Basic dynamic logic
DRAFT

Gerard R. Renardel de Lavalette

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1 Introduction

The essential idea behind dynamic logic is: the meaning of a formula is not an information state (ie. a collection of possible worlds) but an *information state transformer*. This idea has been worked out in quite a number of logics. In an attempt to unify some propositional logics with dynamic semantics, I introduce in this note BDL (basic dynamic logic). It has a straightforward semantics in terms of elements of $\wp(W) \to \wp(W)$, where W is the collection of possible worlds. The following logics will be embedded in BDL:

- update logic UL ([3]);
- modal assertion logic mal_P, in which assertions about formulae in UL can be made ([1]);
- conscious K45-modal logic CK45 ([2], 4.5);
- eliminative K-modal logic EK having the same language as CK45 ([2], 4.4).

2 Preliminaries

P is a collection of propositional atoms. $W = W_P = \wp(P)$ is the collection of possible worlds. Moreover, $R \subseteq W^2$, i.e. R is a binary relation on W, and $\Sigma := \wp(W)$.
For the interpretation of atoms, we use γ:

$$\begin{array}{lcl} \gamma & : & P \to \Sigma \\ \gamma(p) & = & \{w \in W \mid p \in w\} \end{array}$$

Two ways of function application for relations are introduced:

$$\begin{array}{lcl} \cdot\langle\cdot\rangle, \cdot[\cdot] & : & \wp(X) \to \wp(X \times Y) \to \wp(Y) \\ A\langle R\rangle & = & \{y \in Y \mid \exists x(xRy \wedge x \in A)\} \\ A[R] & = & \{y \in Y \mid \forall x(xRy \to x \in A)\} \end{array}$$

Some consequences of these definitions:

$$\begin{array}{rcl} A\langle R\rangle & = & Y - (X - A)[R] \\ \emptyset\langle R\rangle & = & \emptyset \\ X[R] & = & Y \\ A \neq \emptyset \to A\langle X \times Y\rangle & = & Y \\ A \neq X \to A[X \times Y] & = & \emptyset \end{array}$$

In the sequel, several interpretations of logical languages into a semantical domain (usually $\Sigma \to \Sigma$) will be defined, using the notations $[\![\cdot]\!]$ and $||\cdot||$. To distinguish these, I add the name of the interpreted language as a subscript, whenever appropriate.

3 Logics with dynamic semantics

3.1 Update logic

Update logic UL is defined in [3] by

$$\begin{array}{lrcl} & \varphi & ::= & \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \\ \mathsf{UL} & \psi & ::= & \varphi \mid \mathbf{might}\varphi \end{array}$$

Semantics:

$$\begin{array}{lcl} \cdot[\![\cdot]\!] & : & \Sigma \to \mathsf{UL} \to \Sigma \\ \\ \sigma[\![p]\!] & = & \sigma \cap \gamma(p) \\ \sigma[\![\neg\varphi]\!] & = & \sigma - \sigma[\![\varphi]\!] \\ \sigma[\![\varphi \wedge \psi]\!] & = & \sigma[\![\varphi]\!] \cup \sigma[\![\psi]\!] \\ \sigma[\![\mathbf{might}\varphi]\!] & = & \sigma \cap (\sigma[\![\pi]\!])\langle W^2 \rangle \end{array}$$

3.2 Modal assertion logic

For the definition of mal_P, I follow [1]. The syntax of UL is reformulated in L_P with two extensions: nested occurrence of **might** is allowed, and ';' is added. mal_P is the assertion language over L_P.

$$\begin{array}{lrcl} \mathsf{L}_P & \pi & ::= & \bot \mid p \mid \neg\pi \mid \mathbf{might}\pi \mid \pi \cup \pi \mid \pi;\pi \\ \mathsf{mal}_P & \varphi & ::= & \bot \mid p \mid \neg\varphi \mid \Diamond\varphi \mid \varphi \wedge \varphi \mid \langle\pi\rangle\varphi \end{array}$$

As to the semantics of mal_P, I still follow [1], adopting the notation of [2]. So σ (instead of I) ranges over Σ, and I write $\sigma[\![\pi]\!]$ for $[\![\pi]\!]I$, and $\sigma||\varphi||$ for $||\varphi||_I$.

$$\begin{array}{lcl} \cdot[\![\cdot]\!] & : & \Sigma \to \mathsf{L}_P \to \Sigma \\ \cdot||\cdot|| & : & \Sigma \to \mathsf{mal}_P \to \Sigma \\ \\ \sigma[\![\bot]\!] & = & \emptyset \\ \sigma[\![p]\!] & = & \sigma \cap \gamma(p) \\ \sigma[\![\neg\pi]\!] & = & \sigma - \sigma[\![\pi]\!] \\ \sigma[\![\mathbf{might}\pi]\!] & = & \sigma \cap (\sigma[\![\pi]\!])\langle W^2 \rangle \\ \sigma[\![\pi_1 \cup \pi_2]\!] & = & \sigma[\![\pi_1]\!] \cup \sigma[\![\pi_2]\!] \\ \sigma[\![\pi_1;\pi_2]\!] & = & (\sigma[\![\pi_1]\!])[\![\pi_2]\!] \\ \\ \sigma||\bot|| & = & \emptyset \\ \sigma||p|| & = & \sigma \cap \gamma(p) \\ \sigma||\neg\varphi|| & = & \sigma - \sigma||\varphi|| \\ \sigma||\Diamond\varphi|| & = & \sigma \cap (\sigma||\varphi||)\langle W^2 \rangle \\ \sigma||\varphi \wedge \psi|| & = & \sigma||\varphi|| \cap \sigma||\psi|| \\ \sigma||\langle\pi\rangle\varphi|| & = & (\sigma[\![\pi]\!])||\psi|| \end{array}$$

3.3 Dynamic epistemic logics

In [2], Ch.4, the two dynamic epistemic logics $\mathsf{CK45}$ (conscious K45) and EK (eliminative K) are defined, both with the following language DEL:

$$\mathsf{DEL} \quad \varphi \quad ::= \quad p \mid \neg\varphi \mid \Box\varphi \mid \varphi \wedge \varphi \mid [\varphi]\varphi$$

For the semantics of CK45, I deviate slightly from the notation of [2], 4.5.5: instead of $(w, \sigma) \models \varphi$, I write $w \in \sigma||\varphi||$. There are two interpretations:

$$
\begin{array}{lcl}
\cdot[\![\cdot]\!] & : & \Sigma \to \mathsf{CK45} \to \Sigma \\
\cdot||\cdot|| & : & \Sigma \to \mathsf{CK45} \to \Sigma
\end{array}
$$

$$
\begin{array}{lcl}
\sigma[\![p]\!] & = & \sigma \cap \gamma(p) \\
\sigma[\![\neg\varphi]\!] & = & \sigma - \sigma[\![\varphi]\!] \\
\sigma[\![\Box\varphi]\!] & = & \sigma \cap ((W - \sigma) \cup \sigma[\![\varphi]\!])[W^2] \\
\sigma[\![\varphi \wedge \psi]\!] & = & \sigma[\![\varphi]\!] \cap \sigma[\![\psi]\!] \\
\sigma[\![[\varphi]\psi]\!] & = & \sigma \cap [W^2]((W - \sigma) \cup (\sigma[\![\varphi]\!])||\psi||)
\end{array}
$$

$$
\begin{array}{lcl}
\sigma||p|| & = & \gamma(p) \\
\sigma||\neg\varphi|| & = & W - \sigma||\varphi|| \\
\sigma||\Box\varphi|| & = & ((W - \sigma) \cup \sigma||\varphi||)[W^2] \\
\sigma||\varphi \wedge \psi|| & = & \sigma||\varphi|| \cap \sigma||\psi|| \\
\sigma||[\varphi]\psi|| & = & (\sigma[\![\varphi]\!])||\psi||
\end{array}
$$

Some consequences:

$$
\begin{array}{lcl}
\sigma[\![\Box\varphi]\!] & = & \sigma - (\sigma - \sigma[\![\varphi]\!])\langle W^2\rangle \\
\sigma[\![\Diamond\varphi]\!] & = & \sigma \cap (\sigma[\![\varphi]\!])\langle W^2\rangle \\
\sigma[\![[\varphi]\psi]\!] & = & \sigma - (\sigma - (\sigma[\![\varphi]\!])||\psi||)\langle W^2\rangle \\
\sigma[\![\langle\varphi\rangle\psi]\!] & = & \sigma \cap ((\sigma[\![\varphi]\!])||\psi||)\langle W^2\rangle
\end{array}
$$

$$
\begin{array}{lcl}
\sigma||\Diamond\varphi|| & = & (\sigma \cap \sigma||\varphi||)\langle W^2\rangle \\
\sigma||\langle\varphi\rangle\psi|| & = & (\sigma[\![\varphi]\!])||\psi||
\end{array}
$$

To compare these two interpretations, Groeneveld introduces the following mapping:

$$
\begin{array}{lcl}
\cdot^{\mathsf{s}} & : & \mathsf{CK45} \to \mathsf{CK45}
\end{array}
$$

$$
\begin{array}{lcl}
p^{\mathsf{s}} & = & p \\
(\neg\varphi)^{\mathsf{s}} & = & \neg(\varphi^{\mathsf{s}}) \\
(\Box\varphi)^{\mathsf{s}} & = & \Box(\varphi^{\mathsf{s}}) \\
(\varphi \wedge \psi)^{\mathsf{s}} & = & \varphi^{\mathsf{s}} \wedge \psi^{\mathsf{s}} \\
([\varphi]\psi)^{\mathsf{s}} & = & \Box([\varphi]\psi)
\end{array}
$$

and establishes $\sigma[\![\varphi]\!] = \sigma \cap \sigma||\varphi^{\mathsf{s}}||$.

EK has a more complicated semantics than CK45, involving a higher-order information structure $S = \bigcup\{S_n \mid n \in \omega\}$, defined by

$$
\begin{array}{lcl}
S_0 & = & W \\
I_{n+1} & = & \wp(S_n) \\
S_{n+1} & = & \{v \in (S_0 \times I_1 \times \ldots \times I_{n+1}) \mid EXT(v)\}
\end{array}
$$

where EXT is a predicate enforcing a kind of internal consistency. It is shown in [2] (4.4.5, the Extension Lemma) that each situation $s \in S$ is determined by its first and last component. As a consequence, S is isomorphic to $T = \bigcup\{T_n \mid n \in \omega\}$, defined by

$$
\begin{array}{lcl}
T_0 & = & W \\
T_{n+1} & = & W \times \wp(T_n)
\end{array}
$$

Substructures of T are obtained by embedding Kripke structures as follows: let $K = (W, R, \gamma)$ be a Kripke model, then $f_n : W \to T_n$, the nth embedding of K, is defined inductively by

$$\begin{array}{lcl} f_0(w) & = & \gamma(w) \\ f_{n+1}(w) & = & (\gamma(w), \{f_n(w') \mid wRw'\}) \end{array}$$

The essential clauses of the definition of the semantics are ([2], 4.4.28):

$$\begin{array}{lcl} s \models \Diamond\varphi & = & \exists t \in s_k(t \models \varphi) \\ s \models [\varphi]\psi & = & s \bullet_n \{t \in s_n \mid t \models \varphi\} \models \psi \end{array}$$

where $s \in S_k$, s_k is the kth component of s, $n \leq k$ is the modal degree of φ, and $s \bullet_n i$ is: s revised by i at level n.

An axiomatisation of EK is given in [2], 4.4.32. The most interesting principle is Consistent Elimination & Minimality (CEM):

$$[\varphi]\Diamond\psi \equiv \Diamond(\varphi \wedge \psi)$$

4 Basic dynamic logic

In dynamic semantics, the meaning of a formula is an information state transformer. So we are talking about automorphisms of some collection of sets, and it is appropriate to put a few natural primitives in the language: constant functions, the identity function, the complement function, intersection, composition, functions generated by a relation. In this way BDL is obtained.

$$\mathsf{BDL} \quad \varphi \quad ::= \quad p \mid I \mid \neg \mid \Diamond \mid \varphi \wedge \varphi \mid \varphi;\varphi$$

Some abbreviations:

$$\begin{array}{lcl} \bot & := & \mathsf{Id} \wedge \neg \\ \neg\varphi & := & \varphi;\neg \\ \Diamond\varphi & := & \varphi;\Diamond \\ \langle\varphi\rangle\psi & := & \varphi;\psi \end{array}$$

Furthermore, there are the usual definitions of $\top, \vee, \rightarrow, \leftrightarrow, \Box$ and $[\varphi]\psi$.

Semantics:

$$\begin{array}{lcl} \cdot[\![\cdot]\!] & : & \Sigma \to \mathsf{BDL} \to \Sigma \\ \sigma[\![p]\!] & = & \gamma(p) \\ \sigma[\![\mathsf{Id}]\!] & = & \sigma \\ \sigma[\![\neg]\!] & = & W - \sigma \\ \sigma[\![\Diamond]\!] & = & \sigma\langle R\rangle \\ \sigma[\![\varphi \wedge \psi]\!] & = & \sigma[\![\varphi]\!] \cap \sigma[\![\psi]\!] \\ \sigma[\![\varphi;\psi]\!] & = & (\sigma[\![\varphi]\!])[\![\psi]\!] \end{array}$$

As a consequence:

$$\begin{array}{lcl} \sigma[\![\top]\!] & = & W \\ \sigma[\![\neg\varphi]\!] & = & W - \sigma[\![\varphi]\!] \\ \sigma[\![\Diamond\varphi]\!] & = & (\sigma[\![\varphi]\!])\langle R\rangle \\ \sigma[\![\langle\varphi\rangle\psi]\!] & = & (\sigma[\![\varphi]\!])[\![\psi]\!] \end{array}$$

It is natural to say that formulae with the same semantics are equivalent:

$$\varphi \equiv \psi \quad := \quad \forall\sigma \in \Sigma(\sigma[\![\varphi]\!] = \sigma[\![\psi]\!])$$

Now some properties can be formulated:

$$
\begin{array}{rcl}
\varphi; p & \equiv & p \\
\varphi; \mathsf{Id} & \equiv & \varphi \\
\varphi; (\psi \wedge \chi) & \equiv & (\varphi; \psi) \wedge (\varphi; \chi) \\
\varphi; (\psi; \chi) & \equiv & (\varphi; \psi); \chi \\
(\varphi \vee \psi); \Diamond & \equiv & (\varphi; \Diamond) \vee (\psi; \Diamond) \\
\varphi; \neg\psi & \equiv & \neg(\varphi; \psi) \\
\varphi; \Diamond\psi & \equiv & \Diamond(\varphi; \psi) \\
\neg; \neg & \equiv & \mathsf{Id} \\
\mathsf{Id}; \varphi & \equiv & \varphi
\end{array}
$$

The first four properties suggest the eliminability of ';'. Indeed BDL is a definitional extension of the sublanguage BDL^-, defined by

$$\mathsf{BDL}^- \quad \varphi \quad ::= \quad p \mid \mathsf{Id} \mid \neg\varphi \mid \Diamond\varphi \mid \varphi \wedge \varphi$$

The reduction of BDL to BDL^- is given by

$$
\begin{array}{lcl}
\cdot^{\mathsf{E}} & : & \mathsf{BDL} \to \mathsf{BDL}^- \\
\\
p^{\mathsf{E}} & = & p \\
\mathsf{Id}^{\mathsf{E}} & = & \mathsf{Id} \\
\neg^{\mathsf{E}} & = & \neg\mathsf{Id} \\
\Diamond^{\mathsf{E}} & = & \Diamond\mathsf{Id} \\
(\varphi \wedge \psi)^{\mathsf{E}} & = & \varphi^{\mathsf{E}} \wedge \psi^{\mathsf{E}} \\
(\varphi; p)^{\mathsf{E}} & = & p \\
(\varphi; \mathsf{Id})^{\mathsf{E}} & = & \varphi^{\mathsf{E}} \\
(\varphi; \neg)^{\mathsf{E}} & = & \neg(\varphi^{\mathsf{E}}) \\
(\varphi; \Diamond)^{\mathsf{E}} & = & \Diamond(\varphi^{\mathsf{E}}) \\
(\varphi; (\psi \wedge \chi))^{\mathsf{E}} & = & (\varphi; \psi)^{\mathsf{E}} \wedge (\varphi; \chi)^{\mathsf{E}} \\
(\varphi; (\psi; \chi))^{\mathsf{E}} & = & ((\varphi; \psi); \chi)^{\mathsf{E}}
\end{array}
$$

The correctness of this definition is proved using the following complexity measure on BDL:

$$\delta : \mathsf{BDL} \to N$$

$$
\begin{array}{l}
\delta(p) = \delta(\mathsf{Id}) = \delta(\neg) = \delta(\Diamond) = 0 \\
\delta(\varphi \wedge \psi) = \delta\varphi + \delta\psi + 1 \\
\delta(\varphi; \psi) = \delta\varphi + 2 \cdot \delta\psi + 1
\end{array}
$$

It is easily verified that, in the definition of $\cdot^{\mathsf{E}}$, the complexity is lowered. Now $\varphi \equiv \varphi^{\mathsf{E}}$ can be proved with induction over $\delta\varphi$.

4.1 Interpretation of UL and mal_P into BDL

UL is part of mal_P so we directly turn to the latter and define

$$
\begin{array}{lcl}
\cdot^{\mathsf{I}} & : & \mathsf{L}_P \to \mathsf{BDL} \\
\cdot^{\mathsf{I}} & : & \mathsf{mal}_P \to \mathsf{BDL}
\end{array}
$$

$$
\begin{array}{lcl}
\bot^{\mathsf{I}} & = & \bot \\
p^{\mathsf{I}} & = & \mathsf{Id} \wedge p \\
(\neg\pi)^{\mathsf{I}} & = & \mathsf{Id} \wedge \neg(\pi^{\mathsf{I}}) \\
(\neg\varphi)^{\mathsf{I}} & = & \mathsf{Id} \wedge \neg(\varphi^{\mathsf{I}}) \\
(\mathbf{might}\pi)^{\mathsf{I}} & = & \mathsf{Id} \wedge \Diamond(\pi^{\mathsf{I}}) \\
(\Diamond\varphi)^{\mathsf{I}} & = & \mathsf{Id} \wedge \Diamond(\varphi^{\mathsf{I}}) \\
(\pi_1 \cup \pi_2)^{\mathsf{I}} & = & {\pi_1}^{\mathsf{I}} \vee {\pi_2}^{\mathsf{I}} \\
(\varphi \wedge \psi)^{\mathsf{I}} & = & \varphi^{\mathsf{I}} \wedge \psi^{\mathsf{I}} \\
(\pi_1; \pi_2)^{\mathsf{I}} & = & {\pi_1}^{\mathsf{I}}; {\pi_2}^{\mathsf{I}} \\
(\langle\pi\rangle\varphi)^{\mathsf{I}} & = & \pi^{\mathsf{I}}; \varphi^{\mathsf{I}}
\end{array}
$$

Some consequences:

$$
\begin{array}{lcl}
\top^{\mathsf{I}} & = & \mathsf{Id} \\
(\Box\varphi)^{\mathsf{I}} & = & \mathsf{Id} \wedge \Box(\mathsf{Id} \to \varphi^{\mathsf{I}}) \\
(\varphi \vee \psi)^{\mathsf{I}} & = & \varphi^{\mathsf{I}} \vee \psi^{\mathsf{I}} \\
(\varphi \to \psi)^{\mathsf{I}} & = & \mathsf{Id} \wedge (\varphi^{\mathsf{I}} \to \psi^{\mathsf{I}}) \\
(\varphi \leftrightarrow \psi)^{\mathsf{I}} & = & \mathsf{Id} \wedge (\varphi^{\mathsf{I}} \leftrightarrow \psi^{\mathsf{I}}) \\
([\pi]\varphi)^{\mathsf{I}} & = & \mathsf{Id} \wedge \pi^{\mathsf{I}}; (\mathsf{Id} \to \varphi^{\mathsf{I}})
\end{array}
$$

Now, if $R := W^2$ in the semantics of BDL, then:

$$
\begin{array}{lcl}
[\![\pi^{\mathsf{I}}]\!]_{\mathsf{BDL}} & = & [\![\pi]\!]_{\mathsf{L}_P} \\
[\![\varphi^{\mathsf{I}}]\!]_{\mathsf{BDL}} & = & ||\varphi||_{\mathsf{mal}_P}
\end{array}
$$

$$
\begin{array}{rcl}
\pi^{\mathsf{I}} & \equiv & \mathsf{Id} \wedge \pi^{\mathsf{I}} \\
\varphi^{\mathsf{I}} & \equiv & \mathsf{Id} \wedge \varphi^{\mathsf{I}}
\end{array}
$$

$$
\begin{array}{rcl}
(\varphi \downarrow \psi)^{\mathsf{I}} & \equiv & \psi^{\mathsf{I}}; \varphi^{\mathsf{I}} \\
(\mathbf{wp}(\pi, \varphi))^{\mathsf{I}} & \equiv & \pi^{\mathsf{I}}; \varphi^{\mathsf{I}} \\
(\mathbf{nsc}(\varphi, \pi))^{\mathsf{I}} & \equiv & \varphi^{\mathsf{I}}; \pi^{\mathsf{I}}
\end{array}
$$

4.2 Interpretation of CK45 in BDL

To embed CK45 into BDL, I use two mappings:

$$
\begin{array}{lcl}
\cdot^{\mathsf{P}} & : & \mathsf{CK45} \to \mathsf{BDL} \\
\cdot^{\mathsf{F}} & : & \mathsf{CK45} \to \mathsf{BDL}
\end{array}
$$

$$
\begin{array}{lcl}
p^{\mathsf{P}} & = & \mathsf{Id} \wedge p \\
(\neg\varphi)^{\mathsf{P}} & = & \mathsf{Id} \wedge \neg(\varphi^{\mathsf{P}}) \\
(\Box\varphi)^{\mathsf{P}} & = & \mathsf{Id} \wedge \Box(\mathsf{Id} \to \varphi^{\mathsf{P}}) \\
(\varphi \wedge \psi)^{\mathsf{P}} & = & \varphi^{\mathsf{P}} \wedge \psi^{\mathsf{P}} \\
([\varphi]\psi)^{\mathsf{P}} & = & \mathsf{Id} \wedge \Box(\mathsf{Id} \to \varphi^{\mathsf{P}}; \psi^{\mathsf{F}})
\end{array}
$$

$$
\begin{array}{lcl}
p^{\mathsf{F}} & = & p \\
(\neg\varphi)^{\mathsf{F}} & = & \neg(\varphi^{\mathsf{F}}) \\
(\Box\varphi)^{\mathsf{F}} & = & \Box(\mathsf{Id} \to \varphi^{\mathsf{F}}) \\
(\varphi \wedge \psi)^{\mathsf{F}} & = & \varphi^{\mathsf{F}} \wedge \psi^{\mathsf{F}} \\
([\varphi]\psi)^{\mathsf{F}} & = & \varphi^{\mathsf{P}}; \psi^{\mathsf{F}}
\end{array}
$$

Some consequences:

$$
\begin{array}{rcl}
(\Diamond\varphi)^{\mathsf{P}} & = & \mathsf{Id} \wedge \Diamond\varphi^{\mathsf{P}} \\
(\langle\varphi\rangle\psi)^{\mathsf{P}} & = & \mathsf{Id} \wedge \Diamond(\mathsf{Id} \wedge \varphi^{\mathsf{P}}; \psi^{\mathsf{F}}) \\
(\Diamond\varphi)^{\mathsf{F}} & = & \Diamond(\mathsf{Id} \wedge \varphi^{\mathsf{F}}) \\
(\langle\varphi\rangle\psi)^{\mathsf{F}} & = & \varphi^{\mathsf{P}}; \psi^{\mathsf{F}}
\end{array}
$$

$$
\varphi^{\mathsf{P}} \equiv \mathsf{Id} \wedge (\varphi^{\mathsf{s}})^{\mathsf{F}}
$$

$$
\begin{array}{rcl}
[\![\varphi^{\mathsf{P}}]\!]_{\mathsf{BDL}} & = & [\![\varphi]\!]_{\mathsf{CK45}} \\
[\![\varphi^{\mathsf{F}}]\!]_{\mathsf{BDL}} & = & ||\varphi||_{\mathsf{CK45}}
\end{array}
$$

4.3 Interpretation of EK in BDL

How to compare EK with BDL, which has a semantics in terms of Kripke models? Firstly, we observe that the semantics of update formulae involves the modification of the situation. Secondly, the clause for the modal operator suggests an accessibility relation. This relation can be defined globally by $\{(s,t) \mid \forall k(t_0,\ldots,t_k) \in s_{k+1})\}$ (as is done in [2], 4.4.9). But it is also possible to consider each situation as a local model with a local accessibility relation. From this viewpoint, the clause for update formulae reads as a *modification of the accessibility relation* (which is also suggested by CEM). This leads to a semantics definition of EK in terms of Kripke models $K = (W, R, \gamma)$ of the form $Q, w \models \varphi$ where $Q \subseteq W^2$:

$$
\begin{array}{rcl}
Q, w \models \Diamond\varphi & = & \exists w'(wQw' \& R, w' \models \varphi) \\
Q, w \models [\varphi]\psi & = & \{(u,v) \mid uQv \ \& \ R, v \models \varphi\}, w \models \psi
\end{array}
$$

Under this interpretation, we have the equivalence of $[\varphi]\Diamond\psi$ and $\Diamond(\varphi \wedge \psi)$, for both evaluate to

$$
\exists w'(wQw' \ \& \ R, w' \models \varphi \ \& \ R, w' \models \psi)
$$

Closer inspection of the update clause reveals that the change in the relation comes down to restricting its range. With this idea in mind, we can reformulate the interpretation by replacing the relation parameter by a subset parameter $\sigma \subseteq W$,i.e. we put $\sigma, w \models \varphi \ := \ R \cap (W \times \sigma), w \models \varphi$. Now

$$
\begin{array}{rcl}
\sigma, w \models \Diamond\varphi & = & \exists w'(wRw' \ \& \ w' \in \sigma \& W, w' \models \varphi) \\
\sigma, w \models [\varphi]\psi & = & \sigma \cap \{v \mid W, v \models \varphi\}, w \models \psi
\end{array}
$$

or, writing $\sigma[\![\varphi]\!] := \{w \mid \sigma, w \models \varphi\}$:

$$
\begin{array}{rcl}
\sigma[\![\Diamond\varphi]\!] & = & (\sigma \cap W[\![\varphi]\!])\langle R\rangle] \\
\sigma[\![[\varphi]\psi]\!] & = & (\sigma \cap W[\![\varphi]\!])[\![\psi]\!]
\end{array}
$$

This suggests the following embedding of EK into BDL:

$$
.^{\mathsf{E}} \quad : \quad \mathsf{EK} \to \mathsf{BDL}
$$

$$
\begin{array}{rcl}
p^{\mathsf{E}} & = & p \\
(\neg\varphi)^{\mathsf{E}} & = & \neg(\varphi^{\mathsf{E}}) \\
(\varphi \wedge \psi)^{\mathsf{E}} & = & \varphi^{\mathsf{E}} \wedge \psi^{\mathsf{E}} \\
(\Diamond\varphi)^{\mathsf{E}} & = & (\mathsf{Id} \wedge \top; \varphi^{\mathsf{E}}); \Diamond \\
([\varphi]\psi)^{\mathsf{E}} & = & (\mathsf{Id} \wedge \top; \varphi^{\mathsf{E}}); \psi^{\mathsf{E}}
\end{array}
$$

References

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