Recognition of the bifurcation type of resonance in mildly degenerate Hopf-Neĭmark-Sacker families

H.W. Broer S.J. Holtman
G. Vegter
Institute for Mathematics and Computer Science
University of Groningen, The Netherlands
Email: {H.W.Broer,S.J.Holtman,G.Vegter}@math.rug.nl

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Abstract

The present paper deals with families of diffeomorphisms a fixed point of which undergoes a Hopf-Neĭmark-Sacker bifurcation with its characteristic array of resonance tongues organizing the alteration of periodic and quasi-periodic dynamics. Our interest is with the periodic dynamics as this corresponds to subharmonic periodic solutions in the case of flows. We zoom in on the shape of one such tongue, as a subset of the resonance bifurcation diagram, briefly reviewing the classical nondegenerate case, but then turning to a next case of degeneracy. It has already been established that the generic tongue geometry involves both tongues and flames. A description of this can be given in terms of contact-equivalence singularity theory, equivariant under an appropriate cyclic group given by the resonance at hand. At an intermediate stage of the theory a Lyapunov-Schmidt reduction is applied. This gives a finite classification of such bifurcation diagrams. The present paper solves the ensuing recognition problem, which aims to classify any given generic family of diffeomorphisms with respect to this.

Contents

1 Introduction

Resonance tongues. We continue our study [2] of resonance tongues and their boundaries for nondegenerate and (certain) degenerate Hopf bifurcations of maps. Such bifurcations occur if one of the maps — the central singularity — in such a family exhibits *resonant dynamics* near a fixed point where the eigenvalues cross the unit circle. The present study zooms in on the case where this occurs at a q-th root of unity. Resonance tongues are regions in parameter space corresponding to the occurrence of periodic orbits of period q near the fixed point of the central singularity, and the tongue boundaries correspond to the appearance or disappearance of such periodic orbits, typically through a saddle-node bifurcation. In the generic case of weak resonance, i.e., when $q \geq 5$, a pair of q-periodic orbits appears or disappears upon passage of the tongue boundaries in parameter space. Such bifurcations occur in generic two parameter families. If $q \geq 7$ we encounter degenerate situations in which the Hopf coefficient (i.e., the coefficient of the third order term in the normal form) is zero, which gives rise to the appearance or disappearance of up to four q-periodic orbits near the Hopf bifurcation. These bifurcations occur in generic four parameter families.

Main results. In [2] we present normal forms for such families of maps by applying $\mathbb{Z}_q$-equivariant singularity theory. These normal forms, depending on two and four parameters, respectively, determine the geometry of the resonance tongues in generic families. In this paper we solve the *recognition problem* for families of maps exhibiting a weak resonance, i.e., we derive a finite set of conditions distinguishing families of maps with topologically different resonance tongues. As usual, these conditions are polynomial equalities and inequalities in a low-order jet of the map at the bifurcation point. These conditions are obtained via *Lyapunov-Schmidt reduction*, a procedure mapping a family of maps near $p : q$-resonance to a family of $\mathbb{Z}_q$-invariant functions, such that the zeros of the latter family correspond to fixed points or periodic orbits of the former family. The discriminant set of the reduced family is a stratified subset of parameter space, separating open regions in parameter space corresponding to different numbers of zeros. Therefore, this discriminant set corresponds to different numbers of q-periodic orbits of the family of maps, so it coincides with the boundaries of the resonance tongues.

Our main contribution is two-fold: (i) The derivation of an algorithm that computes explicit expressions for the Lyapunov-Schmidt reduction of such a given family of maps; (ii) Using these expressions to solve the recognition problem, based on $\mathbb{Z}_q$-equivariant singularity theory along the lines of our earlier work [2] and [11]. The main results are presented in Section 2, in which we also presents several case studies illustrating our approach. Section 3 contains the details of the Lyapunov-Schmidt reduction algorithm, and the output in a characteristic case. Finally, in Section 4 we apply

$\mathbb{Z}_q$-equivariant singularity theory to derive conditions characterizing a class of generic and mildly degenerate families of planar diffeomorphisms, and, therefore, solving the recognition problem for such families. In the companion paper [4] we analyze the complete geometry of the resonance set of this mildly degenereate family via 4D-tomography, i.e., using 3D-cross sections of 4D parameter space.

Related work. In many applications, the family of planar diffeomorphisms is obtained from a family of vector fields by taking a suitable Poincaré map corresponding to a section transverse to a periodic orbit. The eigenvalues of the derivative of the Poincaré map then are the Floquet exponents of the periodic orbit. The corresponding bifurcation of the periodic orbit, in particular related to subharmonic periodic solutions and invariant tori, usually is referred to as the Neĭmark-Sacker bifurcation, compare with [12, 16]. In particular this approach also works for non-autonomous systems of differential equations, depending periodically on time.

The above approach directly applies in this setting and a natural question then is, how the recognition problem can be solved at the level of continuous systems. In a forthcoming paper, based on examples, we shall return to this problem. Moreover, further properties of the dynamics and bifurcations related to the various subharmonic solutions will be discussed. This includes stability and the behaviour of the stable and unstable manifolds (e.g., invariant circles, hetero- and homoclinic tangle) and bifurcations, compare with [6, 8] and references therein.

This kind of resonance scenario occurs in many other situations as well. A toy model for the array of tongues of the Hopf-Neĭmark-Sacker bifurcation is formed by the Arnol'd family of circle maps [1], given by $x \mapsto x + 2\pi\alpha + \beta \sin x$. Here in the (α, β)-plane tongues appear with their tips in $(\alpha, \beta) = (\frac{p}{q}, 0)$ and stretching out into the regions $\beta \neq 0$. See Figure 1. The order of tangency of the tongue boundaries at the tip, as $q \to \infty$ has the same asymptotics as in the Hopf-Neĭmark-Sacker case. Also compare with [12, 13] and [2, 3].

The research program of Peckam et al. reflected in [17, 18, 20–22] views resonance 'tongues' as projections on a 'traditional' parameter plane of (saddle-node) bifurcation sets in the product of parameter and phase space. This approach has the same spirit as ours and many interesting geometric properties of 'resonance tongues' are discovered and explained in this way. We note that the earlier result [21] on higher order degeneracies in a period-doubling uses $\mathbb{Z}_2$-equivariant singularity theory.

Particularly we like to mention the results of [22] concerning a class of oscillators with doubly periodic forcing. It turns out that these systems can have coexistence of periodic attractors (of the same period), giving rise to 'secondary' saddle-node lines, sometimes enclosing a flame-like shape.

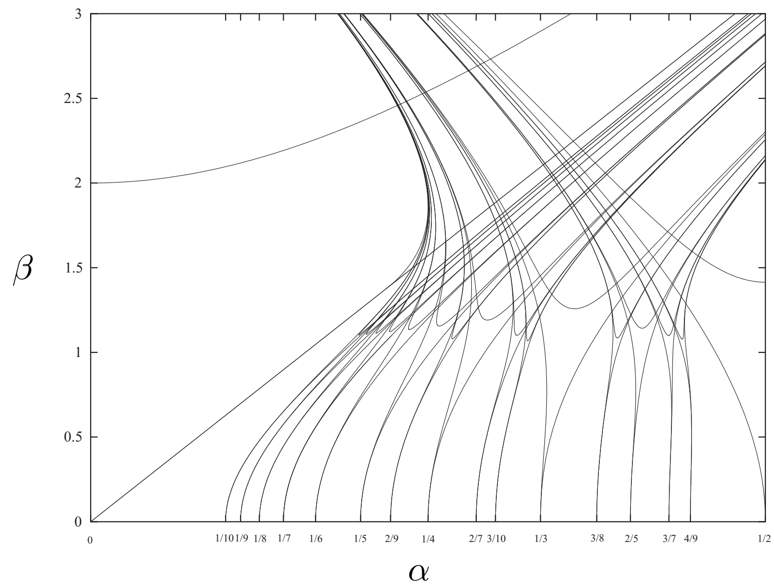

Figure 1: Resonance tongues in the Arnol'd family [6].

For background regarding weak and strong resonances we refer to Takens [23], Newhouse *et al.* [19], Arnol'd [1], Krauskopf [15] and Broer, Golubitsky and Vegter [2,3]. Resonance also is studied in Hamiltonian or reversible settings, etc., compare with Broer and Vegter [10] or Vanderbauwhede [24].

2 Main results

The main results of this paper concern a Lyapunov-Schmidt reduction for the problem of detecting resonant periodic orbits and the precise formulation of the Versality Conditions that determine the classification of bifurcation diagrams. We also consider a few case studies.

2.1 Lyapunov-Schmidt reduction

We consider a family of local diffeomorphisms on the plane, depending on a finite number of parameters, such that for some parameter the diffeomorphism has a fixed point with a pair of q-th roots of unity as eigenvalues. Such a family may also originate from reduction of a higher dimensional family to a two-dimensional center manifold. For convenience we identify the plane with $\mathbb{C}$. If the eigenvalues of the linear part at the fixed point are of the form ω and $\overline{\omega}$, with $\omega = e^{2\pi i p/q}$, the family of diffeomorphisms can be brought into the form

$$P_{a,\varrho}(z) = (\omega + \varrho)\, z + \sum_{2 \le i+j < q} a_{ij}\, z^i\, \overline{z}^j + O(|z|^q), \tag{1}$$

after translating the fixed point of $P_{a,\varrho}$ to the origin, and *after* bringing the linear part of $P_{a,\varrho}$ at this fixed point into Jordan Normal Form. From now on we assume that $\omega = e^{2\pi i p/q}$, with p and q relatively prime, $q > 1$, unless otherwise stated. The complex coefficients ϱ and a_{ij} are considered as parameters. Note that the linear part $z \mapsto (\omega + \varrho)\, z$ of $P_{a,\varrho}$ at the fixed point $z = 0$ is a universal unfolding of the linear part of $P_{a,0}$ (for all a). The parameter ϱ measures detuning. Lyapunov-Schmidt reduction of the family $P_{a,\varrho}$ yields a $\mathbb{Z}_q$-invariant family $G_{a,\varrho} : \mathbb{C} \to \mathbb{C}$ of the form

$$G_{a,\varrho}(z) = z\, K(|z|^2, a, \varrho) + L(a, \varrho)\, \overline{z}^{q-1} + O(|z|^q), \tag{2}$$

where $u \mapsto K(u, a, \varrho)$ is a polynomial in u of degree less than $(q-1)/2$ with coefficients depending on a_{ij} and ϱ, with $K(0, 0, \varrho) = \varrho$. (We discuss Lyapunov-Schmidt reduction in more detail in Section 3.) Since the q-periodic points of the family of maps $P_{a,\varrho}$ correspond to the zeros of the reduced family of functions $G_{a,\varrho}$, the boundary of the resonance tongues of the family of maps corresponds to the *discriminant set* $\Sigma(G)$ of the family G defined by $G(z, a, \varrho) = G_{a,\varrho}(z)$, i.e., the set of parameters (a, ϱ) for which there is a $z \in \mathbb{C}$ such that

$$G_{a,\varrho}(z) = 0 \text{ and } \operatorname{Rank} dG_{a,\varrho}(z) \leq 1.$$

Therefore, determining an explicit form of the Lyapunov-Schmidt reduced family $G_{a,\varrho}$ is a key step towards the explicit computation of the resonance tongues. This is precisely the *first part of our main result*, i.e., the computation of the functions $K(u, a, \varrho)$ and $L(a, \varrho)$ in (2) as a function of the coefficients ϱ and a_{ij} in (1).

Theorem 1. (Lyapunov-Schmidt reduction) *Let $P_{a,\varrho} : \mathbb{C} \to \mathbb{C}$ be a family of diffeomorphisms of the form*

$$P_{a,\varrho}(z) = (\omega + \varrho)\, z + Q(z, a),$$

where $\omega = e^{2\pi i p/q}$, with p and q relatively prime and $q > 1$, and

$$Q(z, a) = \sum_{2 \leq i+j < q} a_{ij}\, z^i\, \overline{z}^j + O(|z|^q). \tag{3}$$

1. Lyapunov-Schmidt reduction yields a family G of $\mathbb{Z}_q$-invariant functions of the form

$$G(z, a, \varrho) = \varrho\, z + z\, B(|z|^2, a, \varrho) + C(a, \varrho)\, \overline{z}^{q-1} + O(|z|^q), \tag{4}$$

where $B(u, a, \varrho)$ is a polynomial in u of degree less than $(q-1)/2$, without constant term, i.e., a polynomial of the form

$$B(u, a, \varrho) = b_1(a, \varrho)\, u + b_2(a, \varrho)\, u^2 + O(u^3).$$

Expressions for $B(u, a, \varrho)$ and $C(a, \varrho)$ are computed by the Lyapunov-Schmidt reduction algorithm, presented in Section 3. An example of these expressions is given at the end of that section.

2. If $Q(z, a)$ is $\mathbb{Z}_q$-equivariant in the sense that $Q(\omega z) = \omega Q(z)$, then Lyapunov-Schmidt reduction yields a $\mathbb{Z}_q$-equivariant family G of the form

$$G(z, a, \varrho) = \varrho z + Q(z, a).$$

Section 3 contains the proof of this theorem. In fact, we will present an *algorithm* performing Lyapunov-Schmidt reduction for families of the form (1).

With regard to the second part of the theorem note that equivariance of Q, at least up to order q, is equivalent to all coefficients a_{ij} being zero, except possibly for $j = i - 1$ and $(i, j) = (0, q - 1)$.

2.2 The recognition problem

$\mathbb{Z}_q$-equivariant contact equivalence. We use equivariant singularity theory to determine the zero set of $\mathbb{Z}_q$-invariant families G_μ of the form (2) obtained by Lyapunov-Schmidt reduction. In [2] we used equivariant singularity theory to obtain normal forms for the simplest $\mathbb{Z}_q$-equivariant germs obtained from Lyapunov-Schmidt reduction. Here we focus on the *recognition problem* for families of germs, i.e., we derive conditions guaranteeing that a given family unfolds one of these simple germs, and does so 'generically'. To this end we first recall some notions and properties of equivariant singularity theory that we use in this paper. See also [14, Chapter III].

We derive invariant conditions characterizing the orbits of such families under the group of $\mathbb{Z}_q$-equivariant *contact transformations*. This group consists of all pairs (S, Z), where $Z : \mathbb{C} \to \mathbb{C}$ is a $\mathbb{Z}_q$-equivariant (local) change of coordinates, i.e.,

$$Z(\gamma z) = \gamma Z(z)$$

for $\gamma \in \mathbb{Z}_q$, and $S(z) : \mathbb{C} \to \mathbb{C}$ is a real linear map for each z that satisfies

$$S(\gamma z)\,\gamma = \gamma\, S(z),$$

for $\gamma \in \mathbb{Z}_q$. The group of $\mathbb{Z}_q$-equivariant contact transformations acts on the ring of functions (or, rather, germs): the group element (S, Z) maps the germ g onto the germ h defined by

$$h(z) = S(z)\, g(Z(z)). \tag{5}$$

In this case g and h are $\mathbb{Z}_q$-*contact equivalent*. Note that Z maps the zero set of h to the zero set of g, an important feature in our approach of resonances.

A k-parameter *unfolding* of a germ $g : \mathbb{C} \to \mathbb{C}$ is a map $G : \mathbb{C} \times \mathbb{R}^k \to \mathbb{C}$ such that $G(z, 0) = g(z)$. The germ g is the *central singularity* of the family G. Such an unfolding is $\mathbb{Z}_q$-equivariant if every germ G_μ, defined by $G_\mu(z) = G(z,\mu)$, is $\mathbb{Z}_q$-equivariant. Two k-parameter $\mathbb{Z}_q$-equivariant unfoldings G and H of the germ g are called $\mathbb{Z}_q$-contact equivalent if there is a k-parameter family (S_μ, Z_μ) of $\mathbb{Z}_q$-equivariant contact transformations mapping G_μ onto H_μ, i.e.:

$$H_\mu(z) = S_\mu(z)\, G_\mu(Z_\mu(z)),$$

such that $S_0(z) = 1$ and $Z_0(z) = z$. An l-parameter $\mathbb{Z}_q$-equivariant unfolding H of a $\mathbb{Z}_q$-equivariant germ g *factors through* a k-parameter $\mathbb{Z}_q$-equivariant unfolding G of g if there is a *reparametrization* $\varphi : \mathbb{R}^l \to \mathbb{R}^k$, with $\varphi(0) = 0$, such that the unfoldings H_ν and $G_{\varphi(\nu)}$ are $\mathbb{Z}_q$-contact equivalent.

In this case, $Z_{\varphi(\nu)}$ maps the zero set of H_ν onto the zero set of $G_{\varphi(\nu)}$, and so the discriminant set $\Sigma(H)$ is obtained by pulling back the discriminant set of $\Sigma(G)$ by the reparametrization φ. Given the reduced family G, our strategy will be to compute a *normal form*, i.e., a 'simple family' in the orbit under the group of contact transformations. Although not unique, such a normal form can usually be chosen to be a low-degree polynomial family H, the discriminant set $\Sigma(H)$ of which can be determined by a straightforward calculation. This notion of 'simple family' is made precise by considering *versal unfoldings*, i.e., unfoldings through which every other unfolding (of the same germ) factors. An unfolding is called *universal* if it is versal with a minimal number of parameters. This minimal number of parameters is the *codimension* of the germ it unfolds. Since resonance tongues correspond to discriminant sets of Lyapunov-Schmidt reduced functions, it is obvious why it is important to study (uni)versal unfoldings of these reduced functions.

Versal unfoldings of $\mathbb{Z}_q$-equivariant functions. The *second part of our main result* gives precise conditions characterizing $\mathbb{Z}_q$-invariant families with generic, or mildly degenerate singularities. These conditions solve the *recognition problem* for such families.

Theorem 2. (Versality conditions) *Let $G : \mathbb{C} \times \mathbb{R}^k \to \mathbb{C}$ be a $\mathbb{Z}_q$-invariant family of the form*

$$G(z,\mu) = z\, K(|z|^2, \mu) + L(\mu)\, \overline{z}^{q-1} + O(|z|^q),$$

with $K(0,0) = 0$ and $L(0) \neq 0$. Let $g = G(\cdot, 0)$ be the central singularity of the family G.

1. If $K_u(0,0) \neq 0$, $q \geq 5$, and there are at least two parameters, i.e., if $k \geq 2$, then G is a versal unfolding of g if

$$\left. \frac{\partial(\operatorname{Re} K(0,\mu), \operatorname{Im} K(0,\mu))}{\partial(\mu_1, \mu_2)} \right|_{\mu=0} \neq 0, \qquad (6)$$

possibly after permutation of the parameters (the generic case). G is a universal unfolding if $k = 2$.

2. If $K_u(0,0) = 0$, $q \geq 7$, and there are at least four parameters, i.e., if $k \geq 4$, then G is a versal unfolding of g if $K_{uu}(0,0) \neq 0$, and

$$\left.\frac{\partial(\operatorname{Re} K(0,\mu), \operatorname{Im} K(0,\mu), \operatorname{Re} K_u(0,\mu), \operatorname{Im} K_u(0,\mu))}{\partial(\mu_1,\mu_2,\mu_3,\mu_4)}\right|_{\mu=0} \neq 0, \tag{7}$$

possibly after permutation of the parameters (the mildly degenerate case). G is a universal unfolding of g if $k = 4$.

Remark. Gradual violation of the nondegeneracy conditions like $L(0) \neq 0$ gives rise to a familiar endless sequence of bifurcations of ever higher codimension.

We prove this result in Section 4, which also contains the corresponding results for the strong resonances $q = 3$ and $q = 4$.

Theorem 2 allows us to derive explicit expressions for universal unfoldings of certain $\mathbb{Z}_q$-equivariant germs.

Corollary 3. *Let g be a $\mathbb{Z}_q$-equivariant germ of the form*

$$g(z) = z\,k(|z|^2) + d\,\overline{z}^{q-1} + O(|z|^q),$$

with $k(0) = 0$ and $d \neq 0$. A universal unfolding of g is of the form

$$G(z,\mu_1,\mu_2) = g(z) + (\mu_1 + i\mu_2)\,z, \qquad \text{if } k'(0) \neq 0; \tag{8}$$

$$G(z,\mu_1,\mu_2,\mu_3,\mu_4) = g(z) + (\mu_1 + i\mu_2)\,z + (\mu_3 + i\mu_4)\,z\,|z|^2, \qquad \text{if } k'(0) = 0 \text{ and } k''(0) \neq 0. \tag{9}$$

Remark. In [2] we also derived normal forms $h(z)$ for the central singularity g in these cases:

$$h(z) = z\,|z|^2 + \overline{z}^{q-1}, \text{ in case (8)};$$

$$h(z) = z\,|z|^4 + \overline{z}^{q-1}, \text{ in case (9)}.$$

Germs of this kind, and their universal unfoldings, will show up in the case studies presented next.

The recognition problem for families of planar diffeomorphisms. We now apply Theorem 2 on versality of general unfoldings of $\mathbb{Z}_q$-equivariant germs to obtain conditions solving the *recognition problem for families of planar diffeomorphisms* of the form $P_{a,\varrho}(z) = (\omega + \varrho)\,z + Q(z,a)$, where Q is of the form (3), with q-periodic orbits bifurcating from a fixed point $z = 0$ at $(a,\varrho) = (a^*,0)$. The reduced family $G(z,a,\varrho)$ is of the form (4), i.e.,

$$G(z,a,\varrho) = zK(|z|^2,a,\varrho) + C(a,\varrho)\,\overline{z}^{q-1} + O(|z|^q),$$

where

$$K(u,a,\mu) = \varrho + b_1(a,\varrho)\,u + b_2(a,\varrho)\,u^2 + O(u^3).$$

In the terminology of Theorem 2, the parameters are $\mu = (a,\varrho)$, ranging over a neighborhood of $\mu^* = (a^*,0)$.

Corollary 4. *Assume that the coefficient $C(a^*, 0)$ of the term $\overline{z}^{q-1}$ in (4) is nonzero.*

1. If $K(0, \mu^*) = b_1(a^*, 0) \neq 0$ and $q \geq 5$, then the family $G(z, a, \varrho)$ is a versal unfolding of the germ $z \mapsto G(z, a^*, 0)$.
2. If $b_1(a^*, 0) = 0$, but $K_u(0, \mu^*) = b_2(a^*, 0) \neq 0$ and $q \geq 7$, then the family $G(z, a, \varrho)$ is a versal unfolding of the germ $z \mapsto G(z, a^*, 0)$ if, moreover, there are two additional real parameters, μ_3 and μ_4 say, such that

$$\left. \frac{\partial(\operatorname{Re} b_2(a^*, 0), \operatorname{Im} b_2(a^*, 0))}{\partial(\mu_3, \mu_4)} \right|_{\mu=\mu^*} \neq 0. \tag{10}$$

Here μ_3 and μ_4 are two distinct real parameters from the set

$$\{\operatorname{Im} a_{ij}, \operatorname{Re} a_{ij} \mid 2 \leq i + j < q\}$$

corresponding to the coefficients of the nonlinear part Q of P.

Proof. The condition $C(a^*, 0) \neq 0$ is equivalent to assuming that $L(\mu^*) \neq 0$ in terms of Theorem 2.

1. First consider the *generic case*, corresponding to $K(0, \mu^*) = b_1(a^*, 0) \neq 0$ and $q \geq 5$. With regard to the parameters $\mu_1 = \operatorname{Re} \varrho$ and $\mu_2 = \operatorname{Im} \varrho$, we have

$$\left. \frac{\partial(\operatorname{Re} K(0, \mu), \operatorname{Im} K(0, \mu))}{\partial(\mu_1, \mu_2)} \right|_{\mu=\mu^*} = 1 \neq 0.$$

Hence the fact that the linear part of the family $P_{a,\varrho}$ at $z = 0$ is in Jordan Normal Form, and is, moreover, a versal unfolding of the linear part for $\varrho = 0$, implies that the condition for versality, cf. (6) in Theorem 2.1, is trivially satisfied in this case.

2. In the *mildly degenerate case*, corresponding to $K(0, \mu^*) = b_1(a^*, 0) = 0$, $K_u(0, \mu^*) = b_2(a^*, 0) \neq 0$ and $q \geq 7$, the versality condition (7) reduces to (10). □

2.3 Case studies

To illustrate our main result we present some examples in which we determine resonance tongues of planar families of diffeomorphisms a fixed point of which undergoes a Hopf-Neĭmark-Sacker bifurcation with its characteristic array of resonance tongues organizing the alteration of periodic and quasi-periodic dynamics. We zoom in on the shape of one such tongue, as a subset of the resonance bifurcation diagram, briefly reviewing the classical nondegenerate case, but then turning to a next case of degeneracy. For further examples of such bifurcations we refer to [7], [8], [9] and [16].

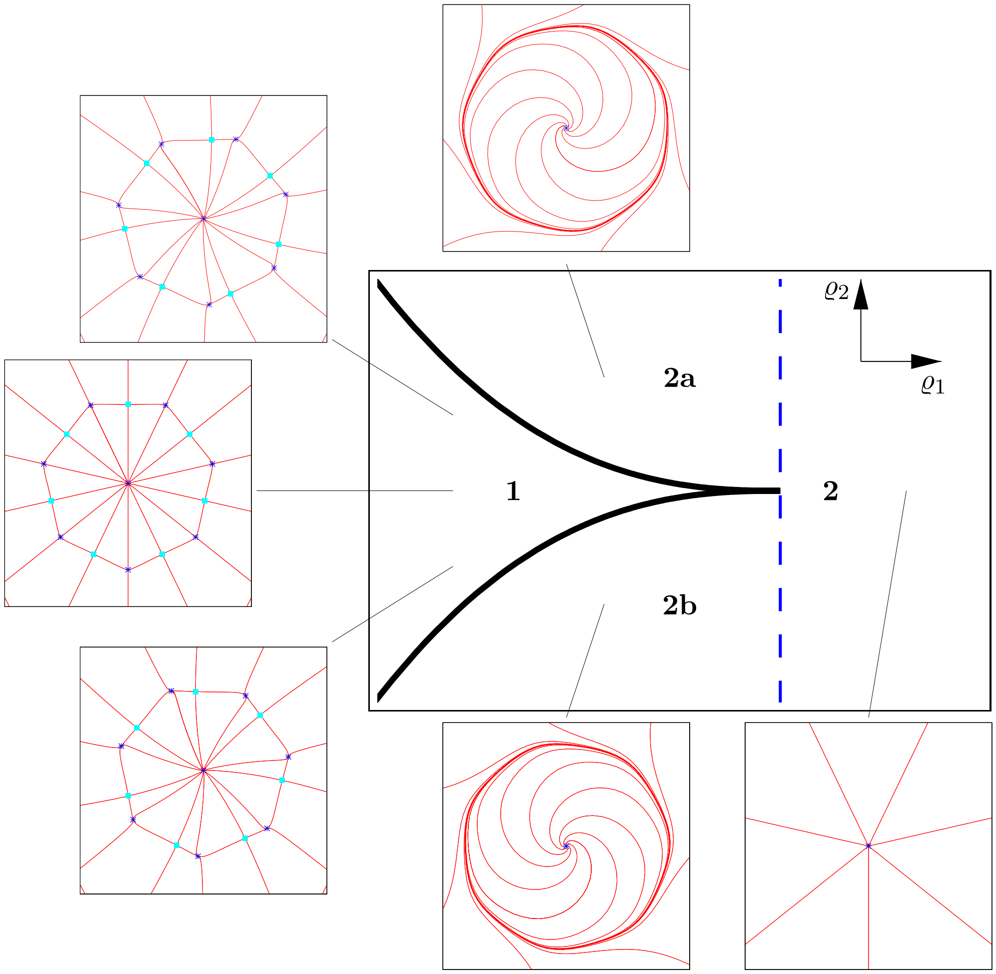

Figure 2: The non-degenerate resonance tongue in the ϱ-plane for $q = 7$ and $b = d = 1$ (central picture), with phase portraits for various values of the detuning parameter ϱ in the complement of the tongue (regions 1 and 2). For ϱ in region 1 the map P has a fixed point surrounded by two period-7 orbits. Curves between the periodic points are their stable or unstable manifolds. If ϱ is changed such that it crosses the tongue boundary, the two period-7 orbits disappear in a saddle-node bifurcation. Regions 2a and 2b correspond to phase portraits with a single fixed point enclosed in an invariant circle. The complete bifurcation diagram consists of the resonance tongue and a Hopf line, given by $\varrho_1 = 0$: when the parameter crosses this line into region 2 the invariant circle disappears.

Generic equivariant families. The first example is a non-degenerate planar family of diffeomorphisms near a $p : q$-resonance with a $\mathbb{Z}_q$-equivariant $(q-1)$-jet, i.e., a family $P(z, \varrho)$, given by

$$P(z, \varrho) = (\omega + \varrho)\, z + b\, z\, |z|^2 + d\bar{z}^{q-1} + O(|z|^q), \tag{11}$$

where $\varrho \in \mathbb{C}$ is a complex parameter ranging over a neighborhood of $0 \in \mathbb{C}$, b and d are non-zero complex constants, and $\omega = e^{2\pi i p/q}$, with p and $q \geq 5$

coprime. Theorem 1, and the subsequent observations, show that Lyapunov-Schmidt reduction yields the reduced family

$$G(z, \varrho) = \varrho\, z + b\, z\, |z|^2 + d\bar{z}^{q-1} + O(|z|^q).$$

A straightforward derivation along the lines of [2, Section 4] shows that, after a parameter transformation, given by $\sigma = b\varrho$, the discriminant set of this family, and, hence, the tongue boundary of the family (11), is a familiar $\frac{q-2}{2}$ cusp in the σ-plane. More precisely, the tongue boundary is of the form

$$\sigma_2^2 = \frac{|d|^2}{|b|^2}\,(-\sigma_1)^{q-2} + O(\sigma_1^{2q-4}).$$

The resonance tongue boundary, and some characteristic phase portraits, are plotted in Figure 2 for $q = 7$. The complete bifurcation set consists of the tongue boundary and a Hopf line, corresponding to the appearance or disappearance of an invariant circle. This Hopf line is determined by other (standard) methods from bifurcation theory. See the caption of Figure 2 for further details.

A higher dimensional example: reduction to a center manifold. Our second example is a family of diffeomorphisms in three-space, with two conjugate 5-th roots of unity as eigenvalues, and a third real eigenvalue off the unit circle. We determine for which values of the parameter resonances of order 5 may occur, and show how the resonance tongues can be determined by restricting to a two-dimensional center manifold.

The example is inspired by a related system studied in [8], where the resonance tongues are determined by first computing a Takens Normal form of the family of maps. Then the fixed points corresponding to this type of resonance are located, and finally an expression for the tongue boundary is obtained by looking for saddle-node bifurcations of the period five points.

We follow an alternative approach by applying the methods developed in this paper to determine the resonance tongues via Lyapunov-Schmidt reduction. The family of maps $F_\tau : \mathbb{R}\times\mathbb{C} \to \mathbb{R}\times\mathbb{C}$ is given by

$$F_\tau(x, z) = (F_1(x, z), F_2(x, z, \tau)),$$

with

$$\begin{aligned} F_1(x, z) &= x + b^2 - x^2 - |z|^2,\\ F_2(x, z, \tau) &= z\,(\omega + \tau - a\,x - x^2) + d\,\overline{z}^4. \end{aligned}$$

Here τ is a complex parameter, to be specified later on, b is a non-zero real constant, and a and d are complex constants. Furthermore, $\omega = 2\pi i p/5$, with $1 \le p < 5$. The map has fixed points at $(\pm b, 0)$, with one real eigenvalue equal to $1 - 2b$, which is off the unit circle since $b \neq 0$. We are interested in the occurrence of $p : 5$ resonances at these fixed points. To this end we focus on the fixed point $(b, 0)$, and impose the condition that the linear part of F_τ of this fixed point also has two eigenvalues ω and $\overline{\omega}$, lying on the unit circle. A short calculation shows that this situation occurs for $\tau = a\,b + b^2$. For convenience we introduce the parameter ϱ, defined by $\varrho = \tau - a\,b - b^2$, ranging over a neighborhood of $0 \in \mathbb{C}$. At the fixed point $(b, 0)$ the family F_τ has a center manifold of the form

$$x = \varphi(z, \varrho) = b + c(\varrho)\,|z|^2 + O(|z|^3).$$

We determine the unknown real coefficient $c(\varrho)$ from the invariance condition

$$F_1(\varphi(z, \varrho), z) = \varphi(F_2(\varphi(z, \varrho), z)).$$

A short computation shows that

$$c(\varrho) = \frac{1}{1 - 2b - |\omega + \varrho|^2} = -\frac{1}{2b} + O(|\varrho|).$$

Restricted to the invariant manifold the map F_ϱ is of the form

$$z \mapsto F_2(z, \varphi(z, \varrho)) = (\omega + \varrho)\,z + (\frac{a}{2b} - 1 + O(|\varrho|))\,z\,|z|^2 + d\,\overline{z}^4 + O(|z|^5),$$

parametrized by the complex parameter ϱ. Note that this family is of the form (11), provided $a \neq 2b$, so the analysis of the first case applies to this system as well. It leads to a standard cusp shaped resonance tongue like the one depicted in Figure 2.

Mildly degenerate equivariant families. As in the first case study, our third example is a planar family $P(z, \varrho, \sigma)$ in normal form, i.e.,

$$P(z, \varrho, \sigma) = (\omega + \varrho)\,z + \sigma\,z\,|z|^2 + c\,z\,|z|^4 + d\bar{z}^{q-1} + O(|z|^q), \qquad (12)$$

where ϱ and σ are complex parameters ranging over a small neighborhood of $0 \in \mathbb{C}$, and c and d are nonzero complex constants. Note that this family is slightly more degenerate than (11) since also the coefficient σ of the third order term is a small parameter. Furthermore, we require $q \geq 7$, cf Theorem 2. Lyapunov-Schmidt reduction yields the reduced family

$$G(z, \varrho, \sigma) = \varrho\,z + \sigma\,z\,|z|^2 + c\,z\,|z|^4 + d\bar{z}^{q-1} + O(|z|^q). \qquad (13)$$

In [2] we present a full description of the resonance tongue, i.e., of the discriminant set of the family G, which is now an algebraic hypersurface in four-dimensional parameter space. In Figure 3 we depict a 2-dimensional intersection of this parameter space for $q = 7$, together with some phase portraits. Note that, again, the resonance tongues do not represent all local bifurcations of the family (12). In a companion paper we will describe the complete bifurcation set.

Remark. The case studies presented in this section all start from a rather simple expression (normal form) of the family of diffeomorphisms. The Lyapunov-Schmidt algorithm, to be presented in Section 2.1, reduces arbitrarily complicated expressions. See the example at the end of Section 2.1.

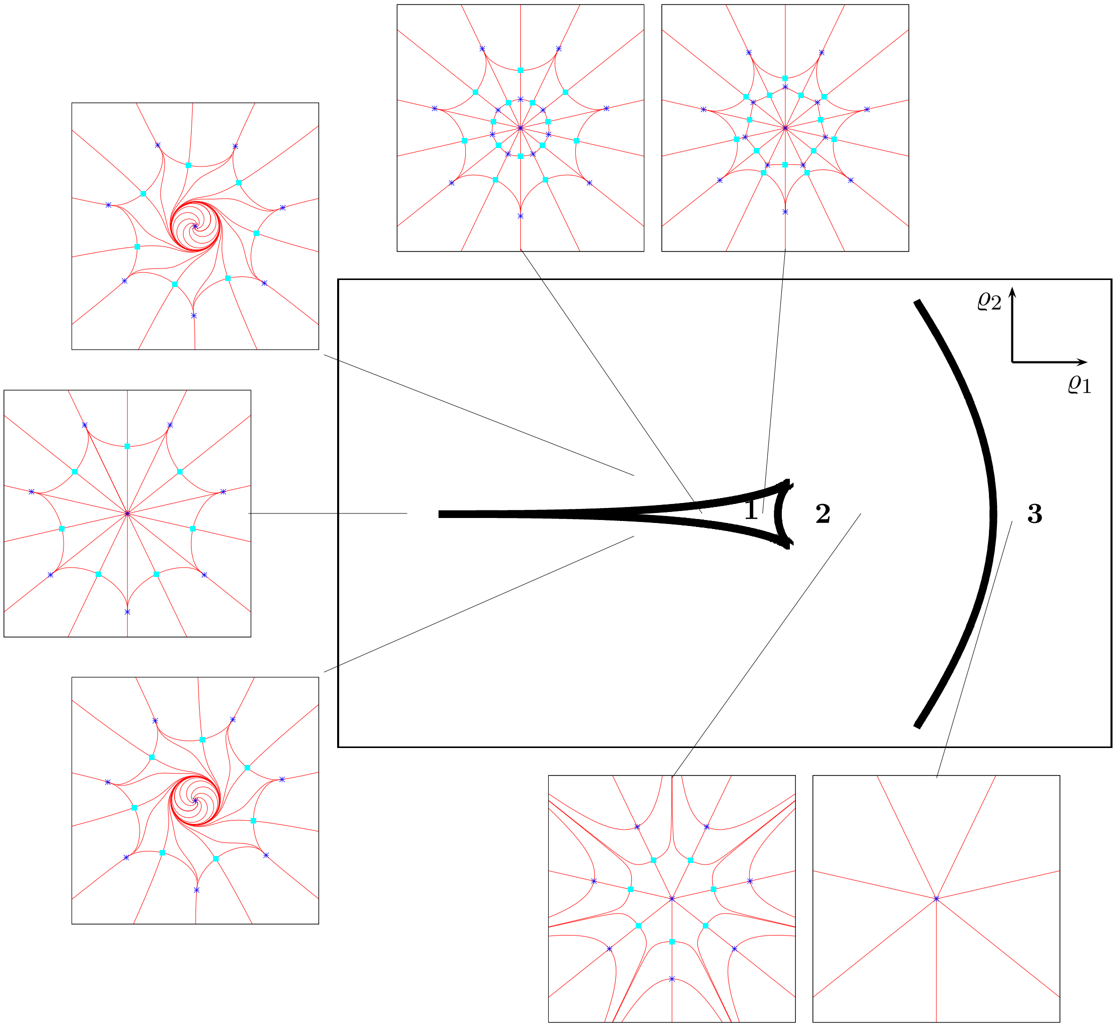

Figure 3: The resonance tongue for the mildly-degenerate family (12) in a 2D cross section σ = constant of ϱ, σ-space for $q = 7$ (central picture), with phase portraits for various values of the parameter ϱ in the complement of the tongue (regions 1, 2 and 3). The origing $\varrho = 0$ is at the tip of the triangular region 1. Regions 1, 2 and 3 correspond to the occurrence of 4, 2, and 0 period-7 orbits. The 4 period-7 orbits lie on two disjoint invariant circles. If $\operatorname{Im} \varrho$ varies such that ϱ crosses the boundary of region 1, two period-7 orbits on the inner circle disappear in a saddle-node bifurcation. On the other hand, if $\operatorname{Re} \varrho$ increases such that ϱ crosses boundary between region 2 and region 3, a periodic orbit of the inner circle and one of the outer circle disappear in a saddle-node bifurcation, destroying both invariant circles. Crossing the boundary between region 2 and 3 yields another saddle-node bifurcation destroying the remaining periodic orbits.

3 An algorithm for Lyapunov-Schmidt reduction

The algorithm for Lyapunov-Schmidt reduction[1] performs three steps. First it transforms the problem of finding q-periodic orbits of maps $P : \mathbb{C} \to \mathbb{C}$, bifurcating from a fixed point, into the problem of finding zeros of an

[1]A `C++` implementation of this algorithm, and some sample output, can be downloaded from `http://www.math.rug.nl/~sijbo/`.

associated map $\widehat{P} : \mathbb{C}^q \to \mathbb{C}^q$. Second, under certain conditions on the linear part of P at the fixed point, the problem is reduced to finding the zeros of a reduced function $G : \mathbb{C} \to \mathbb{C}$. Third, it determines the function G to any desired finite order by successive approximation of the solution of an implicit equation.

Detecting periodic points. The first step of the algorithm is the construction a map $\widehat{P}$, the zeros of which correspond to the q-periodic orbits of $P : V \to V$, where V is some finite dimensional vector space. In our setting $V = \mathbb{C}$ and P is a family of maps given by (1). To this end, let $x_1, \ldots, x_q$ be a q-periodic orbit of P. This orbit is the zero of the map $\widehat{P} : V^q \to V^q$, defined by

$$\widehat{P}(x_1, \ldots, x_q) = (P(x_1) - x_2, \ldots, P(x_{q-1}) - x_q, P(x_q) - x_1). \tag{14}$$

We assume that the periodic orbit bifurcates from a fixed point at the origin, implying $\widehat{P}(0) = 0$. Moreover, this map is $\mathbb{Z}_q$-equivariant, because it commutes with $\sigma : V^q \to V^q$, defined by

$$\sigma(x_1, \ldots, x_q) = (x_2, \ldots, x_q, x_1).$$

Clearly, σ generates an action of $\mathbb{Z}_q$ on V^q. The search for q-periodic orbits of P is transformed into computing the zeros of the $\mathbb{Z}_q$-equivariant map $\widehat{P}$, locally near $0 \in V^q$.

An algorithm for Lyapunov-Schmidt reduction. Starting in a more general setting, let $\Phi : W \to W$ be a smooth map on a vector space V, having the origin as a fixed point. (In our setting, $\Phi = \widehat{P}$ and $W = V^q$.) Moreover, assume that $\mathrm{d}_0\Phi$ is semi-simple, i.e.,

$$W = \ker(\mathrm{d}_0\Phi) \oplus \mathrm{R}(\mathrm{d}_0\Phi),$$

where $\ker(\mathrm{d}_0\Phi)$ is the kernel and $\mathrm{R}(\mathrm{d}_0\Phi)$ the range of $\mathrm{d}_0\Phi$. Let E be the projection onto $\mathrm{R}(\mathrm{d}_0\Phi)$ Then $I - E$ is the projection onto $\ker(\mathrm{d}_0\Phi)$. Using the variables $s \in \ker(\mathrm{d}_0\Phi)$ and $t \in \mathrm{R}(\mathrm{d}_0\Phi)$, it follows that

$$\Phi(s, t) = 0$$

if and only if

$$E\Phi(s, t) = 0 \text{ and } (I - E)\Phi(s, t) = 0. \tag{15}$$

The Implicit Function Theorem implies there is a unique map $s \mapsto t(s)$ near $s = 0$, such that $E\Phi(s, t(s)) = 0$. The solution $t(s)$ can be substituted in $(I - E)\Phi(s, t)$, giving $G(s) \equiv (I - E)\Phi(s, t(s))$. Consequently, the study of zeros of $\Phi : W \to W$ has been reduced to the study of zeros of $G : \ker(\mathrm{d}_0\Phi) \to \ker(\mathrm{d}_0\Phi)$. It should be noted that Lyapunov-Schmidt reduction preserves any equivariance of Φ when appropriate coordinates are used, cf. [14, Chapter VII.3]. We use the coordinates on W in which $\mathrm{d}_0\Phi$ and the symmetry generating map σ are both diagonal.

To determine the reduced function $G(s) = (I - E)\Phi(s, t(s))$ up to a required order, first $t(s)$ is determined by successive approximation of higher order terms. To this end, rewrite $E\,\Phi(s,t)$ as

$$E\,\Phi(s,t) = (As + Bt) + O_2(s,t) + \cdots + O_k(s,t) + O((s^2 + t^2)^{(k+1)/2}),$$

where $A = E\,\mathrm{d}_0\Phi|_{\ker(\mathrm{d}_0\Phi)}$, $B = E\,\mathrm{d}_0\Phi|_{\mathrm{R}(\mathrm{d}_0\Phi)}$ and $O_l(s,t)$, $1 < l \leq k$, denotes the homogeneous part of $E\Phi(s,t)$ of total order l in s and t. A first order approximation $t^{(1)}(s)$ of $t(s)$ is given by

$$t^{(1)}(s) = -B^{-1}As.$$

Assume the algorithm has computed the approximation $t^{(k-1)}(s)$ of $t(s)$ up to order k, i.e.,

$$t(s) = t^{(k-1)}(s) + O(s^k).$$

The approximation of $t(s)$ up to order $k+1$ is then given by

$$t^{(k)}(s) = -B^{-1}(As + O_2(s, t^{(k-1)}(s)) + \cdots + O_k(s, t^{(k-1)}(s)))|_k,$$

where $\ldots|_k$ denote truncation of all terms of order greater than k. Hence, the approximation of the reduced function up to order $k+1$ is given by

$$G(s) = (I - E)\Phi(s, t^{(k)}(s)) + O(s^{k+1}).$$

Lyapunov-Schmidt reduction for planar families of diffeomorphisms. We now prove Theorem 1 by applying the algorithm for Lyapunov-Schmidt reduction to the family (1), i.e.,

$$P_{a,\varrho}(z) = (\omega + \varrho)\,z + Q(z,a),$$

with

$$Q(z,a) = \sum_{2 \leq i+j < q} a_{ij}\, z^i\, \overline{z}^j + O(|z|^q).$$

As before, $\omega = e^{2\pi i p/q}$, with p and q coprime integers.

1. As for the first part of the proof, observe the map $\widehat{P}_\varrho : \mathbb{C}^q \to \mathbb{C}^q$, defined by

$$\widehat{P}_\varrho(z_1, \ldots, z_q) = (P(z_1) - z_2, \ldots, P(z_{q-1}) - z_q, P(z_q) - z_1),$$

is of the form

$$\widehat{P}_\varrho = \hat{L}_\varrho + \hat{Q},$$

where $\hat{L}_\varrho : \mathbb{C}^q \to \mathbb{C}^q$ is the linear map with matrix

$$\hat{L}_\varrho = \begin{pmatrix} \omega+\varrho & -1 & 0 & \cdots & 0 & 0 \\ 0 & \omega+\varrho & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega+\varrho & -1 \\ -1 & 0 & 0 & \cdots & 0 & \omega+\varrho \end{pmatrix},$$

and $\hat{Q}$ is the map defined by

$$\hat{Q}(z_1, \dots, z_q) = (\, Q(z_1), Q(z_2), \dots, Q(z_q)\,).$$

Note that we abuse notation by denoting a linear map and its matrix by the same symbol. The linear transformation $A : \mathbb{C}^q \to \mathbb{C}^q$ bringing $\hat{L}_\varrho$ into diagonal form is the Vandermonde map, with matrix entries

$$A_{ij} = \omega^{(i-1)j}.$$

Using $\omega^q = 1$ it is easy to prove that its inverse has entries

$$A^{-1}_{ij} = \tfrac{1}{q}\, \omega^{(q-i)(j-1)},$$

and that

$$A^{-1}\, \hat{L}_\varrho\, A = \begin{pmatrix} \varrho & & & \\ & \omega-\omega^2+\varrho & & \\ & & \ddots & \\ & & & \omega-\omega^q+\varrho \end{pmatrix}. \qquad (16)$$

Using the linear map with matrix A yields convenient coordinates for solving equations (15). To this end, let the map $\Phi_\varrho : \mathbb{C}^q \to \mathbb{C}^q$ be defined by

$$\Phi_\varrho = A^{-1}\, \widehat{P}_\varrho\, A.$$

For $\varrho = 0$ the linear part $\mathrm{d}_0\Phi_0$ at $0 \in \mathbb{C}^q$ has matrix (16), so its kernel is

$$\ker \mathrm{d}_0\Phi_0 = \{(z_1, z_2, \dots, z_q) \mid z_2 = \cdots = z_q = 0\}.$$

Furthermore, the projection onto $\mathrm{Im}\, \mathrm{d}_0\Phi_0$ is the map $E : \mathbb{C}^q \to \mathbb{C}^{q-1}$, given by

$$E(z_1, z_2, \dots, z_q) = (z_2, \dots, z_q).$$

Using these coordinates, a straightforward application of the successive approximation algorithm for Lyapunov-Schmidt reduction up to order q yields an explicit expression for the function G, introduced in the first part of Theorem 1.

2. To prove the second part of Theorem 1, stating that Lyapunov-Schmidt reduction of a family that is already $\mathbb{Z}_q$-equivariant is a trivial operation. We shall prove that

$$\Phi_\varrho(z, 0, \dots, 0) = (\varrho\, z + Q(z), 0, \dots, 0). \tag{17}$$

Assuming (17) holds, the system

$$E\, \Phi_\varrho(z_1, z_2, \dots, z_q) = (0, \dots, 0) \in \mathbb{C}^{q-1}$$

has solution $z_2 = \cdots = z_q = 0$, locally near $0 \in \mathbb{C}^q$. According to the Implicit Function Theorem this solution is locally unique. Therefore, Lyapunov-Schmidt reduction yields the family $G_\varrho : \mathbb{C} \to \mathbb{C}$, given by

$$G_\varrho(z) = \Phi_\varrho(z, 0, \dots, 0)_1 = \varrho\, z + Q(z).$$

So it remains to prove (17). It follows from

$$\Phi_\varrho = A^{-1}\, \hat{L}_\varrho\, A + A^{-1}\, \hat{Q}\, A$$

and (16) that we only have to prove

$$A^{-1}\, \hat{Q}\, A(z, 0, \dots, 0) = (Q(z), 0, \dots, 0).$$

This identity follows from the following computation:

$$\begin{aligned} A^{-1}\, \hat{Q}\, A(z, 0, \dots, 0) &= A^{-1}\, \hat{Q}(z\, A(1, 0, \dots, 0)) \\ &= A^{-1}\, \hat{Q}(z, \omega\, z, \dots, \omega^{q-1}\, z)) \\ &= A^{-1}(Q(z), Q(\omega\, z) \dots, Q(\omega^{q-1}\, z)) \\ &= A^{-1}(Q(z), \omega\, Q(z), \dots, \omega^{q-1}\, Q(z)) \\ &= Q(z)\, A^{-1}(1, \omega\, \dots, \omega^{q-1}) \\ &= Q(z)\, (1, 0, \dots, 0) \\ &= (Q(z), 0, \dots, 0). \end{aligned}$$

This concludes the proof of Theorem 1. □

Example output of the Lyapunov-Schmidt algorithm. To show the kind of output the algorithm generates we consider resonances of order 5, in a family of planar diffeomorphisms. More precisely, consider

$$P_{a,\varrho}(z) = (\omega + \varrho)\, z + \sum_{2 \le i+j < 5} a_{ij}\, z^i\, \overline{z}^j + O(|z|^5).$$

where $\omega = e^{2\pi i p/5}$, with $1 \le p \le 4$. Then the reduced function G is of the form

$$G(z, a, \varrho) = \varrho\, z + (\, B(a) + O(\varrho)\,)\, z\, |z|^2 + (\, C(a) + O(\varrho)\,)\, \overline{z}^4 + O(|z|^5),$$

cf Theorem 1. The Lyapunov-Schmidt reduction algorithm computes the following expressions for $B(a)$ and $C(a)$:

$$B(a) = a_{21} + B_1\, a_{20}a_{11} + B_2\,|a_{11}|^2 + B_3\,|a_{02}|^2,$$

with

$$\begin{aligned} B_1 =& \tfrac{1}{5}\left(6\omega^3 + 7\omega^2 + 8\omega + 9\right),\\ B_2 =& -\tfrac{1}{5}\left(\omega^3 + 2\omega^2 + 3\omega - 1\right),\\ B_3 =& \tfrac{2}{5}\left(3\omega^3 + \omega^2 - \omega + 2\right),\end{aligned}$$

and

$$\begin{aligned} C(a) = a_{04} + C_1 a_{11}a_{03} + C_2 a_{02}\overline{a_{30}} + C_3 a_{11}^2 a_{02} \\ + C_4 a_{11}a_{02}\overline{a_{20}} + C_5 a_{02}^2\overline{a_{11}} + C_6 a_{02}\overline{a_{20}}^2,\end{aligned}$$

with

$$\begin{aligned} C_1 =& \tfrac{1}{5}\left(4\omega^3 + 3\omega^2 + 2\omega + 1\right),\\ C_2 =& \tfrac{2}{5}\left(3\omega^3 + \omega^2 - \omega + 2\right),\\ C_3 =& \tfrac{1}{5}(\omega - 1)2(\omega + 1),\\ C_4 =& \tfrac{1}{5}\left(-4\omega^3 - 3\omega^2 - 2\omega + 4\right),\\ C_5 =& \tfrac{2}{5}\left(\omega^3 + \omega^2 + 3\right),\\ C_6 =& \tfrac{1}{5}\left(-2\omega^3 + 8\omega^2 + 5\omega + 9\right).\end{aligned}$$

4 Recognition problem for planar families

The (Uni)versal Unfolding Theorem [2, 14].

The germs considered in this paper are elements of $\mathcal{E}(\mathbb{Z}_q)$, the ring of all germs of $\mathbb{Z}_q$-equivariant functions $\mathbb{C} \to \mathbb{C}$ at $0 \in \mathbb{C}$.

The $\mathbb{Z}_q$-*tangent space* of a germ $g \in \mathcal{E}(\mathbb{Z}_q)$ consists of all germs of the form $\frac{d}{dt} g_t(z)\big|_{t=0}$, where g_t is a one-parameter family of germs $\mathcal{E}(\mathbb{Z}_q)$ that are $\mathbb{Z}_q$-contact equivalent to $g_0 = g$. In other words, there is a one-parameter family (S_t, Z_t) of $\mathbb{Z}_q$-equivariant contact transformations such that

$$g_t(z) = S_t(z)\, g(Z_t(z)),$$

with $S_0(z) = 1$ and $Z_0(z) = z$. Note that $t \mapsto g_t$ is a curve in the orbit of g under the action of the group of $\mathbb{Z}_q$-equivariant contact transformations. The $\mathbb{Z}_q$-tangent space of the germ g is denoted by $T(g)$.

The *(Uni)versal Unfolding Theorem* states that a k-parameter unfolding G of a germ g is *versal* if

$$\mathcal{E}(\mathbb{Z}_q) = T(g) + \mathbb{R}\,\{\left.\frac{\partial G}{\partial \mu_1}\right|_{\mu=0} + \cdots + \left.\frac{\partial G}{\partial \mu_k}\right|_{\mu=0}\},$$

and that g is *universal* if, moreover, the number k of parameters in G equals the codimension of $T(g)$, i.e., the dimension of the real vector space $\mathcal{E}(\mathbb{Z}_q)\,/\,T(g)$.

The Schwarz Finitude Theorem [14], states that every $g \in \mathcal{E}(\mathbb{Z}_q)$ is of the form

$$g(z,\bar{z}) = K(u,v)\,z + L(u,v)\,\bar{z}^{q-1},$$

where $u = z\bar{z}$ and $v = z^{q-1} + \bar{z}^{q-1}$, and K, L are uniquely define complex-valued function germs. In other words, every $g \in \mathcal{E}(\mathbb{Z}_q)$ can be identified with the pair $(K,L) \in \mathcal{E}^2_{u,v}$, where $\mathcal{E}_{u,v}$ is the ring of smooth complex-valued germs depending on the real variables u and v. This identification was used in [2] to determine the tangent space of the 'simplest' $\mathbb{Z}_q$-equivariant germs. These results will also be crucial in the proof of Theorem 2.

Proof of Theorem 2. We consider $\mathbb{Z}_q$-equivariant families G, which are of the form

$$G(z,\mu) = K(u,v,\mu)\,z + L(u,v,\mu)\,\overline{z}^{q-1},$$

for parameter values μ near $0 \in \mathbb{R}^k$, with $K(0,0,0) = 0$ and $L(0,0,0) \neq 0$, unfolding the central singularity g given by

$$g(z) = K_0(u,v)\,z + L_0(u,v)\,\overline{z}^{q-1},$$

where $K_0(u,v) = K(u,v,0)$ and $L_0(u,v) = L(u,v,0)$. To prove that G is a versal unfolding of g under the conditions (6) and (7), respectively, we use the (Uni)versal Unfolding Theorem. In view of this theorem it is sufficient to show that

$$\mathcal{E}(\mathbb{Z}_q) = T(g) + \mathbb{R}\{\left.\frac{\partial G}{\partial \mu_1}\right|_{\mu=0}, \dots, \left.\frac{\partial G}{\partial \mu_k}\right|_{\mu=0}\} \tag{18}$$

where $T(g)$ is the $\mathbb{Z}_q$-tangent space of g.

Recall that we identify $\mathcal{E}(\mathbb{Z}_q)$ with $\mathcal{E}^2_{u,v}$, and $g \in \mathcal{E}^2_{u,v}(\mathbb{Z}_q)$ with the pair $(K_0,L_0) \in \mathcal{E}^2_{u,v}$. Since $L_0(0,0) \neq 0$, it follows that the second component of (K_0,L_0) generates $\mathcal{E}_{u,v}$. Since $K_0(0,0) = 0$, the first component belongs to the maximal ideal $\mathcal{M}$ of $\mathcal{E}_{u,v}$. So,

$$T(g) \subset \mathcal{M}\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathcal{E}_{u,v}\begin{pmatrix}0\\1\end{pmatrix},$$

for all central singularities g we consider.

We now focus on the two kinds of families G addressed in Theorem 2. Here we use the results of [2, Appendix A] regarding the $\mathbb{Z}_q$-tangent space of $T(g)$, for various $\mathbb{Z}_q$-equivariant germs g.

1. If $q \geq 5$ and $K(0,0,0) \neq 0$, then

$$T(g) = \mathcal{M}\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathcal{E}_{u,v}\begin{pmatrix}0\\1\end{pmatrix},$$

cf [2, Appendix A]. Therefore, a complement of $T(g)$ is the two-dimensional real vector space V defined by

$$V = \mathbb{R}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}i\\0\end{pmatrix}\right\}.$$

Indeed, the projection $\Pi : \mathcal{E}^2_{u,v} \to V$ is given by

$$\Pi(M,N) = \big(\operatorname{Re} M(0,0), \operatorname{Im} M(0,0)\big),$$

since, for $M \in \mathcal{E}_{u,v}$:

$$M(u,v) = \operatorname{Re} M(0,0) + i\, \operatorname{Im} M(0,0) \mod \mathcal{M}.$$

Therefore, (18) holds if $k \geq 2$, and there are two parameters, μ_1 and μ_2, say, such that

$$\left.\frac{\partial(\operatorname{Re} K(0,0,\mu), \operatorname{Im} K(0,0,\mu))}{\partial(\mu_1,\mu_2)}\right|_{\mu=0} \neq 0.$$

2. If $q \geq 7$ and $K(0,0,0) = 0$, but $K_u(0,0,0) \neq 0$, then

$$T(g) = \mathcal{M}^2\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathcal{E}_{u,v}\begin{pmatrix}v\\0\end{pmatrix} \oplus \mathcal{E}_{u,v}\begin{pmatrix}0\\1\end{pmatrix},$$

cf [2, Appendix A]. Therefore, a complement of $T(g)$ is the four-dimensional real vector space V defined by

$$V = \mathbb{R}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}i\\0\end{pmatrix}, \begin{pmatrix}u\\0\end{pmatrix}, \begin{pmatrix}iu\\0\end{pmatrix}\right\}.$$

In this case, the projection $\Pi : \mathcal{E}^2_{u,v} \to V$ is given by

$$\Pi(M,N) = \big(\operatorname{Re} M(0,0), \operatorname{Im} M(0,0), \operatorname{Re} M_u(0,0), \operatorname{Im} M_u(0,0)\big),$$

since, for $M \in \mathcal{E}_{u,v}$:

$$\begin{aligned} M(u,v) = \operatorname{Re} M(0,0) + i\, \operatorname{Im} M(0,0) + \operatorname{Re} M_u(0,0)\, u \\ + i\, \operatorname{Im} M_u(0,0)\, u \qquad \mod \mathcal{M}^2 + \langle v\rangle. \end{aligned}$$

Therefore, (18) holds if $k \geq 4$, and there are four parameters, μ_1, μ_2, μ_3 and μ_4, say, such that

$$\left.\frac{\partial(\operatorname{Re} K(0,0,\mu), \operatorname{Im} K(0,0,\mu), \operatorname{Re} K_u(0,0,\mu), \operatorname{Im} K_u(0,0,\mu))}{\partial(\mu_1,\mu_2,\mu_3,\mu_4)}\right|_{\mu=0} \neq 0.$$

This concludes the proof of Theorem 2. □

Although we do not address the cases $q = 3$ and $q = 4$ in our leading examples, the versality of unfoldings can also be established in these cases, based on the expressions for the $\mathbb{Z}_q$-tangent space of the central singularity derived for these cases in [2]. The precise conditions, solving the recognition problem, are summarized in the following result.

Lemma 5. *1. If $q = 3$, the family G is a versal unfolding of g if $k \geq 2$, and if there are two parameters, μ_1 and μ_2 say, such that*

$$\left.\frac{\partial(\operatorname{Re} K(0,0,\mu), \operatorname{Im} K(0,0,\mu))}{\partial(\mu_1, \mu_2)}\right|_{\mu=0} \neq 0,$$

irrespective of K, i.e., irrespective of whether $K_u(0,0)$ is non-zero or not.

2. If $q = 4$, the family G is a versal unfolding of g if $a = |K_u(0,0)/L(0,0)| \neq 0, 1$, if $k \geq 3$, and if there are three parameters, μ_1, μ_2 and μ_3 say, such that

$$\left.\frac{\partial(\operatorname{Re} K(0,0,\mu), \operatorname{Im} K(0,0,\mu), \operatorname{Re} K_u(0,0,\mu))}{\partial(\mu_1, \mu_2, \mu_3)}\right|_{\mu=0} \neq 0.$$

Proof. 1. If $q = 3$, we have $T(g) = \mathcal{M}\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathcal{E}_{u,v}\begin{pmatrix}0\\1\end{pmatrix}$, irrespective of K, that is, irrespective of whether $K_u(0,0)$ is non-zero or not. In this case the family $G(z,\mu)$ is a generic unfolding of g if

$$\left.\frac{\partial(\operatorname{Re} K(0,0,\mu), \operatorname{Im} K(0,0,\mu))}{\partial(\mu_1, \mu_2)}\right|_{\mu=0} \neq 0.$$

2. The case $q = 4$ is interesting in the sense that a *moduli parameter* appears: the tangent space is equal to

$$T(g) = \mathcal{M}^2\begin{pmatrix}1\\0\end{pmatrix} \oplus \mathcal{M}\begin{pmatrix}0\\1\end{pmatrix} \oplus \mathbb{R}\left\{\begin{pmatrix}v\\0\end{pmatrix}, \begin{pmatrix}iv\\0\end{pmatrix}, \begin{pmatrix}0\\i\end{pmatrix}, \begin{pmatrix}au\\1\end{pmatrix}, \begin{pmatrix}iau\\0\end{pmatrix}\right\},$$

with $a = |K_u(0,0)/L(0,0)|$. Therefore, if $|a| \neq 0, 1$ a complement of $T(g)$ is the three-dimensional real vector space V defined by

$$V = \mathbb{R}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}i\\0\end{pmatrix}, \begin{pmatrix}u\\0\end{pmatrix}\right\},$$

since, for $M \in \mathcal{E}_{u,v}$:

$$M(u,v) = \operatorname{Re} M(0,0) + i\, \operatorname{Im} M(0,0) + \operatorname{Re} M_u(0,0)\, u \mod T(g).$$

Therefore, (18) holds if $k \geq 3$, and there are three parameters, μ_1, μ_2 and μ_3, say, such that

$$\left.\frac{\partial(\operatorname{Re} K(0,0,\mu), \operatorname{Im} K(0,0,\mu), \operatorname{Re} K_u(0,0,\mu))}{\partial(\mu_1, \mu_2, \mu_3)}\right|_{\mu=0} \neq 0.$$

□

Lemma 5 yields explicit expressions for universal unfoldings of $\mathbb{Z}_q$-equivariant germs g of the form

$$g(z) = z\, k(|z|^2) + d\,\overline{z}^{q-1} + O(|z|^q),$$

with $k(0) = 0$ and $d \neq 0$, also for the cases $q = 3$ and $q = 4$. These are of the form

$$G(z, \mu_1, \mu_2) = g(z) + (\mu_1 + i\mu_2)\, z, \qquad \text{if } q = 3;$$

$$G(z, \mu_1, \mu_2, \mu_3) = g(z) + (\mu_1 + i\mu_2)\, z + \mu_3\, z\, |z|^2, \qquad \text{if } q = 4 \text{ and } |k'(0)/l(0)| \neq 0, 1.$$

Normal forms for $\mathbb{Z}_q$-equivariant germs. In [2] we determined normal forms for generic and midly degenerate $\mathbb{Z}_q$-equivariant germs of functions $\mathbb{C} \to \mathbb{C}$. These normal forms are low-degree polynomials in $z, \overline{z}$. For completeness, we recall these results here.

The *tangent space constant theorem* states that, for $g \in \mathcal{E}(\mathbb{Z}_q)$, the germ $g + tp$ is equivalent to g for all $p \in \mathcal{E}(\mathbb{Z}_q)$ if

$$T(g + tp) = T(g),$$

for $t \in [0, 1]$. Using this theorem, in [2] we obtained the following classification of the simplest $\mathbb{Z}_q$-germs

$$g(z, \bar{z}) = K(u, v)z + L(u, v)\bar{z}^{q-1},$$

in case $K(0, 0) = 0$, and $L(0, 0) \neq 0$:

1. If $K_u(0, 0) \neq 0$, then a normal form of the germ g is:

$$\begin{cases} \overline{z}^2, & \text{if } q = 3; \\ a\, z\, |z|^2 + \overline{z}^3, & \text{if } q = 4, \text{ with } a = |K_u(0,0)/L(0,0)| \neq 0, 1; \\ z\, |z|^2 + \overline{z}^{q-1}, & \text{if } q \geq 5. \end{cases}$$

The parameter a in the case $q = 4$ is a modulus: the germs corresponding to different values of this modulus belong to different orbits under the group of $\mathbb{Z}_q$-equivariant contact transformations. We refer to our earlier paper [2] for further details, and for a derivation of this fact.

2. If $K_u(0, 0) = 0$ and $K_{uu}(0, 0) \neq 0$, then a normal form of g is:

$$z\, |z|^4 + \overline{z}^{q-1}, \qquad \text{if } q \geq 7.$$

5 Conclusion and future work

The reduction algorithm, and the solution to the recognition problem presented in this paper allow for a classification of the resonance sets of generic and mildly degenerate families of planar diffeomorphisms a fixed point of which undergoes a Hopf-Neĭmark-Sacker bifurcation with its characteristic array of resonance tongues organizing the alteration of periodic and quasi-periodic dynamics. With this approach the shape of the boundary of the resonance set can be determined up to diffeomorphism. More precisely, we are able to determine a normal form for the family. The resonance tongues appearing in normal forms of generic families are easy to analyze; See Figure 3. In the mildly degenerate case the boundary of the resonance set is an algebraic subset of four-dimensional space. We are currently exploring methods to analyze and visualize this set, which will be published in a companion paper.

To determine the boundary of the resonance set of the actual family, and not just of its normal form, our algorithm needs to be extended with a module computing the parametrization involved in the transformation that brings the LS-reduced family into normal form under $\mathbb{Z}_q$-equivariant contact equivalence. Such an extension would bring our current work under the same paradigm of our earlier volume [5], where we used fine-tuned Gröbner basis methods to compute both the normal form and the transformation bringing the system at hand into this normal form. The current context, in which all germs are $\mathbb{Z}_q$-equivariant, is more complicated, so further work needs to be done in order to fully solve the recognition problem.

We also plan to extend our work to study resonance sets in families of *continuous systems*. The development of effective methods requires a combination of symbolic and numeric methods.

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