for $\varepsilon = 10^{-1}$ can be explained by the fact that the higher order terms are not taken into consideration for computing the theoretical bound. This causes the anomaly for relatively higher values of ε.



(a) Conic spline approximation



(b) Parabolic spline approximation

Figure 5. Plot of the approximations of a part of Cayley's sextic for ε ranging from 10^{-1} to 10^{-8}.

7. Future work

It would be interesting to determine the constants in the approximation order of some of the existing methods for geometric Hermite interpolation (Floater [14], Schaback [26]), using the methods of this paper. Another open problem is to determine more terms in the asymptotic expansions of the complexity of optimal parabolic and conic splines derived in Section 5, like Ludwig [17] extends the complexity bound of the linear spline approximation of Fejes Tóth [13].

ε	Parabolic Exp./Th.	Conic Exp./ Th.
10^{-1}	6	4/2
10^{-2}	8	4
10^{-3}	14	6
10^{-4}	24	8
10^{-5}	44	12
10^{-6}	76	18
10^{-7}	134	28
10^{-8}	238	44

TABLE 2. The complexity of the parabolic spline and the conic spline approximating Cayley's sextic. The theoretical complexity matches exactly with the complexity measured in experiments (except for $\varepsilon = 10^{-1}$ in the conic case), for various values of the maximal Hausdorff distance ε.

To enable certified computation of conic arcs with guaranteed bounds on the Hausdorff distance we would have to derive sharp upper bounds on the Hausdorff distance between a curve and a bitangent conic, extending the *asymptotic* expression for these error bounds for short curves, as given in Theorem 5.1. Such a certified method could lead to robust computation of geometric structures for curved objects, like its Voronoi Diagram. In this approach the curved object would first be approximated by conic splines, after which the Voronoi Diagram of the conic arcs of these splines would be computed. The number of elements of such a conic spline would be orders of magnitude smaller than the number of line segments needed to approximate the curved object with the same accuracy. Deciding whether this feature outweighs the added complexity of the geometric primitives in the computation of the Voronoi Diagram would have to be the goal of extensive experiments.

Appendix A. Proof of Proposition 2.3

Proof. Identity (2.4) implies $\gamma'(r) = \Gamma(u(r))$, where $\Gamma(u) = \frac{1}{\varphi(u)}\,\dot{\alpha}(u)$. We denote differentiation with respect to u by a dot, like in $\dot{\alpha}$, and differentiation with respect to r by a dash, like in γ'. Then $\gamma''(r) = u'(r)\,\dot{\Gamma}(u(r))$, and $\gamma'''(r) = u''(r)\,\dot{\Gamma}(u(r)) + u'(r)^2\,\ddot{\Gamma}(u(r))$. From the definition of Γ we obtain

$$\dot{\Gamma} = -\frac{\dot{\varphi}}{\varphi^2}\,\dot{\alpha} + \frac{1}{\varphi}\,\ddot{\alpha}, \quad \text{and} \quad \ddot{\Gamma} = \left(2\frac{\dot{\varphi}^2}{\varphi^3} - \frac{\ddot{\varphi}}{\varphi^2}\right)\dot{\alpha} - 2\frac{\dot{\varphi}}{\varphi^2}\,\ddot{\alpha} + \frac{1}{\varphi}\,\dddot{\alpha}.$$

Furthermore, since $u'(r) = \frac{1}{\varphi(u(r))}$, it follows that $u''(r) = -\frac{\dot{\varphi}(u(r))}{\varphi(u(r))^3}$. Therefore,

$$\gamma''(r) = -\frac{\dot{\varphi}}{\varphi^3}\,\dot{\alpha} + \frac{1}{\varphi^2}\ddot{\alpha}, \quad \text{and} \quad \gamma'''(r) = \left(3\frac{\dot{\varphi}^2}{\varphi^5} - \frac{\ddot{\varphi}}{\varphi^4}\right)\dot{\alpha} - 3\frac{\dot{\varphi}}{\varphi^4}\,\ddot{\alpha} + \frac{1}{\varphi^3}\,\dddot{\alpha},$$

where we adopt the convention that φ, α, and their derivatives are evaluated at $u = u(r)$. Hence, the affine curvature of α at $u \in I$ is given by

$$\begin{aligned} k(u) &= [\gamma'', \gamma'''] \\ &= \frac{1}{\varphi^5}\,[\ddot{\alpha}, \dddot{\alpha}] - \left(3\,\frac{\dot{\varphi}^2}{\varphi^7} - \frac{\ddot{\varphi}}{\varphi^6}\right)[\dot{\alpha}, \ddot{\alpha}] + 3\,\frac{\dot{\varphi}^2}{\varphi^7}\,[\dot{\alpha}, \ddot{\alpha}] - \frac{\dot{\varphi}}{\varphi^6}\,[\dot{\alpha}, \dddot{\alpha}] \\ &= \frac{1}{\varphi^5}\,[\ddot{\alpha}, \dddot{\alpha}] + \frac{\ddot{\varphi}}{\varphi^6}\,[\dot{\alpha}, \ddot{\alpha}] - \frac{\dot{\varphi}}{\varphi^6}\,[\dot{\alpha}, \dddot{\alpha}]. \end{aligned}$$

From (2.3) it follows that $[\dot{\alpha}, \ddot{\alpha}] = \varphi^3$ and $[\dot{\alpha}, \dddot{\alpha}] = 3\,\varphi^2\,\dot{\varphi}$. Using the latter identity we obtain expression (2.12) for the affine curvature of α. □

Appendix B. Divided differences and the Division Property

Recall that, for a real-valued function f defined on an interval I and points $x_0, x_1, \ldots, x_n \in I$, the *$n$-th divided difference* $[x_0, \ldots, x_n]f$ is defined as the coefficient of x^n in the polynomial of degree n that interpolates f at $x_0, x_1, \ldots, x_n$. This definition is equivalent to the well-known recursive definition; see [7, Chapter 4] or [25, Chapter 5]. The interpolating polynomial can be written in the *Newton form*

$$p(x) = f(x_0) + (x - x_0)\,[x_0, x_1]f + \cdots + (x - x_0)\cdots(x - x_{n-1})\,[x_0, \ldots, x_n]f. \tag{B.1}$$

The n-th divided difference is well defined if the points are distinct. However, if f is sufficiently differentiable on I, then the n-th divided difference is also defined if some of the points coincide. More precisely, if f is a C^n-function, then the n-th divided difference has the following integral representation, known as the *Hermite-Genocchi* identity:

$$[x_0, x_1, \cdots, x_n]\,f = \int_{\Sigma^n} f^{(n)}(t_0 x_0 + t_1 x_1 + \cdots + t_n x_n)\,dt_1 \cdots dt_n,$$

where $t_0 = 1 - \sum_{i=1}^n t_i$, and the domain of integration is the standard $\Sigma^n = \{(t_1, \ldots, t_n) \mid t_1 + \cdots + t_n \leqslant 1, t_i \geqslant 0, \text{ for } i = 0, 1, \ldots, n\}$. For a proof we refer to [4, Chapter 1], [20] or [21]. The Hermite-Genocchi identity implies that $[x_0, x_1, \cdots, x_n]\,f$ is symmetric and continuous in $(x_0, x_1, \ldots, x_n)$. If f is a C^m-function, with $m \geqslant n$, this divided difference is a C^{m-n}-function of $(x_0, x_1, \ldots, x_n)$. Furthermore, if $x_i = \xi$ for $i = 0, \ldots, n$, then

$$[\underbrace{\xi, \ldots, \xi}_{n+1}]\,f = \frac{1}{n!}\,f^{(n)}(\xi). \tag{B.2}$$

Furthermore, taking $x_0 = \cdots = x_{n-1} = \xi$, and $x_n = x$, we see that

$$[\underbrace{\xi, \dots, \xi}_{n}, x]\, f = \frac{1}{(n-1)!} \int_{u=0}^{1} (1-u)^{n-1} f^{(n)}\big((1-u)\xi + ux\big)\, du. \qquad \text{(B.3)}$$

The key result used in this paper is the following 'Newton development' of a function f, akin to the Taylor series expansion.

Lemma B.1. *Let $f : I \to \mathbb{R}$ be a C^m-function defined on an interval $I \subset \mathbb{R}$, and let $x_0, \dots, x_{n-1} \in I$. Then*

$$f(x) = f(x_0) + \sum_{k=1}^{n-1} \prod_{i=0}^{k-1} (x - x_i)\, [x_0, \dots, x_k]\, f + \prod_{i=0}^{n-1} (x - x_i)\, [x_0, x_1, \cdots, x_{n-1}, x]\, f.$$

If $m \geqslant n$, then $[x_0, x_1, \cdots, x_{n-1}, x]\, f$ is a C^{n-m}-function of x. Furthermore, if $x_0 = \ldots = x_{n-1} = \xi$, then the preceding identity reduces to the Taylor expansion with integral remainder:

$$f(x) = f(\xi) + \textstyle\sum_{k=1}^{n-1} \frac{(x-\xi)^k}{k!}\, f^{(k)}(\xi) + \frac{(x-\xi)^n}{(n-1)!} \int_{u=0}^{1} (1-u)^{n-1}\, f^{(n)}(ux + (1-u)\xi)\, du.$$

The result follows from the observation that the polynomial p, defined by (B.1), interpolates f at $x_0, \dots, x_n$, so in particular $f(x_n) = p(x_n)$. Taking $x_n = x$ yields the first identity. The Taylor expansion follows using identities (B.2) and (B.3).

Since $[x_1, \dots, x_k]\, f = 0$ if $f(x_i) = 0$, $1 \leqslant i \leqslant k$, a straightforward consequence of Newton's expansion (Lemma B.1) is the following.

Lemma B.2 (Division Property). *Let $I \subset \mathbb{R}$ be an interval containing points $x_1, \dots, x_n$, not necessarily distinct, and let $f : I \to \mathbb{R}$ be a C^m-function, $m \geqslant n$, having a zero at x_i, for $1 \leqslant i \leqslant n$. Then*

$$f(x) = \prod_{i=1}^{n} (x - x_i)\, [x_1, \dots, x_n, x]\, f.$$

where the divided difference $[x_1, \dots, x_n, x]\, f$ is a C^{m-n}-function of x.

Appendix C. Approximation of n-flat functions

In this section we derive error bounds for univariate real functions with multiple zeros at the endpoints of some small interval $[0, r]$. To stress that the error also depends on the size of the interval we consider a one-parameter family of functions $(u, r) \mapsto f(u, r)$, where r is a small positive parameter. We look for a bound of the error

$$\max\nolimits_{0 \leqslant u \leqslant r} |f(u, r)|.$$

To obtain asymptotic bounds for this error as r goes to zero, we assume that the function f is defined on a neighborhood of $(0, 0)$ in $\mathbb{R} \times \mathbb{R}$.

Lemma C.1. *Let $I \subset \mathbb{R}$ be an interval which is a neighborhood of $0 \in \mathbb{R}$.*

1. Let $f : I \times I \to \mathbb{R}$ be a C^m-function such that the function $u \mapsto f(u,r)$ has an n-fold zero at $u = 0$ and at $u = r$, with $2n + 2 \leqslant m$. Then

$$\max_{0 \leqslant u \leqslant r} |f(u,r)| = \frac{1}{2^{2n}\,(2n)!} \left| \frac{\partial^{2n} f}{\partial u^{2n}}(0,0) \right| r^{2n} + O(r^{2n+1}).$$

2. Let $f : I \times I \times I \to \mathbb{R}$ be a C^m-function such that the function $u \mapsto f(u,s,r)$ has an n-fold zero at $u = 0$ and at $u = r$, and an additional single zero at $u = s$, with $2n + 3 \leqslant m$. Let

$$\delta(s,r) = \max_{0 \leqslant u \leqslant r} |f(u,s,r)|.$$

Then δ is a continuous function, and

$$\min_{0 \leqslant s \leqslant r} \delta(s,r) = \frac{c_n}{(2n+1)!} \left| \frac{\partial^{2n+1} f}{\partial u^{2n+1}}(0,0,0) \right| r^{2n+1} + O(r^{2n+2}), \tag{C.1}$$

where

$$c_n = \frac{n^n}{2^{n+1}\,(2n+1)^{n+\frac{1}{2}}}.$$

Moreover, the minimum in (C.1) *is attained at $s = s_0(r)$, where s_0 is a C^{m-2n+1}-function, with $s_0(0) = \frac{1}{2}$.*

Proof. 1. We prove that, for $r > 0$ sufficiently small, the function $u \mapsto f(u,r)$ has a unique extremum in the interior of the interval $(0,r)$. According to the Division Property (see Appendix B, Lemma B.2), there is a C^{m-2n}-function $F : I \times I \to \mathbb{R}$ such that $f(u,r) = u^n\,(r-u)^n\,F(u,r)$. Observe that $\dfrac{\partial^{2n} f}{\partial u^{2n}}(0,0) = (-1)^n\,(2n)!\,F(0,0)$.

Note that the 'model function' $g(u) = u^n\,(r-u)^n\,F(0,0)$ has its extreme value $\frac{1}{2^{2n}}F(0,0)$ on $0 \leqslant u \leqslant r$ at $u = \frac{1}{2}r$. We shall prove that the function $f(u,r)$ has its extreme value at $u = \frac{1}{2}r + O(r^2)$. To this end we apply the Implicit Function Theorem to solve the equation $\dfrac{\partial f}{\partial u}(u,r) = 0$.

Since $0 \leqslant u \leqslant r$, we scale the variable u by introducing the variable x such that $u = rx$, with $0 \leqslant x \leqslant 1$, and observe that $f(rx,r) = r^{2n}\,\tilde{f}(x,r)$, with $\tilde{f}(x,r) = x^n(1-x)^n\,F(rx,r)$. Therefore,

$$\frac{\partial \tilde{f}}{\partial x}(x,r) = n x^{n-1}(1-x)^{n-1}\,E(x,r),$$

where $E(x,r) = (1-2x)\,F(0,0) + O(r)$, uniformly in $0 \leqslant x \leqslant 1$. Since $x \mapsto \dfrac{\partial \tilde{f}}{\partial x}(x,r)$ has an $(n-1)$-fold zero at $x = 0$ and $x = r$, the Division Property allows us to conclude that E is a C^{m-2n+1}-function. Since $E(\frac{1}{2},0) = 0$, and $\dfrac{\partial E}{\partial x}(\frac{1}{2},0) = -2F(0,0) \neq 0$, the Implicit Function Theorem tells us that there is a unique

C^{m-2n+1}-function $r \mapsto x(r)$ with $x(0) = \frac{1}{2}$ and $\dfrac{\partial \tilde{f}}{\partial x}(x(r), r) = 0$. Therefore, $\tilde{f}(\cdot, r)$ has a unique extremum at $x = x(r)$. Hence,

$$\begin{aligned}
\max\nolimits_{0 \leqslant u \leqslant r} |f(u,r)| &= |\tilde{f}(x(r), r)|\, r^{2n} \\
&= |\tilde{f}(\tfrac{1}{2}, 0)|\, r^{2n} + O(r^{2n+1}) \\
&= \frac{|F(0,0)|}{2^{2n}}\, r^{2n} + O(r^{2n+1}) \\
&= \frac{1}{2^{2n}\,(2n)!} \left| \frac{\partial^{2n} f}{\partial u^{2n}}(0,0) \right| r^{2n} + O(r^{2n+1}).
\end{aligned}$$

2. The proof of the second part goes along the same lines, but is slightly more complicated due to the occurrence of *two* critical points of the function $f(\cdot, s, r)$ in the interior of the interval $(0, r)$. Again, the Division Property guarantees the existence of a C^{m-2n-1}-function $F : I \times I \times I \to \mathbb{R}$ such that $f(u, s, r) = u^n\,(r - u)^n\,(s - u)\,F(u, s, r)$.

The 'model function' $g(u) = u^n\,(r - u)^n\,(s - u)\,F(0,0,0)$ has two critical points for $0 \leqslant u \leqslant r$: one on the interval $[0, s]$ and one on the interval $[s, r]$. The derivative of this function is of the form

$$g'(u) = u^{n-1}\,(r-u)^{n-1}\,\big(-(2n+1)\,u^2 + (2ns+n+1)\,u - ns\big)\,F(0,0,0).$$

A straightforward calculation shows that g' has two zeros $u_\pm(s)$, and that the critical values of g at these zeros are equal iff $s = \frac{1}{2}$. In the remaining part of the proof we show that the function $f(\cdot, s, r)$ has its extreme values at $u = u_\pm(s) + O(r^2)$, again by applying the Implicit Function Theorem to solve the equation $\dfrac{\partial f}{\partial u}(u, s, r) = 0$.

The critical values of $f(\cdot, s, r)$**.** Putting $u = rx$ and $s = ry$, with $0 \leqslant x, y \leqslant 1$, we obtain $f(rx, ry, r) = r^{2n+1}\,\tilde{f}(x, y, r)$, with $\tilde{f}(x, y, r) = x^n\,(1-x)^n\,(x - y)\,F(rx, ry, r)$. To determine the critical points of $x \mapsto \tilde{f}(x, y, r)$ on the interval $(0, 1)$, we observe that

$$\frac{\partial \tilde{f}}{\partial x}(x, y, r) = x^{n-1}\,(1-x)^{n-1}\,Q(x, y, r), \tag{C.2}$$

where Q is a function of the form

$$Q(x, y, r) = \big(-(2n+1)\,x^2 + (2ny+n+1)\,x - ny\big)\,F(0,0,0) + O(r),$$

uniformly in $x, y \in [0, 1]$. Since $\dfrac{\partial \tilde{f}}{\partial x}$ is a C^{m-1}-function such that $x \mapsto \dfrac{\partial \tilde{f}}{\partial x}(x, r)$ has $(n-1)$-fold zeros at $x = 0$ and $x = 1$, the Division Property allows us to conclude that Q, determined by (C.2), is a C^{m-2n+1}-function.

Assume $F(0,0,0) > 0$ (the case $F(0,0,0) < 0$ goes accordingly). Then, if $0 < y < 1$, the function $x \mapsto \tilde{f}(x, y, 0)$ has one minimum at $x = x_-^0(y)$ and one maximum at $x = x_+^0(y)$, where $x_\pm^0$ are the zeros of the quadratic function $x \mapsto Q(x, y, 0)$. Since $\dfrac{\partial Q}{\partial x}(x_\pm^0(y), y, 0) \neq 0$, the Implicit Function Theorem guarantees the existence of C^{m-2n+1}-functions $x_\pm : I \times I \to \mathbb{R}$, with $x_-(y,r) < x_+(y,r)$, such that $x_\pm(y,0) = x^0_\pm(y)$, and $Q(x_\pm(y,r),y,r) = 0$. So, in view of (C.2), the function $x \mapsto \tilde{f}(x,y,r)$ has one minimum at $x = x_-(y,r)$, and one maximum at $x = x_+(y,r)$. Putting

$$\tilde{\delta}(y,r) = \max\nolimits_{0\leqslant x\leqslant 1} |\tilde{f}(x,y,r)|, \tag{C.3}$$

we see that

$$\tilde{\delta}(y,r) = \max\big(|\tilde{f}(x_-(y,r),y,r)|, |\tilde{f}(x_+(y,r),y,r)|\big).$$

The minimax norm of the family $\{f(\cdot,s,r) \mid s \in [0,r]\}$**.** For fixed x and r, with $0 < x < 1$ and $r > 0$ sufficiently small, the function $y \mapsto \tilde{f}(x,y,r)$ is decreasing. See Figure 6. This follows from the observation that

$$\frac{\partial \tilde{f}}{\partial y}(x,y,r) = -x^n(1-x)^n E(x,y,r),$$

with $E(x,y,r) = F(0,0,0) + O(r)$, uniformly in $x, y \in [0,1]$. Therefore, there is a $\varrho_0 > 0$ such that, for $0 \leqslant r \leqslant \varrho_0$, we have $E(x,y,r) > 0$ for $0 \leqslant x, y \leqslant 1$, and hence $\dfrac{\partial \tilde{f}}{\partial y}(x,y,r) < 0$.

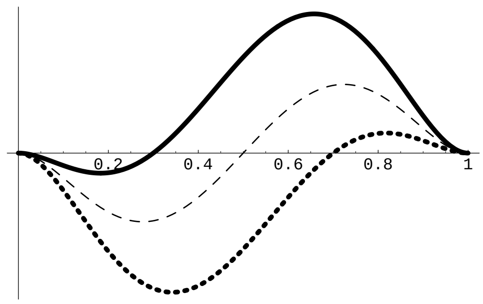


FIGURE 6. Graph of the function $x \mapsto \tilde{f}(x,y,r)$, for r fixed and $y = y_0$ (solid), $y = y_1$ (dashed), and $y = y_2$ (dotted), with $y_0 < y_1 < y_2$.

From this observation it follows that, for fixed r and y ranging from 0 to 1, the graphs of the functions $x \mapsto f(x,y,r)$ are disjoint, except at their endpoints. See again Figure 6. Therefore, the function $y \mapsto \tilde{\delta}(y,r)$ attains its minimum iff

$$\Delta(y,r) = 0,$$

where $\Delta(y,r) = \tilde{f}(x_-(y,r),y,r) + \tilde{f}(x_+(y,r),y,r)$.

Claim: There is a C^{m-2n+1}-function y_0, such that, for $0 \leqslant r \leqslant \varrho_0$:

$$\Delta(y,r) = 0 \text{ iff } y = y_0(r).$$

and $y_0(r) = \frac{1}{2} + O(r)$.

To prove this claim, we first prove that $\Delta(\frac{1}{2},0)=0$. To see this, observe that

$$\tilde{f}(x,\tfrac{1}{2},0)=-\tilde{f}(1-x,\tfrac{1}{2},0),$$

so

$$\frac{\partial \tilde{f}}{\partial x}(x,\tfrac{1}{2},0)=\frac{\partial \tilde{f}}{\partial x}(1-x,\tfrac{1}{2},0).$$

Therefore, $x_+(\frac{1}{2},0)=1-x_-(\frac{1}{2},0)$, and hence $\Delta(\frac{1}{2},0)=0$. Since

$$\frac{\partial \Delta}{\partial y}(y,0)=\frac{\partial \tilde{f}}{\partial y}(x_-(y,0),y,0)+\frac{\partial \tilde{f}}{\partial y}(x_+(y,0),y,0)<0,$$

the function $y\mapsto\Delta(y,0)$ has a *unique zero* at $y=\frac{1}{2}$. Furthermore, the Implicit Function Theorem guarantees the existence of a C^{m-2n+1}-function y_0 with $\Delta(y_0(r),r)=0$, and $y_0(0)=\frac{1}{2}$.

In view of (C.3) we have

$$\begin{aligned}\min\nolimits_{0\leqslant y\leqslant 1}\tilde{\delta}(y,r)&=|\tilde{f}(x_\pm(y_0(r),r),y_0(r),r)|\\&=|\tilde{f}(x_\pm(\tfrac{1}{2},0),\tfrac{1}{2},0)|+O(r)\\&=\max\nolimits_{0\leqslant x\leqslant 1}|x^n(1-x)^n(x-\tfrac{1}{2})|+O(r)\\&=c_n+O(r).\end{aligned}$$

Finally, $\min_{0\leqslant s\leqslant r}\delta(s,r)=r^{2n+1}\min_{0\leqslant y\leqslant 1}\tilde{\delta}(y,r)=c_n\,r^{2n+1}+O(r^{2n+2})$. The minimum is attained at $s=s_0(r)=r\,y_0(r)$. Obviously, s_0 is a C^{m-2n+1}-function. This concludes the proof of the second part of the Lemma. □

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