On the Linear Independence of Truncated Hierarchical Generating Systems

Urška Zore[a], Bert Jüttler[a,*], Jiří Kosinka[b]

[a] *Institute of Applied Geometry, Johannes Kepler University, Linz, Austria*
[b] *Johann Bernoulli Institute, University of Groningen, The Netherlands*

Abstract

Motivated by the necessity to perform adaptive refinement in geometric design and numerical simulation, the construction of hierarchical splines from generating systems spanning nested spaces has been recently studied in several publications. Linear independence can be guaranteed with the help of the local linear independence of the spline basis at each level. The present paper extends this framework in several ways. Firstly, we consider spline functions that are defined on domain manifolds, while the existing constructions are limited to domains that are open subsets of $\mathbb{R}^d$. Secondly, we generalize the approach to generating systems containing functions which are not necessarily non-negative. Thirdly, we present a more general approach to guarantee linear independence and present a refinement algorithm that maintains this property. The three extensions of the framework are then used in several relevant applications: doubly hierarchical B-splines, hierarchical Zwart-Powell elements, and three different types of hierarchical subdivision splines.

Keywords: Multi-level spline space, hierarchical generating system, truncation, doubly hierarchical B-splines, Zwart-Powell elements, hierarchical subdivision splines.

1. Introduction

The powerful framework of Isogeometric Analysis [6] facilitates the exchange of data between various software tools used for geometric design (CAD systems) and for analysis (numerical simulation). The use of B-splines and NURBS not only for modeling but also for analysis offers advantages over traditional finite element functions, such as increased smoothness, faster convergence, improved stability, and, most importantly, it eliminates the need for model (re)meshing.

However, since multivariate spline representations are based on tensor-product constructions, they suffer from two major limitations: local refinement is not possible and only trivial box-like topologies are supported. Various generalizations of tensor-product

*Corresponding author. Tel. +43(0)732 2468 4080
Email addresses: `urska.zore@jku.at` (Urška Zore), `bert.juettler@jku.at` (Bert Jüttler), `j.kosinka@rug.nl` (Jiří Kosinka)

splines, such as T-splines, hierarchical splines, LR splines, and PHT splines, have been introduced in order to facilitate *local adaptivity*.

T-splines [20, 30] are splines defined by local knot vectors. Hierarchical B-splines [35] are obtained by combining selected B-splines from a sequence of nested spline spaces. LR splines [14] are constructed by repeatedly splitting tensor-product B-splines, starting from an initial set defined on a mesh of tensor-product topology. PHT splines [19] are based on the full space of piece-wise polynomial functions on a given T-mesh, which is equipped with a suitable basis.

In the context of this paper, we are particularly interested in truncated hierarchical B-splines (THB-splines) [11, 12]. This spline basis provides various useful properties in the mathematical framework of hierarchical splines, such as partition of unity, strong stability, full approximation power, and efficient implementation [16, 34]. This approach has recently been extended to more general hierarchies of spline spaces [39].

As the domains of tensor-product spline functions are boxes in parameter space, it becomes more challenging to create suitable representations for domains of *general manifold topology*. A typical solution consists in using multi-patch representations with reduced smoothness across their interfaces [17, 31]. This may, however, lead to artifacts in numerical solutions at the interfaces. Moreover, global high-order smoothness is advantageous for solving higher-order problems, such as the biharmonic equation.

The framework of T-splines [32] includes extraordinary vertices (i.e., vertices where other than four patches meet), which are essential for dealing with general topologies, but the mathematical properties of the isogeometric functions in the vicinity of these points are not well understood.

Typically, the presence of extraordinary vertices leads to a reduced order of smoothness, approximation, and convergence rate [24]. Nevertheless, the use of subdivision functions seems to be one of the most promising approaches [2, 5, 13, 15], mainly due to their support for arbitrary manifold topology, built-in refinement relations, and the widespread use of subdivision representations in applications, especially in Computer Graphics [8].

Using trimmed NURBS representations is a different approach, which is adopted by CAD systems. When used in Isogeometric Analysis, these representations require special treatment [29]. Alternatively it is possible to convert them to subdivision surfaces [33].

Among other results, the present paper describes an extension of the THB construction to functions generated by subdivision algorithms, which are defined on domain manifolds (see [26]). Subdivision algorithms create sequences of nested spaces and are therefore suited for invoking the multi-resolution framework, e.g., when performing hierarchical editing [22]. This is in fact analogous to hierarchical B-spline refinement, which was originally formulated by Forsey and Bartels [10]. Their construction was later augmented by introducing a *basis* [18], thereby facilitating its application in adaptive surface reconstruction and Isogeometric Analysis. It is desirable to introduce a similar basis for hierarchical subdivision splines, and we address this issue in our paper.

While preparing an earlier version of this manuscript [40], we became aware of the recent papers [37, 38], which report on a parallel development of another group of authors. These papers focus on numerical simulation with truncated hierarchical Catmull-Clark subdivision functions and provide sufficient conditions for linear independence of the corresponding basis functions.

In contrast, the present paper presents a generalized framework for the construction of hierarchical spline spaces, which extends the existing results [11, 12, 34, 39] in several ways. Firstly, we consider spline functions that are defined on *domain manifolds*, while the existing constructions are limited to domains that are open subsets of $\mathbb{R}^d$. Secondly, we generalize the approach to generating systems (that span the nested spline spaces) containing functions which are *not* required to take only *non-negative values*. Thirdly, we present a less restrictive approach to *guarantee linear independence* than using local linear independence, and we present a refinement algorithm that maintains this property. These theoretical results are then used in five relevant applications: doubly hierarchical B-splines, hierarchical Zwart-Powell elements, and three different types of hierarchical subdivision splines.

The method for guaranteeing linear independence is based on catalogs containing subdomains with linearly independent restricted generating systems. In order to obtain these catalogs we use techniques that were pioneered by Peters and Wu [27] when analyzing linear independence for Catmull-Clark and Loop subdivision, which was done independently from the investigation of hierarchical splines. We formulate these techniques in our general framework, making them applicable to other hierarchical spline constructions as well. These include doubly hierarchical B-splines, hierarchical Zwart-Powell elements, and Butterfly subdivision splines.

The remainder of the paper is organized as follows. Section 2 extends the construction of (truncated) hierarchical generating systems presented in [39] to spaces of functions defined on domain manifolds and to generating systems containing functions that are not necessarily non-negative. Section 3 derives an algorithm to perform adaptive refinement while maintaining linear independence. Five different applications of the general framework are presented in Section 4. These include doubly hierarchical B-splines and hierarchical Zwart-Powell elements, but also three hierarchical subdivision splines: Catmull-Clark, Loop, and Butterfly. Finally we conclude the paper.

2. Hierarchical generating systems

The framework of hierarchical generating systems was established in [39] for nested spaces spanned by systems of non-negative functions in $C(\Omega)$, which are defined on an open subset $\Omega \subset \mathbb{R}^d$. Local linear independence was used to certify their linear independence. We extend this framework in three ways: Firstly, we use more general parameter domains Ω, which can be d-dimensional manifolds. Secondly, we consider functions that possibly have negative values. Thirdly, we ensure linear independence of the hierarchical generating system by sufficient conditions that are substantially weaker than local linear independence.

2.1. Selecting functions from generating systems on manifolds

Recall that a d-dimensional manifold $\mathfrak{M}$ (or d-manifold for short) is a topological space that locally resembles the d-dimensional Euclidean space, i.e., each point has a neighborhood homeomorphic to the Euclidean space of dimension d. More precisely, each point of the manifold has a neighborhood homeomorphic to an open ball in $\mathbb{R}^d$.

Open subsets in $\mathbb{R}^d$ are d-manifolds. Curves and surfaces in $\mathbb{R}^3$ can be equipped with a manifold structure, provided that they do not possess self-intersections or singularities. The notion of manifolds also encompasses more abstract objects, some of which will be described later in Section 4.

Given a d-manifold $\mathfrak{M}$, we consider real-valued functions $f : \mathfrak{M} \to \mathbb{R}$ with domain $\mathfrak{M}$. Due to the topological structure of $\mathfrak{M}$ it is possible to extend the notion of *continuity* to functions on manifolds, and we consider only continuous functions in the remainder of this paper. The *support* of f, denoted by $\operatorname{supp} f$, is the closure of the set of all points in $\mathfrak{M}$ where f is non-zero.

A *generating system* G on a domain manifold $\mathfrak{M}$ is a finite system of continuous functions with domain $\mathfrak{M}$. In contrast to [39], we do not restrict ourselves to non-negative functions. We express G as the column vector

$$G = [\gamma_i]_{i\in\mathcal{I}},$$

where the elements are indexed by a finite set $\mathcal{I}$. The coefficients of the functions in

$$\operatorname{span}(G) = \mathbb{R}^{\mathcal{I}} G = \{\mathbf{c}\, G \mid \mathbf{c} \in \mathbb{R}^{\mathcal{I}}\}$$

will be collected in row vectors $\mathbf{c} = [c_i]^T_{i\in\mathcal{I}}$. Generating systems that satisfy

$$\mathbb{1} G = 1,$$

with $\mathbb{1} = (1, \ldots, 1) \in \mathbb{R}^{\mathcal{I}}$, are said to be *normalized*.

We consider an infinite sequence of generating systems with a refinement property. More precisely, we consider generating systems $G^\ell = [\gamma_i^\ell]_{i\in\mathcal{I}^\ell}$, $\ell = 0, 1, \ldots$, with index sets $\mathcal{I}^\ell$, where the additional index ℓ is called the *level*. We assume that there exist matrices

$$R^{\ell+1} = [r_{ij}^{\ell+1}]_{i\in\mathcal{I}^\ell, j\in\mathcal{I}^{\ell+1}}$$

such that the generating systems satisfy the refinement equation

$$G^\ell = R^{\ell+1} G^{\ell+1}, \quad \gamma_i^\ell = \sum_{j\in\mathcal{I}^{\ell+1}} r_{ij}^{\ell+1} \gamma_j^{\ell+1}, \quad i \in \mathcal{I}^\ell. \tag{1}$$

Each function of G^ℓ is expressed as a linear combination of functions of level $\ell+1$. In order to ensure the property of affine invariance, which is satisfied for normalized generating systems, we assume that the ***columns of the matrices $R^{\ell+1}$ sum to 1***,

$$\sum_{i\in\mathcal{I}^\ell} r_{ij}^{\ell+1} = 1 \quad \forall j \in \mathcal{I}^{\ell+1}. \tag{CS1}$$

In contrast to [39], there are no restrictions on the signs of the matrix elements.

Hierarchical generating systems can now be defined with the help of a *subdomain hierarchy* that contains N levels, where N is a non-negative integer. The subdomain hierarchy is a decreasing sequence $[\mathfrak{M}^\ell]_{\ell=0,\ldots,N-1}$ of N *open subsets* of $\mathfrak{M}$ satisfying

$$\mathfrak{M} = \mathfrak{M}^0 \supseteq \mathfrak{M}^1 \supseteq \cdots \supseteq \mathfrak{M}^{N-1} \supseteq \mathfrak{M}^N = \emptyset, \tag{2}$$

where the auxiliary set $\mathfrak{M}^N = \emptyset$ is introduced to simplify the notation later on. The open subsets $\mathfrak{M}^\ell$ are also d-submanifolds with respect to the topology inherited from $\mathfrak{M}$.

The *differences* between the domain manifold $\mathfrak{M}$ and the subdomains $\mathfrak{M}^\ell$ define the complementary hierarchy

$$\mathfrak{D}^\ell = \mathfrak{M} \setminus \mathfrak{M}^{\ell+1}, \quad \ell = 0, \ldots, N-1,$$

satisfying

$$\mathfrak{D}^0 \subseteq \mathfrak{D}^1 \subseteq \cdots \subseteq \mathfrak{D}^{N-1} = \mathfrak{M}.$$

Note that the difference subdomains are not necessarily manifolds themselves since they may not be open.

The relation between supports and subdomains is used to split each index set $\mathcal{I}^\ell$ into three disjoint subsets,

$$\begin{aligned} \mathcal{A}^\ell &= \{i \mid \operatorname{supp} \gamma_i^\ell \not\subseteq \overline{\mathfrak{M}^\ell}\}, \\ \mathcal{B}^\ell &= \{i \mid \operatorname{supp} \gamma_i^\ell \subseteq \overline{\mathfrak{M}^\ell} \ \wedge \ \operatorname{supp} \gamma_i^\ell \not\subseteq \overline{\mathfrak{M}^{\ell+1}}\}, \\ \mathcal{C}^\ell &= \{i \mid \operatorname{supp} \gamma_i^\ell \subseteq \overline{\mathfrak{M}^{\ell+1}}\}. \end{aligned}$$

In order to define the hierarchical generating system, we select the functions γ_i^ℓ with $i \in \mathcal{B}^\ell$ and collect them in vectors

$$\hat{G}^\ell = [\gamma_i^\ell]_{i \in \mathcal{B}^\ell}.$$

More precisely, $\hat{G}^\ell$ collects all the functions, called *selected functions*, from G^ℓ whose support is contained in the closure of the level ℓ subdomain $\overline{\mathfrak{M}^\ell}$ but not in $\overline{\mathfrak{M}^{\ell+1}}$.

The *hierarchical generating system* is obtained by concatenating all these vectors $\hat{G}^\ell$,

$$K = \left[\hat{G}^\ell\right]_{\ell=0,\ldots,N-1} = \left[\gamma_i^\ell\right]_{i \in \mathcal{B}^\ell, \ell=0,\ldots,N-1}. \tag{3}$$

By choosing the symbol K we refer to the inventor of this selection mechanism [18].

2.2. Nested hierarchical refinement

In order to be useful in practice, the adaptive refinement using hierarchical generating systems should produce nested spaces. More precisely, if one starts from a given subdomain sequence, then enlarging all subdomains should produce a hierarchical generating system, which spans a space that contains the previous one. This also includes the case of adding a new level of refinement since one may see this as enlarging a previously empty subdomain at the finest level while keeping the remaining subdomains unchanged.

In addition, it is desirable to obtain a generating system that forms a partition of unity since this is an essential property for geometric modeling. The hierarchical generating system K does not possess this property. We introduce an assumption on the domain hierarchy that ensures nested refinement and allows us to invoke the truncation mechanism [11] for restoring the normalization.

A function γ_i^ℓ is said to *refine to* $\gamma_j^{\ell+1}$ if the corresponding entry $r_{ij}^{\ell+1}$ in the refinement matrix $R^{\ell+1}$ is non-zero. Note that the refinement matrix is not necessarily unique since we do not assume linear independence of the generating systems. In the case of non-unique refinement matrices, this notion refers to a certain fixed choice.

We assume that the domain hierarchy satisfies the condition of ***s**elected **f**unction **r**eplacement*: If the closure of the subdomain $\mathfrak{M}^{\ell+1}$ contains the support of a function of level ℓ, then it also contains the supports of all the functions of the next level which this function refines to,

$$\forall i \in \mathcal{C}^\ell \, \forall j \in \mathcal{I}^{\ell+1} \colon r_{ij}^{\ell+1} \neq 0 \;\Rightarrow\; j \in \mathcal{B}^{\ell+1} \cup \mathcal{C}^{\ell+1}. \tag{SFR}$$

Consequently, if a function of level ℓ is not selected for inclusion into the hierarchical generating system, even though its support is contained in $\overline{\mathfrak{M}^\ell}$, then it can be represented as a linear combination of functions that are selected at higher levels, thus it can be *replaced* by them. Note that this assumption is always satisfied for non-negative generating systems and refinement matrices with only non-negative entries, as considered in [39].

If SFR is satisfied for all levels, the hierarchical refinement creates *nested hierarchical spaces* as described in the following result, which generalizes [35, Proposition 4] and [39, Proposition 15]:

Proposition 1. *Any subdomain hierarchy $[\mathfrak{M}_+^\ell]_{\ell=0,\dots,N-1}$ which consists of supersets of the subdomains $[\mathfrak{M}^\ell]_{\ell=1,\dots,N-1}$ results in a hierarchical generating system that spans a superspace of the previous one, provided that SFR is satisfied for both subdomain hierarchies.*

Proof. Let $[\mathfrak{M}_+^\ell]_{\ell=0,\dots,N-1}$ denote the sequence of enlarged subdomains,

$$\mathfrak{M} = \mathfrak{M}^0 = \mathfrak{M}_+^0 \quad \text{and} \quad \mathfrak{M}^\ell \subseteq \mathfrak{M}_+^\ell, \quad \ell = 1, \dots, N-1,$$

$\mathcal{A}_+^\ell$, $\mathcal{B}_+^\ell$, $\mathcal{C}_+^\ell$ the associated disjoint subsets of the index set $\mathcal{I}^\ell$, and $\hat{G}_+^\ell$ the corresponding sub-vectors of selected functions. Moreover, let K_+ denote the hierarchical generating system which is defined by the sequence of the enlarged subdomains. We use mathematical induction with respect to decreasing levels $\ell = N-1, \dots, 0$ to show that

$$\gamma_i^\ell \in \operatorname{span}([\hat{G}_+^k]_{k=\ell,\dots,N-1}) \quad \text{for all} \quad i \in \mathcal{B}^\ell \cup \mathcal{C}^\ell. \tag{4}$$

Firstly we consider the highest level $\ell = N-1$ and use $\mathfrak{M}^N = \mathfrak{M}_+^N = \emptyset$ and $\mathcal{C}^{N-1} = \emptyset$ to confirm (4) by noting that for all $i \in \mathcal{B}^{N-1} \cup \mathcal{C}^{N-1}$ we obtain $i \in \mathcal{B}_+^{N-1}$ since these functions satisfy $\operatorname{supp} \gamma_i^{N-1} \subseteq \overline{\mathfrak{M}^{N-1}} \subseteq \overline{\mathfrak{M}_+^{N-1}}$.

Secondly, we suppose that (4) is satisfied for all levels $\ell+1, \dots, N-1$ and consider the next lower level ℓ. We consider an index $i \in \mathcal{B}^\ell \cup \mathcal{C}^\ell$. Eq. (4) is obviously satisfied if $i \in \mathcal{B}_+^\ell$. Otherwise we have

$$i \in (\mathcal{B}^\ell \cup \mathcal{C}^\ell) \setminus \mathcal{B}_+^\ell \subseteq (\mathcal{B}_+^\ell \cup \mathcal{C}_+^\ell) \setminus \mathcal{B}_+^\ell = \mathcal{C}_+^\ell,$$

thus SFR implies $j \in \mathcal{B}_+^{\ell+1} \cup \mathcal{C}_+^{\ell+1}$, and hence

$$\gamma_j^{\ell+1} \in \text{span}([\hat{G}_+^k]_{k=\ell+1,\ldots,N-1})$$

by the induction hypothesis, if $r_{ij}^{\ell+1} \neq 0$. Considering the representation of γ_i^ℓ at level $\ell+1$,

$$\gamma_i^\ell = \sum_j r_{ij}^{\ell+1} \gamma_j^{\ell+1}, \tag{5}$$

and noting that

$$\text{span}([\hat{G}_+^k]_{k=\ell+1,\ldots,N-1}) \subseteq \text{span}([\hat{G}_+^k]_{k=\ell,\ldots,N-1})$$

confirms (4) also in this situation. This completes the proof of (4) for all levels.

Finally we note that (4) applies to functions with $i \in \mathcal{B}^\ell$, i.e., to the selected functions $\hat{G}^\ell$ of all levels. □

2.3. Truncation and preservation of coefficients

We use the three index sets $\mathcal{A}^\ell$, $\mathcal{B}^\ell$, $\mathcal{C}^\ell$ to decompose the coefficient vectors of level ℓ into the sum of three vectors

$$\mathbf{c}^\ell = \mathbf{c}_\mathcal{A}^\ell + \mathbf{c}_\mathcal{B}^\ell + \mathbf{c}_\mathcal{C}^\ell, \tag{6}$$

where the elements of the three vectors $\mathbf{c}_\mathcal{X}^\ell = [c_{\mathcal{X},i}^\ell]_{i \in \mathcal{I}^\ell}$ are defined by

$$c_{\mathcal{X},i}^\ell = \begin{cases} c_i^\ell \text{ if } i \in \mathcal{X}^\ell \\ 0 \text{ otherwise} \end{cases} \qquad \mathcal{X} = \mathcal{A}, \mathcal{B}, \mathcal{C}.$$

An analogous decomposition is performed for the refinement matrices,

$$R^{\ell+1} = \sum_{\mathcal{X},\mathcal{Y}=\mathcal{A},\mathcal{B},\mathcal{C}} R_{\mathcal{X}\mathcal{Y}}^{\ell+1}, \tag{7}$$

where we obtain matrices $R_{\mathcal{X}\mathcal{Y}}^{\ell+1} = [r_{\mathcal{X}\mathcal{Y},ij}^{\ell+1}]_{i \in \mathcal{I}^\ell, j \in \mathcal{I}^{\ell+1}}$ with elements

$$r_{\mathcal{X}\mathcal{Y},ij}^{\ell+1} = \begin{cases} r_{ij}^{\ell+1} \text{ if } i \in \mathcal{X}^\ell \text{ and } j \in \mathcal{Y}^{\ell+1} \\ 0 \text{ otherwise} \end{cases} \qquad \mathcal{X}, \mathcal{Y} = \mathcal{A}, \mathcal{B}, \mathcal{C}.$$

SFR is equivalent to the fact that one among the nine matrices defined in (7) is simply the null matrix, namely

$$R_{\mathcal{C}\mathcal{A}}^{\ell+1} = [0]_{i \in \mathcal{I}^\ell, j \in \mathcal{I}^{\ell+1}}. \tag{SFR'}$$

This assumption enables us to restore partition of unity by a truncation mechanism, which combines the refinement with the elimination of contributions from selected functions of higher levels. More precisely, the *truncation of a coefficient vector of level* $\ell-1$ with respect to level ℓ is defined by

$$\text{trunc}^\ell(\mathbf{c}^{\ell-1}) = \mathbf{c}^{\ell-1}(R_{\mathcal{A}\mathcal{A}}^\ell + R_{\mathcal{B}\mathcal{A}}^\ell).$$

From this definition we obtain immediately

$$\mathrm{trunc}^\ell(\mathbf{c}^{\ell-1})G^\ell = f - \mathbf{c}^{\ell-1}(R^\ell_{\mathcal{AB}} + R^\ell_{\mathcal{BB}} + R^\ell_{\mathcal{CB}} + R^\ell_{\mathcal{AC}} + R^\ell_{\mathcal{BC}} + R^\ell_{\mathcal{CC}})G^\ell \tag{8}$$

if $f = \mathbf{c}^{\ell-1}G^{\ell-1}$. We define the *truncated hierarchical generating system*

$$T = [\tau_i^\ell]_{i\in\mathcal{B}^\ell,\,\ell=0,\dots,N-1} \tag{9}$$

by applying truncation repeatedly to the selected functions,

$$\tau_i^\ell = \mathrm{trunc}^{N-1}(\cdots \mathrm{trunc}^{\ell+1}([\delta_{ij}]^T_{j\in\mathcal{I}^\ell})\cdots)G^{N-1}, \quad i \in \mathcal{B}^\ell,\ \ell = 0,\dots,N-1, \tag{10}$$

where the Kronecker delta δ_{ij} is used to define the characteristic coefficient vector $[\delta_{ij}]^T_{j\in\mathcal{I}^\ell}$ which corresponds to the selected function γ_i^ℓ of level ℓ. Note that the truncation does not affect the values on difference subdomains of lower levels, i.e.,

$$\tau_i^\ell|_{\mathfrak{D}^\ell} = \gamma_i^\ell|_{\mathfrak{D}^\ell} \text{ for } i \in \mathcal{B}^\ell,\ \ell = 0,\dots,N-1. \tag{11}$$

Consider a function that has representations on all difference subdomains

$$f|_{\mathfrak{D}^\ell} = (\mathbf{c}^\ell_{\mathcal{A}} + \mathbf{c}^\ell_{\mathcal{B}})G^\ell|_{\mathfrak{D}^\ell}, \quad \ell = 0,\dots,N-1. \tag{12}$$

We assume ***c**ompatibility of the **r**epresentations at different **l**evels*,

$$\mathbf{c}^\ell_{\mathcal{A}} = (\mathbf{c}^{\ell-1}_{\mathcal{A}} + \mathbf{c}^{\ell-1}_{\mathcal{B}})(R^\ell_{\mathcal{AA}} + R^\ell_{\mathcal{BA}}), \quad \ell = 1,\dots,N-1, \tag{CRL}$$

and use it to define the *hierarchical spline space with compatible representations*

$$\mathbb{H} = \{f \in C(\mathfrak{M}) \mid\ \exists[\mathbf{c}^\ell]_{\ell=0,\dots,N-1} \text{ such that Eq. (12) and CRL are satisfied}\}.$$

It consists of functions f with the property that their restriction to each difference subdomain can be represented with respect to the associated generating system. In addition, we require CRL, which is automatically satisfied for linearly independent generating systems. Indeed, the decomposition of the coefficient vectors (6) and refinement matrices (7) gives unique representations

$$\mathbf{c}^\ell_{\mathcal{A}} = (\mathbf{c}^{\ell-1}_{\mathcal{A}} + \mathbf{c}^{\ell-1}_{\mathcal{B}} + \mathbf{c}^{\ell-1}_{\mathcal{C}})(R^\ell_{\mathcal{AA}} + R^\ell_{\mathcal{BA}} + R^\ell_{\mathcal{CA}})$$

if the generating systems are linearly independent, while using SFR leads to CRL.

CRL is essential for proving the following characterization result, which generalizes Theorem 2 of [34]:

Proposition 2 (Preservation of coefficients)**.** *Any function $f \in \mathbb{H}$ possesses a representation*

$$f = \sum_{\ell=0}^{N-1}\sum_{i\in\mathcal{B}^\ell} c_i^\ell \tau_i^\ell \tag{13}$$

with respect to the truncated hierarchical generating system T that preserves the coefficients $[c_i^\ell]_{i\in\mathcal{B}^\ell}$ of the local representations (12) *if SFR is satisfied.*

We prepare the proof by deriving an auxiliary result:

Lemma 3. *Given a function* $f \in \mathbb{H}$ *with representations* (12), *we define a sequence of coefficient vectors* $[\mathbf{d}^\ell]_{\ell=0,\ldots,N-1}$ *by*

$$\mathbf{d}^\ell = \begin{cases} \mathbf{c}^0_\mathcal{B}, & \textit{if} \quad \ell = 0 \\ \operatorname{trunc}^\ell(\mathbf{d}^{\ell-1}) + \mathbf{c}^\ell_\mathcal{B}, & \textit{otherwise.} \end{cases} \tag{14}$$

These vectors satisfy

$$\mathbf{d}^\ell = \mathbf{c}^\ell_\mathcal{A} + \mathbf{c}^\ell_\mathcal{B} \quad \textit{for all} \quad \ell = 0, \ldots, N-1. \tag{15}$$

Proof of Lemma 3. We prove it by induction. The hypothesis is true for $\ell = 0$ as $\mathcal{A}^0 = \emptyset$. We assume that the assumption is satisfied for level $\ell - 1$. We thus obtain

$$\operatorname{trunc}^\ell(\mathbf{d}^{\ell-1}) = (\mathbf{c}^{\ell-1}_\mathcal{A} + \mathbf{c}^{\ell-1}_\mathcal{B})(R^\ell_{\mathcal{A}\mathcal{A}} + R^\ell_{\mathcal{B}\mathcal{A}}) = \mathbf{c}^\ell_\mathcal{A},$$

where the second equality is simply CRL. Using the recursion (14) immediately confirms that (15) is valid for level ℓ as well. □

Now we are ready to prove the preservation of coefficients:

Proof of Proposition 2. First we combine $\mathfrak{D}^{N-1} = \mathfrak{M}$ with equations (12) and (15) to obtain $f = \mathbf{d}^{N-1} G^{N-1}$. Using (14) repeatedly leads to

$$f = \mathbf{d}^{N-1} G^{N-1} = \sum_{\ell=0}^{N-1} \operatorname{trunc}^{N-1}(\cdots \operatorname{trunc}^{\ell+1}(\mathbf{c}^\ell_\mathcal{B}) \cdots) G^{N-1}.$$

We note that truncation is a linear operator and obtain

$$\operatorname{trunc}^{N-1}(\cdots \operatorname{trunc}^{\ell+1}(\mathbf{c}^\ell_\mathcal{B}) \cdots) = \sum_{i \in \mathcal{B}^\ell} c^\ell_i \operatorname{trunc}^{N-1}(\cdots \operatorname{trunc}^{\ell+1}([\delta_{ij}]^T_{j \in \mathcal{I}^\ell}) \cdots)$$

since $\mathbf{c}^\ell_\mathcal{B} = \sum_{i \in \mathcal{B}^\ell} c^\ell_i [\delta_{ij}]^T_{j \in \mathcal{I}^\ell}$. The representation (13) can be established by combining these two observations and comparing the result with the definition (10) of the truncated hierarchical generating system. □

We complete the analysis by showing that $\mathbb{H} = \operatorname{span}(T) = \operatorname{span}(K)$.

Proposition 4. *The hierarchical generating system* K *and the truncated hierarchical generating system* T *span the hierarchical spline space with compatible representations* $\mathbb{H}$ *if SFR is satisfied.*

Proof. Proposition 2 implies that $\mathbb{H} \subseteq \text{span}(T)$. Next, observe that $\text{span}(T) \subseteq \text{span}(K)$. Indeed, each function τ_i^ℓ in T is obtained from the corresponding function γ_i^ℓ in K by subtracting certain linear combinations of selected functions from higher levels, cf. (8) and (10). We complete the proof by showing that $\text{span}(K) \subseteq \mathbb{H}$:

Consider a selected function γ_i^ℓ, $i \in \mathcal{B}^\ell$. Its restrictions $\gamma_i^\ell|_{\mathfrak{D}^k}$ possess representations (12) with coefficients

$$\mathbf{c}^k = \begin{cases} [0]_{i\in\mathcal{I}^k}^T & \text{if } k < \ell, \\ [\delta_{ij}]_{j\in\mathcal{I}^\ell}^T & \text{if } k = \ell, \\ [\delta_{ij}]_{j\in\mathcal{I}^\ell}^T R^{\ell+1}\cdots R^k & \text{if } k > \ell. \end{cases}$$

This is obvious for $k < \ell$ since the restriction is then simply the null function. It is clearly also true for $k = \ell$ since $i \in \mathcal{B}^\ell$. The representations for $k > \ell$ are obtained with the help of the refinement equation (1).

CRL is satisfied for $k \le \ell$ with coefficient vectors $\mathbf{c}_\mathcal{A}^k = \mathbf{0}$ and $\mathbf{c}_\mathcal{A}^{k-1} = \mathbf{c}_\mathcal{B}^{k-1} = \mathbf{0}$. It is also satisfied for $k > \ell$ since the refinement equation (1) gives $\mathbf{c}^k = \mathbf{c}^{k-1}R^k$, which implies

$$\mathbf{c}_\mathcal{A}^k = (\mathbf{c}_\mathcal{A}^{k-1} + \mathbf{c}_\mathcal{B}^{k-1} + \mathbf{c}_\mathcal{C}^{k-1})(R_{\mathcal{A}\mathcal{A}}^k + R_{\mathcal{B}\mathcal{A}}^k + R_{\mathcal{C}\mathcal{A}}^k) = (\mathbf{c}_\mathcal{A}^{k-1} + \mathbf{c}_\mathcal{B}^{k-1})(R_{\mathcal{A}\mathcal{A}}^k + R_{\mathcal{B}\mathcal{A}}^k)$$

according to SFR'. Consequently we have $\gamma_i^\ell \in \mathbb{H}$ for all $i \in \mathcal{B}^\ell$ and for all levels ℓ, and thus $\text{span}(K) \subseteq \mathbb{H}$. □

Finally we identify assumptions that guarantee that the truncated hierarchical generating system is normalized.

Corollary 5. *The truncated hierarchical generating system T forms a partition of unity if all generating systems G^ℓ are normalized and CS1 is satisfied.*

Proof. The assumptions of the corollary guarantee that $1 \in \mathbb{H}$ has compatible representations at all levels, where all coefficients are equal to 1. The preservation of coefficients (Proposition 2) implies that the truncated hierarchical generating system sums to 1. □

3. Linear independence

It has been shown that *local linear independence* of the generating systems G^ℓ on $\Omega \subset \mathbb{R}^d$ is a sufficient condition for the linear independence of the hierarchical generating system [11, 39]. This observation can be extended to functions defined on domain manifolds. Local linear independence, however, is a relatively strong condition, which is not satisfied in certain important applications, such as functions defined by subdivision algorithms. The analysis of weaker sufficient conditions is therefore of interest.

3.1. Linear independence on subsets

We consider the restrictions of the generating systems to certain subsets. Given a subset $\mathfrak{S} \subseteq \mathfrak{M}$, we consider local representations of functions,

$$f|_\mathfrak{S} = \mathbf{c}^\ell\{\mathfrak{S}\}\, G^\ell\{\mathfrak{S}\}.$$

The column vector

$$G^\ell\{\mathfrak{S}\} = [\gamma_i^\ell|_{\mathfrak{S}}]_{i\in\mathcal{I}^\ell,\ \mathrm{int}(\mathrm{supp}\,\gamma_i^\ell)\cap\mathfrak{S}\neq\emptyset}$$

is the sub-vector of G^ℓ which consists of the functions that take non-zero values on $\mathfrak{S}$, restricted to $\mathfrak{S}$. The row vector $\mathbf{c}^\ell\{\mathfrak{S}\}$, which is a sub-vector of $\mathbf{c}^\ell$, contains the coefficients of these functions.

We will also use the local refinement matrices $R^{\ell+1}\{\mathfrak{S}\}$, which are the sub-matrices of the full refinement matrix that express the relationship between the restricted generating systems

$$G^\ell\{\mathfrak{S}\} = R^{\ell+1}\{\mathfrak{S}\}\, G^{\ell+1}\{\mathfrak{S}\}.$$

We use the symbol $\#G^\ell\{\mathfrak{S}\}$ to denote the dimension of the vector $G^\ell\{\mathfrak{S}\}$. For later use we formalize two simple facts about linear independence of functions on subsets.

Lemma 6. *Let G^ℓ be linearly independent on a subset $\mathfrak{S}$ of $\mathfrak{M}$ and let $\mathfrak{S}'$ and $\mathfrak{S}''$ be subsets of $\mathfrak{M}$.*

(i) G^ℓ is linearly independent on $\mathfrak{S}\cup\mathfrak{S}'$ if G^ℓ is linearly independent on $\mathfrak{S}'$.

(ii) G^ℓ is linearly independent on $\mathfrak{S}''$ if $\mathfrak{S}''\supseteq\mathfrak{S}$ and $\#G^\ell\{\mathfrak{S}''\} = \#G^\ell\{\mathfrak{S}\}$.

The linear independence on $\mathfrak{S}''$ is valid even if $\#G^\ell\{\mathfrak{S}''\} \leq \#G^\ell\{\mathfrak{S}\}+1$. However, we will use the weaker form (ii) of the statement, as follows. We define the G^ℓ-*closure*

$$\langle\mathfrak{S}\rangle_\ell = \overline{\mathfrak{M}\setminus\bigcup_{\substack{\gamma\in G^\ell\\ \gamma|_{\mathfrak{S}}=0}}\mathrm{supp}\,\gamma} \tag{16}$$

of a subset $\mathfrak{S}\subseteq\mathfrak{M}$ as the largest subset $\langle\mathfrak{S}\rangle_\ell$ with the property that the vectors $G^\ell\{\mathfrak{S}\}$ and $G^\ell\{\langle\mathfrak{S}\rangle_\ell\}$ are identical. According to part (ii) of Lemma 6, G^ℓ is linearly independent on any subset $\mathfrak{S}''$ satisfying

$$\mathfrak{S}\subseteq\mathfrak{S}''\subseteq\langle\mathfrak{S}\rangle_\ell \tag{17}$$

if it is linearly independent on $\mathfrak{S}$.

We present a simple observation, formulated as a lemma, which helps to analyze whether G^ℓ is linearly independent on a chosen subdomain $\mathfrak{S}$. It summarizes some of the arguments used in [27] to analyze linear independence of Catmull-Clark and Loop subdivision blending functions.

Lemma 7. *G^ℓ is linearly dependent on $\mathfrak{S}$ if the matrix $R\{\mathfrak{S}\}^{\ell+1}\cdots R\{\mathfrak{S}\}^k$ does not have full row rank for some $k\geq\ell$. Otherwise, it is linearly independent on $\mathfrak{S}$ if additionally G^k is linearly independent on $\mathfrak{S}$.*

Proof. Consider a linear combination of $G^\ell\{\mathfrak{S}\}$ representing the null function,

$$0 = \mathbf{c}^\ell\{\mathfrak{S}\}\, G^\ell\{\mathfrak{S}\} = \mathbf{c}^\ell\{\mathfrak{S}\}\, R\{\mathfrak{S}\}^{\ell+1}\cdots R\{\mathfrak{S}\}^k\, G^k\{\mathfrak{S}\}. \tag{18}$$

There exists a non-trivial vector $\mathbf{c}^\ell\{\mathfrak{S}\}$ in $\ker R\{\mathfrak{S}\}^{\ell+1}\cdots R\{\mathfrak{S}\}^k$ if this matrix does not have full row rank. Otherwise one may conclude that $\mathbf{c}^\ell\{\mathfrak{S}\}$ is the row null vector if $G^k\{\mathfrak{S}\}$ is linearly independent on $\mathfrak{S}$. □

3.2. Ensuring linear independence

Generalizing earlier results from [18] and [11], we formulate a sufficient condition for linear independence of hierarchical generating systems:

Theorem 8. *The functions in K or T form a basis for $\mathbb{H}$ if G^ℓ is linearly independent on $\mathfrak{D}^\ell$ for all $\ell = 0, \ldots, N-1$.*

Proof. We need to show that

$$\mathbf{b}\, K = 0 \ \text{ implies } \ \mathbf{b} = \mathbf{0}, \tag{19}$$

where K is understood as a single vector, $\mathbf{b}$ is a row vector of coefficients and $\mathbf{0}$ is a row null vector of the same dimension. We decompose and rearrange the left-hand side of (19) according to the hierarchy of the generating systems,

$$\mathbf{b}^0 \hat{G}^0 \ + \ \ldots \ + \ \mathbf{b}^{N-1} \hat{G}^{N-1} = 0. \tag{20}$$

The vector $\mathbf{b}^\ell$ collects the coefficients of functions in $\hat{G}^\ell$.

The functions in the first term in (20) are the only non-zero functions acting on the difference subdomain $\mathfrak{D}^0$. By assumption, $\hat{G}^0$ is linearly independent on $\mathfrak{D}^0$. It follows that $\mathbf{b}^0 = \mathbf{0}$. In the remaining sum only the functions in the first term $\mathbf{b}^1 \hat{G}^1$ are non-zero on the difference subdomain $\mathfrak{D}^1$. Consequently the above argument can be used repeatedly, eventually exhausting all the terms in (20). Moreover, since truncation does not change the values of the functions on the corresponding difference subdomains (11), this proof applies to the truncated hierarchical generating system T as well. □

For each level ℓ we consider a *catalog* $\mathbf{C}^\ell$, which consists of (some) subdomains $\mathfrak{S} \subseteq \mathfrak{M}$ with the property that the restricted generating system $G^\ell\{\mathfrak{S}\}$ is linearly independent. For many concrete applications, the generating systems G^ℓ are known to be linearly independent on certain subdomains. These subdomains form the initial catalogs. The catalogs can also be enriched using Lemma 7. The construction of the catalogs will be exemplified in the next section.

Once these catalogs are available, Lemma 6 and Theorem 8 lead to a *refinement algorithm* that maintains linear independence of (truncated) hierarchical generating systems created by adaptive refinement:

Algorithm (**r**efinement guaranteeing **l**inear **i**ndependence: ARLI)**.**

`INPUT:` Subdomain hierarchy $[\mathfrak{M}^\ell]_{\ell=0,\ldots,N-1}$ and open subset $\mathfrak{R}$ marked for refinement

$\mathfrak{M}_+^N = \emptyset$

`FOR` ℓ `FROM` $N-2$ `DOWNTO` 0 `DO`

$$\mathfrak{A}^{\ell+1} = \mathfrak{M}^{\ell+1} \cup \mathfrak{M}_+^{\ell+2} \cup (\mathfrak{M}^\ell \cap \mathfrak{R}) \tag{i}$$

$$\mathfrak{d}_+^\ell = \bigcup \{\mathfrak{S} \in \mathbf{C}^\ell \mid \mathfrak{S} \subseteq \mathfrak{M} \setminus \mathfrak{A}^{\ell+1}\} \tag{ii}$$

$$\mathfrak{D}_+^\ell = \langle \mathfrak{d}_+^\ell \rangle_\ell \cap (\mathfrak{M} \setminus \mathfrak{A}^{\ell+1}) \tag{iii}$$

$$\mathfrak{M}_+^{\ell+1} = \mathfrak{M} \setminus \mathfrak{D}_+^\ell \tag{iv}$$

```
END DO
```
$\mathfrak{M}^0_+ = \mathfrak{M}$

`RETURN`: New subdomain hierarchy $[\mathfrak{M}^\ell_+]_{\ell=0,\dots,N-1}$

In step (i) of the algorithm we create a subset $\mathfrak{A}^{\ell+1}$, which is then "enlarged" to $\mathfrak{M}^{\ell+1}_+$ in the following way: We create a set $\mathfrak{D}^\ell_+$ as the largest subset of $\mathfrak{M} \setminus \mathfrak{A}^{\ell+1}$ where G^ℓ is linearly independent. More precisely, in step (ii) we form a set $\mathfrak{d}^\ell_+$ as a union of subsets from the catalog $\mathbf{C}^\ell$. According to Lemma 6, the functions G^ℓ are linearly independent there. In step (iii) we take the G^ℓ-closure of the set created in step (ii) and intersect it with $\mathfrak{M} \setminus \mathfrak{A}^{\ell+1}$. Again, according to Lemma 6, the functions G^ℓ are linearly independent there. Finally we define $\mathfrak{M}^{\ell+1}_+$ in step (iv) as the complement of the newly created $\mathfrak{D}^\ell_+$.

Note that since $\mathfrak{R}$ is open, the set $\mathfrak{A}^{\ell+1}$ is open as well. Moreover, the G^ℓ-closure of $\mathfrak{d}^\ell_+$ intersected with the closed set $\mathfrak{M}\setminus\mathfrak{A}^{\ell+1}$ in step (iii) creates the closed set $\mathfrak{D}^\ell_+$. Consequently, $\mathfrak{M}^{\ell+1}_+$ is again open.

Theorem 9. *The subdomain hierarchy $[\mathfrak{M}^\ell_+]_{\ell=0,\dots,N-1}$ generated by ARLI defines a linearly independent hierarchical generating system K_+, which spans a superspace of the space spanned by K, i.e.,* $\operatorname{span} K \subseteq \operatorname{span} K^+$, *if the generating system G^{N-1} is linearly independent on $\mathfrak{M}$. Moreover, it increases the level of all points that have been marked for refinement, except for points of the maximum level $N-1$.*

Proof. Firstly, let x be a point of level $\ell \le N-1$, i.e., $x \in \mathfrak{M}^\ell$, and suppose it is marked for refinement: $x \in \mathfrak{M}^\ell \cap \mathfrak{R}$. According to step (i) of ARLI, we have $x \in \mathfrak{A}^{\ell+1}$, hence step (iii) gives $x \notin \mathfrak{D}^\ell_+$. We thus conclude that $x \in \mathfrak{M}^{\ell+1}_+$. Note that the level of $x \in \mathfrak{M}^{N-1}$ does not increase, as $\mathfrak{M}^N_+ = \emptyset$.

Secondly, the new subdomains form a nested sequence of open sets, and the first subdomain remains unchanged,

$$\mathfrak{M} = \mathfrak{M}^0_+ \ \text{ and } \ \mathfrak{M}^{\ell+1}_+ \supseteq \mathfrak{M}^{\ell+2}_+, \quad \ell = 0,\dots,N-2.$$

Indeed, $\mathfrak{M}^{\ell+2}_+ \subseteq \mathfrak{A}^{\ell+1}$ by (i) and $\mathfrak{A}^{\ell+1} \subseteq \mathfrak{M}^{\ell+1}_+$ by (iii) and (iv). Note that the algorithm terminates with $\ell = 0$ and assigns $\mathfrak{M}^0_+ = \mathfrak{M} = \mathfrak{M}^0$. As observed before, the new subdomain hierarchy consists of open sets.

Thirdly, combining (i) with (iii) and (iv) confirms $\mathfrak{M}^{\ell+1} \subseteq \mathfrak{A}^{\ell+1} \subseteq \mathfrak{M}^{\ell+1}_+$, thus we can use Proposition 1 to conclude that $\operatorname{span} K \subseteq \operatorname{span} K_+$.

Finally, to prove that the resulting hierarchical generating system K_+ is linearly independent, it suffices to observe that each generating system G^ℓ is linearly independent on the new difference subdomain $\mathfrak{D}^\ell_+ = \mathfrak{M} \setminus \mathfrak{M}^{\ell+1}_+$, see Theorem 8. By construction we obtain $\mathfrak{d}^\ell_+ \subseteq \mathfrak{M} \setminus \mathfrak{A}^{\ell+1}$, hence the new difference subdomain satisfies

$$\mathfrak{d}^\ell_+ \subseteq \mathfrak{D}^\ell_+ \subseteq \langle \mathfrak{d}^\ell_+ \rangle_\ell$$

and the linear independence on the new difference subdomain is guaranteed by (17). □

Examples of applying ARLI will be presented in the following section for various generating systems.

4. Applications

We exemplify the extended framework of hierarchical generating systems by studying five concrete applications. The first two applications (doubly hierarchical B-splines and hierarchical Zwart-Powell elements) consider domain manifolds that are open subsets of $\mathbb{R}^d$, while the three remaining ones (hierarchical Catmull-Clark, Loop and Butterfly subdivision splines) use domains that are *meshes*. The fifth application, the Butterfly scheme, relies on generating systems formed by functions which are not exclusively non-negative.

In all cases, the hierarchical generating systems and truncated hierarchical generating systems possess the properties which were discussed in Section 2, except for the partition of unity property, which is generally only satisfied for the three latter applications (the subdivision splines). Linear independence depends on the choice of the subdomain hierarchies and will be discussed individually.

4.1. Doubly hierarchical B-splines

Doubly hierarchical B-splines are obtained as an instance of the *hierarchy of hierarchies*, which was briefly discussed in [39, p. 555 and Example 20], motivated by the necessity to perform adaptive refinement in the presence of 'features'.

Domain and generating systems. The domain $\mathfrak{M}$ is an open subset of $\mathbb{R}^d$, and the subdomains $\mathfrak{M}^\ell$ form a nested sequence of subsets thereof. The generating systems G^ℓ are hierarchical B-splines [11].

For each level ℓ we consider a sequence of generating systems $[g^{\ell,k}]_{k=0,\ldots,n-1}$, which are simply tensor-product B-splines of degree $\mathbf{p} = [p_i]_{i=1,\ldots,d}$ spanning nested spaces, and an associated subdomain hierarchy $[\mathfrak{m}^{\ell,k}]_{k=0,\ldots,n-1}$. The hierarchical B-splines G^ℓ are defined by using the selection mechanism. They are non-negative and linearly independent. These generating systems span nested spaces

$$\operatorname{span}(G^{\ell-1}) \subseteq \operatorname{span}(G^\ell), \quad \ell = 1, \ldots, N-1,$$

if the hierarchies at all levels ℓ satisfy

$$\mathfrak{m}^{\ell-1,k} \subseteq \mathfrak{m}^{\ell,k} \quad \text{and} \quad \operatorname{span}(g^{\ell-1,k}) \subseteq \operatorname{span}(g^{\ell,k}), \quad k = 0, \ldots, n-1,$$

according to the General Enlargement Theorem [39, Theorem 19]. The same result also implies that the refinement matrices $R^{\ell+1}$ relating the generating systems G^ℓ are non-negative. SFR is thus automatically satisfied since the generating systems G^ℓ are linearly independent and consist of non-negative functions.

The hierarchical generating system K that is defined by the subdomains $\mathfrak{M}^\ell$ and the generating systems G^ℓ is called the system of *doubly hierarchical B-splines*[1].

[1]Note that this is only one of several possibilities to construct generating systems from hierarchies of hierarchical splines. Other possibilities are obtained when considering truncation for each of the individual hierarchies, or for the overall hierarchy. We restrict ourselves to the simplest case.

Linear independence. The linear independence of the doubly hierarchical B-splines K is not automatically guaranteed since the generating systems G^ℓ do not possess the property of local linear independence in general. We may use Theorem 8 and ARLI to construct subdomain hierarchies that generate a linearly independent generating system K. The catalogs $\mathbf{C}^\ell$ contain all elements of the hierarchical grid defined by $[\mathfrak{m}^{\ell,k}]_{k=0,\ldots,n-1}$ that belong to the support of $\prod_{i=1}^{d}(p_i+1)$ B-splines in G^ℓ since this is the dimension of the space of tensor-product polynomials of degree $\mathbf{p}$, which is contained in $\mathrm{span}(G^\ell)$. The catalogs can be enriched with the help of Lemma 7, as shown in the following example.

Example. Consider bivariate doubly hierarchical B-splines of degree $\mathbf{p} = (2, 2)$ defined on $\mathfrak{M} = [0, 4]^2$. We construct them from hierarchical B-splines with two levels ($n = 2$) and subdomains

$$\mathfrak{m}^{\ell,0} = \mathfrak{M} \quad \text{and} \quad \mathfrak{m}^{\ell,1} = [1, 3] \times [2, 4],$$

which are identical for all levels $\ell = 0, \ldots, N-1$.

We use generating systems $g^{\ell,0}$ that consist of biquadratic tensor-product B-splines with knots $2^{-\ell}\mathbb{Z} \times 2^{-\ell}\mathbb{Z}$, see Fig. 1. The generating systems $g^{\ell,1}$ are tensor-product B-splines with the same knots, but with a double knot at $x = 2$. A similar example was used, without analyzing linear independence, in [39, Example 20]. The hierarchical generating systems G^ℓ are linearly independent, and they are even locally linearly independent for all levels $\ell \geq 1$, analogously to [1, Proposition 1]. Thus, the catalogs $\mathbf{C}^\ell$ contain all open subsets of $\mathfrak{M}$ if $\ell \geq 1$.

However, G^0 does not have the property of local linear independence since it contains ten functions that take non-zero values on $[1, 2] \times [2, 3]$, while the dimension of the space of biquadratic polynomials is only 9. By counting the functions that take non-zero values on each of the 12 cells in $\mathfrak{M} \setminus \mathfrak{m}^{0,1}$ we conclude that these cells (and any open subset contained in their union) can be included into the catalog $\mathbf{C}^0$. In addition we use Lemma 7 to include the subsets $[\frac{3}{2}, \frac{5}{2}] \times [\frac{i}{2}, \frac{i+1}{2}]$ for $i = 4, \ldots, 7$, see Fig. 2 (left). Moreover, one may use Lemma 6 to enlarge $\mathbf{C}^0$ further, see Fig. 2 (right).

We use these catalogs for ensuring linear independence by ARLI, see Fig. 3. Applying ARLI to a subdomain hierarchy with two levels and $\mathfrak{M}^1 = \emptyset$ (left) creates the subdomain hierarchy $[\mathfrak{M}^\ell_+]_{\ell=0,1}$ with a linearly independent hierarchical generating system, see Fig. 3, right.

4.2. Hierarchical Zwart-Powell elements

The Zwart-Powell (ZP) element was first introduced in [42], see also [7]. The construction of the hierarchical generating system was analyzed in [39], in particular focusing on its linear dependencies. The question of completeness of the hierarchical generating system of ZP elements was discussed recently in [23].

Domain and generating systems. We consider generating systems which consist of translates of the ZP element, a quadratic C^1-smooth box spline defined on a type-2 (or criss-cross) triangulation in $\mathbb{R}^2$. Such a triangulation is defined by horizontal, vertical and diagonal lines through points with integer coordinates, forming a triangulation with vertices

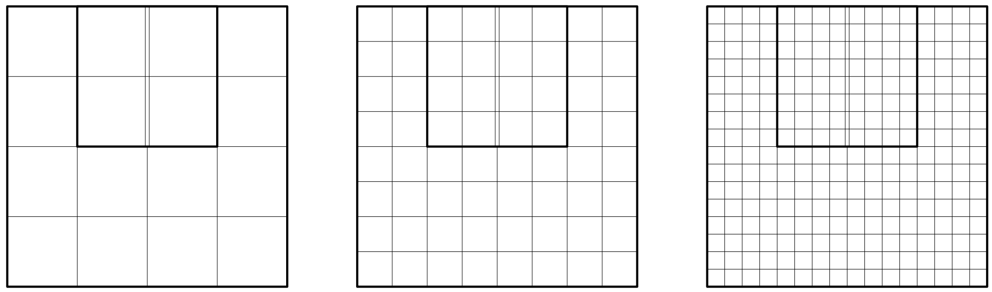


Figure 1: The meshes for $\ell = 0, 1, 2$ (from left to right) which are used to define the doubly hierarchical B-splines in the example in Section 4.1.

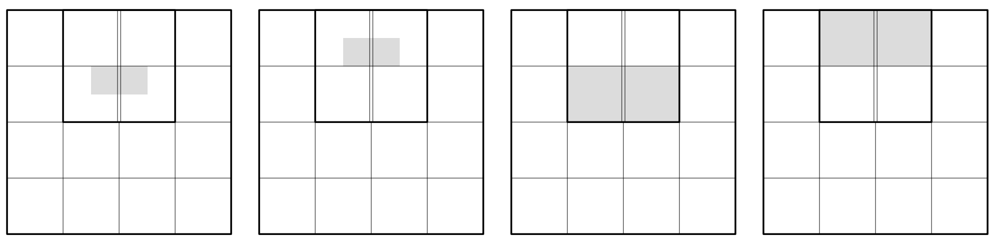


Figure 2: Far left and left: Lemma 7 allows us to include $[\frac{3}{2}, \frac{5}{2}] \times [\frac{i}{2}, \frac{i+1}{2}]$ for $i = 5, 6$ into $\mathbf{C}^0$. Right and far right: We use Lemma 6 to include $[1, 3] \times [j, j+1]$ for $j = 2, 3$.

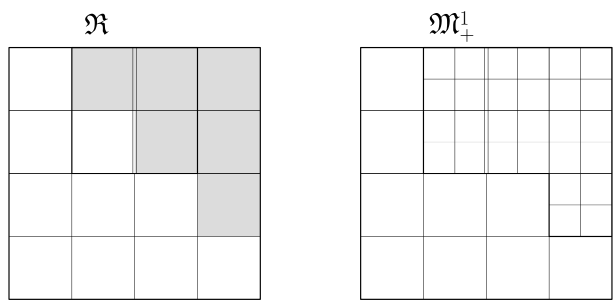


Figure 3: Ensuring linear independence of doubly hierarchical B-splines. Given a region $\mathfrak{R}$ that is marked for refinement (left, gray), the subdomain $\mathfrak{M}^1_+$ is found by ARLI (right).

of valency 4 and 8. Uniform refinement of this grid leads to a sequence of triangulations of levels $\ell = 0, 1, \ldots$, which are defined by the lines

$$x = 2^{-\ell}\mathbb{Z}, \quad y = 2^{-\ell}\mathbb{Z}, \quad x + y = 2^{-\ell}\mathbb{Z}, \quad x - y = 2^{-\ell}\mathbb{Z}.$$

The ZP element is a locally supported function with an octagonal support, which is obtained as the convex hull of the cross formed by five squares, see Fig. 4 (left). The level ℓ translates are centered at the points of the scaled and shifted integer grid with vertices $2^{-\ell}\mathbb{Z}^2 + 2^{-(\ell+1)}(1, 1)$. We consider a domain which is a connected, bounded, and open subset $\mathfrak{M} \subset \mathbb{R}^2$, and the finitely many translates of all levels taking non-zero values on it.

For each level, the system of translates is known to possess one linear dependency relation (also called the chess-board pattern, see Fig. 4, right). The spaces spanned by the

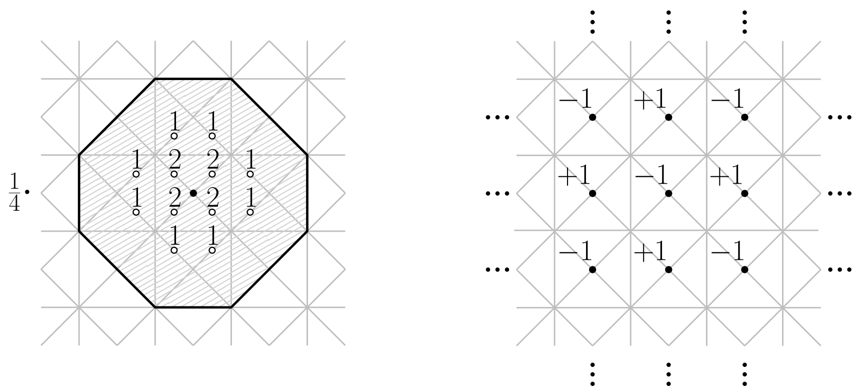


Figure 4: Left: The support of a ZP element defined on a criss-cross triangulation. The center point is denoted by a black dot. The circles denote the functions defined on the finer triangulation with a non-zero coefficient in the representation of the coarse function. Right: The coefficients in the chess-board pattern.

generating systems are nested since each level ℓ translate can be represented as a linear combination of the 12 translates of the next finer level whose supports are contained in the support of the coarser function, using non-negative coefficients, see Fig. 4 (left).

Linear independence is restored by omitting one (arbitrarily chosen) translate at each level, see [36]. The remaining functions of each level ℓ form the generating system G^ℓ.

The (uniquely determined) refinement matrices represent each level ℓ translate as a linear combination of translates of the next level. In most cases, these linear combinations take the form shown in Fig. 4 (left). If one of the level $\ell+1$ translates is the omitted one, however, then the linear combination involves *all* elements of $G^{\ell+1}$ as the omitted function is expressed with the help of the chess-board pattern. Consequently, SFR is not automatically satisfied but requires a careful choice of the subdomain hierarchy.

Linear independence. We consider a nested sequence of subdomains $[\mathfrak{M}^\ell]_{\ell=0,\dots,N-1}$. SFR is satisfied if *the support of the omitted translate of level* $\ell+1$ *is not contained in* $\overline{\mathfrak{M}^{\ell+1}}$ for all levels $\ell = 0,\dots,N-2$. In fact, the 12 functions (see Fig. 4, left) needed to represent γ_i^ℓ, where $i \in \mathcal{C}^\ell$, are then all present in $G^{\ell+1}$ since their supports are contained in $\operatorname{supp}\gamma_i^\ell \subseteq \overline{\mathfrak{M}^{\ell+1}}$.

This condition, however, does not yet suffice to guarantee linear independence of the hierarchical generating system. The catalog $\mathbf{C}^\ell$, which is required to build suitable hierarchies via ARLI, depends on the choice of the omitted translate. The generating system G^ℓ is linearly independent on a connected subset $\mathfrak{S}$ *if this subset possesses a nonempty intersection with the support of the omitted level* ℓ *translate*, see [23, Lemma 5]. The catalog $\mathbf{C}^\ell$ consists of all sets $\mathfrak{S}$ with this property.

Example. We consider a subdomain hierarchy with two levels ($N=2$) and $\mathfrak{M}^1 = \emptyset$, see Fig. 5 (left). We define G^ℓ by omitting the level ℓ translate in the top left corner. For $\ell=0$ and $\ell=1$, the center of its support is shown in Fig. 5, left and right, respectively.

Let $\mathfrak{R}$ be the region marked for refinement (shown in gray in Fig. 5, left). We use ARLI to restore linear independence of the hierarchical generating system. The algorithm

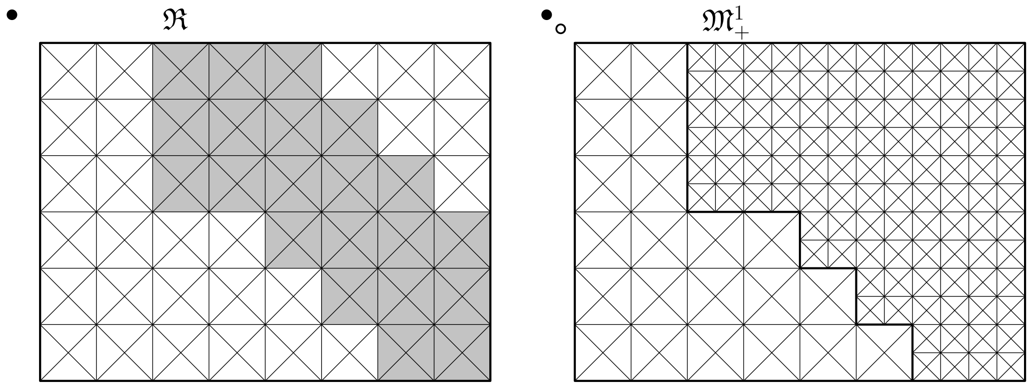


Figure 5: Applying ARLI to maintain linear independence of hierarchical ZP elements. Left: Initial subdomain hierarchy with two levels ($N = 2$) and $\mathfrak{M}^1 = \emptyset$. The region $\mathfrak{R}$ (gray) is marked for refinement. The support center of the omitted level 0 translate is visualized by a black dot. Right: Refined subdomain hierarchy with two levels created by ARLI. The support center of the omitted level 1 translate is visualized by a circle.

creates the new subdomain hierarchy $\mathfrak{M} = \mathfrak{M}^0_+ \supseteq \mathfrak{M}^1_+$, which is shown in Fig. 5 (right). Note that $\mathfrak{M} \setminus \mathfrak{R}$ has two connected components, and the north-east one is not contained in $\mathbf{C}^0$ since it does not intersect the support of the omitted function. Finally we confirm that SFR is satisfied since the support of the omitted function from level 1 (denoted by a circle) is not contained in $\overline{\mathfrak{M}^1_+}$.

4.3. Hierarchical Catmull–Clark subdivision

Catmull–Clark (CC) subdivision [3] (see also the monograph [26] and the references therein) generalizes the construction of bicubic spline surfaces to control meshes of arbitrary topology. It is one of the classical methods for surface design. This subdivision scheme defines refinable generating systems on meshes, which are suitable for defining hierarchical generating systems. Truncated hierarchical CC subdivision was studied recently in [37, 38] and independently in [40].

Domain and generating systems. The domain $\mathfrak{M}$ of CC subdivision splines is a two–dimensional topological manifold $\mathfrak{M}/\!\sim$, which is constructed by gluing together numerous copies of the elementary cell $\square = [0,1]^2$ across edges, see [26]. More precisely, one considers a finite index set $\mathcal{J}$, which defines $\#\mathcal{J}$ copies $\square \times \mathcal{J}$ of the elementary cell, and a symmetric, irreflexive set of edge indices $\mathcal{E} \subset \mathcal{J}^2$. For each edge index $(i,j) \in \mathcal{E}$ we have an associated affine map $\alpha_{ij} : \mathbb{R}^2 \to \mathbb{R}^2$, which is one of the 32 isometries that map the unit square $\square$ to one of its four neighboring squares of size 1. These affine maps identify points on edges via

$$(\mathbf{x}, i) \sim (\alpha_{ij}(\mathbf{x}), j).$$

It is assumed that the identification is consistent (i.e., $\alpha_{ij} = \alpha_{ji}^{-1}$) and that each point in the interior of an edge is identified with exactly one point from another cell.

The number of points a vertex is identified with (i.e., the size of the equivalence class generated by a vertex in $\mathfrak{M}$) is called the *valence* of the vertex. We assume that all valences are larger than 2. Vertices of valence other than four are said to be *extraordinary vertices* (EVs). All cells are assumed to have at most one EV.

For each level $\ell = 0, \ldots, N-1$ we define a subdivision of $\mathfrak{M}$ into cells of level ℓ by applying dyadic subdivision (which splits a square into four smaller squares of equal size) ℓ times to the initial cells $\mathcal{J}$. The generating systems G^ℓ consist of CC blending functions of level ℓ. They are associated to the vertices of level ℓ, and their support consists of the cells in the two-ring neighborhood.

The blending functions satisfy the CC refinement equations taking the form (1), which also relate the coefficients of representations of a function at different levels via

$$f = \mathbf{c}^\ell G^\ell = \underbrace{\mathbf{c}^\ell R^{\ell+1}}_{=\mathbf{c}^{\ell+1}} G^{\ell+1}. \tag{21}$$

The coefficients $c_i^{\ell+1}$ are weighted averages of coefficients c_j^ℓ at neighboring vertices. The weights (i.e., the elements of the refinement matrices) are derived from the subdivision equations of bicubic B-splines, which are complemented by special rules in the vicinity of extraordinary vertices, see [26]. The CC blending functions of level ℓ can be obtained as the limit of the sequence of piecewise bilinear functions (which define the 'control meshes') that interpolate the values c_i^k at the vertices of level k, where these coefficients are determined by the refinement equations (21).

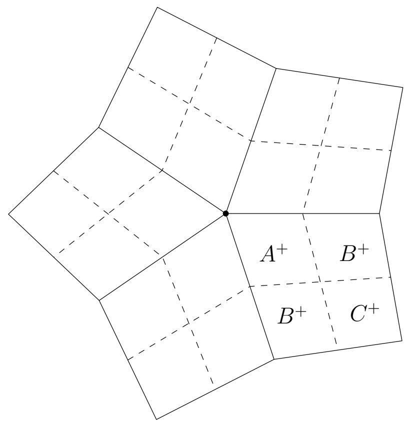
Due to the choice of the weights, the CC blending functions of level ℓ are the non-negative bicubic polynomials on all *regular cells* (i.e., cells without EVs) for all levels $k \geq \ell$. They are simply bicubic B-splines if associated with a vertex that is shared by four regular cells, and consist of an infinite number of non-negative bicubic polynomial segments otherwise.

According to the CC refinement relation, each blending function of level ℓ can be represented as a non-negative linear combination of $1+6\nu$ blending functions of level $\ell+1$, where ν is the valence of the associated vertex, and the supports of the finer functions are contained in the support of the original one. SFR is therefore satisfied for any choice of subdomains $\mathfrak{M}^\ell$.

Linear independence. The CC blending functions of level ℓ are known to be linearly independent on all regular cells of the same level since their restrictions to cells of this type are simply the 16 polynomial segments of bicubic tensor-product B-splines. This observation is the starting point for ensuring linear independence.

We build the catalogs $\mathbf{C}^\ell$, which are required for executing ARLI, by analyzing the linear independence of G^ℓ on cells of levels ℓ and $\ell+1$, using Lemmas 6 and 7. This is similar to the approach taken in [27] for studying linear independence of CC blending functions, and leads to the same results. In contrast to that earlier work, we do not use eigenanalysis of the subdivision matrix; instead we work solely with the observations that were formalized in Lemmas 6 and 7.

The results are summarized in Fig. 6, where we consider cells of level ℓ with one EV, and three different types (depending on their relative location to the EV) of cells of level $\ell+1$. The generating system G^ℓ is linearly independent on cells marked with ✓, and linearly



valence	ℓ	$\ell+1$		
ν		A^+	B^+	C^+
3	✓	×	✓	✓
4 (regular)	✓	✓	✓	✓
5	✓	✓	×	×
6	✓	✓	×	×
7	✓	✓	×	×
8	✓	✓	×	×
9	✓	✓	×	×
10	✓	✓	×	×
11	✓	✓	×	×
12	✓	✓	×	×

Figure 6: Linear independence of CC blending functions of level ℓ on cells of levels ℓ and $\ell+1$ for various values of the valence ν. The case of regular cells is covered by $\nu = 4$.

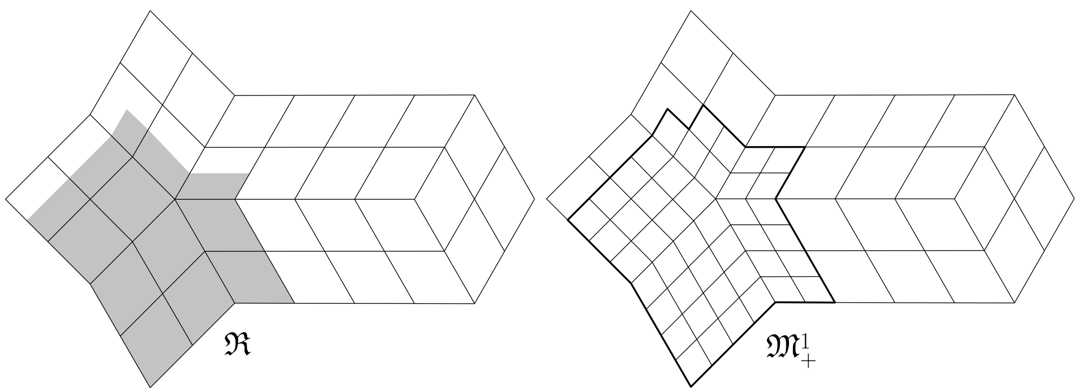


Figure 7: Left: A two-level subdomain hierarchy $\mathfrak{M} = \mathfrak{M}^0 \supseteq \mathfrak{M}^1 = \emptyset$, with a region $\mathfrak{R}$ marked for refinement (shaded in gray). Right: ARLI generates the subdomain $\mathfrak{M}^1_+$ (shown as collection smaller cells).

dependent otherwise. The constructions of Truncated Hierarchical (TH) CC splines in [37, 38] consider cells of level ℓ (i.e., the second column of the table) only.

Example. Consider a two-level subdomain hierarchy ($N = 2$) with $\mathfrak{M}^1 = \emptyset$ and the domain $\mathfrak{M}$, a subset of which is shown in Fig. 7 (left). Notice the presence of EVs of valency 5 and 3. According to the table in Fig. 6, not every subset in the vicinity of the EVs is contained in the catalog $\mathbf{C}^0$, and linear independence needs to be achieved by applying ARLI.

Let $\mathfrak{R}$ be the region marked for refinement (gray-shaded area in Fig. 7, left). Applying ARLI modifies $\mathfrak{R}$ by enlarging it and defines a new subdomain hierarchy $[\mathfrak{M}^\ell_+]_{\ell=0,1}$, with $\mathfrak{M}^1_+$ as shown in Fig. 7 (right). In particular, the 3 cells of level 1 (two of type B^+ and two of type C^+) are added to $\mathfrak{R}$ as they are not contained in the closure of $\mathfrak{d}^0_+$.

4.4. Hierarchical Loop subdivision

Loop subdivision [21, 28] is another popular subdivision scheme for surface design. It operates on triangular (instead of quadrangular) meshes. Since Loop subdivision generates refinable generating systems, defined on two-dimensional domain manifolds, it lends itself to the construction of hierarchical generating systems.

Domain and generating systems. The domain $\mathfrak{M}$ of Loop subdivision splines is a two–dimensional topological manifold $\mathfrak{M}/\sim$, which is constructed in an analogous fashion as in the previous section. There are some subtle differences, however: The cells are copies of an equilateral *triangle* $\triangle$ instead of the unit square. The affine maps associated with the edges are one of the 18 isometries that map $\triangle$ to one of its three neighboring equilateral triangles of the same size. Extraordinary vertices (EVs) have a valence other than *six*.

The generating systems G^ℓ consist of Loop blending functions, which are determined by the refinement matrices. The elements of these matrices are derived from the subdivision equations of the box spline N_{222} of type (2,2,2) on the type-1 triangulation, see [7], which are again complemented by special rules in the vicinity of extraordinary vertices.

The Loop blending functions of level ℓ are non-negative quartic polynomials on all *regular cells* (i.e., cells without EVs) of level $k \geq \ell$. They are the box spline N_{222} if associated with a vertex that is shared by six regular cells, and they consist of an infinite number of non-negative quartic polynomial segments otherwise.

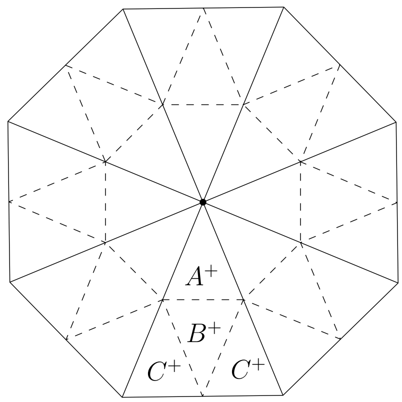
Each blending function of level ℓ can be represented as a non-negative linear combination of $1 + 3\nu$ blending functions of level $\ell + 1$, where ν is the valence of the associated vertex, and the supports of the finer functions are contained in the support of the original one. SFR is again satisfied for any choice of subdomains $\mathfrak{M}^\ell$.

Linear independence. The Loop blending functions of level ℓ are known to be linearly independent on all regular cells of the same level since their restrictions to cells of this type are simply the polynomial segments of the box spline N_{222}. This observation is the starting point for ensuring linear independence.

Once again, we build the catalogs $\mathbf{C}^\ell$, which are required for executing ARLI, by analyzing the linear independence of G^ℓ on cells of levels ℓ and $\ell+1$, using Lemmas 6 and 7. This is similar to the approach taken in [27] for studying linear independence of Loop blending functions, and leads to the same results. Similar to the CC case we work solely with the observations that were formalized in Lemmas 6 and 7 instead of using eigenanalysis.

Fig. 8 summarizes our findings. We consider cells of level ℓ with one EV, and three different types (depending on their relative location to the EV) of cells of level $\ell + 1$. The generating system G^ℓ is linearly independent on cells marked with ✓, and linearly dependent otherwise.

Example. We use the catalog presented in Fig. 8 to perform adaptive refinement with the help of ARLI. We consider a subdomain hierarchy with two levels $\mathfrak{M} = \mathfrak{M}^0 \supseteq \mathfrak{M}^1 = \emptyset$, where $\mathfrak{M}$ is a domain manifold consisting of triangular cells, see Fig. 9 (left). After selecting a region $\mathfrak{R}$ (left, gray) for refinement, the algorithm generates a new subdomain hierarchy $\mathfrak{M} = \mathfrak{M}^0_+ \supseteq \mathfrak{M}^1_+ \supseteq \mathfrak{M}^2_+ = \emptyset$, see Fig. 9 (center). We need to add 8 cells of level 1 to obtain $\mathfrak{M}^1_+$.



valence	ℓ	$\ell+1$		
ν		A^+	B^+	C^+
3	✓	✓	✓	✓
4	✓	✓	✓	✓
5	✓	✓	✓	✓
6 (regular)	✓	✓	✓	✓
7	✓	✓	×	×
8	✓	✓	×	×
9	✓	✓	×	×
10	✓	✓	×	×
11	✓	✓	×	×
12	✓	✓	×	×

Figure 8: Linear independence of Loop blending functions of level ℓ on cells of levels ℓ and $\ell+1$ (types A^+, B^+, C^+) for various values of the valence ν. The case of regular cells is covered by $\nu = 6$.

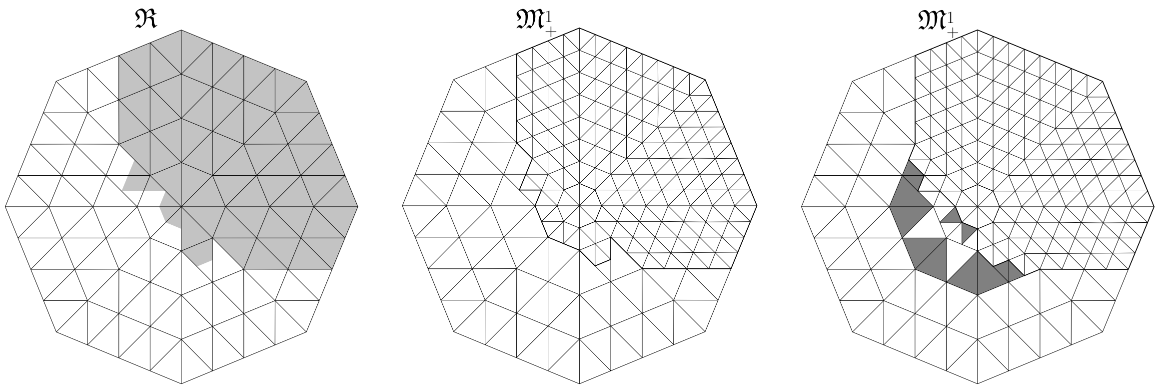


Figure 9: Using ARLI to perform adaptive refinement while maintaining linear independence for Loop (center) and Butterfly (right) blending functions. A region $\mathfrak{R}$, which is marked for refinement (left, gray), is enlarged to $\mathfrak{M}^1_+$ (center and right) by adding certain cells to ensure linear independence. In the case of Butterfly subdivision, some cells (right, dark gray) which do not belong to $\mathbf{C}^0$ individually do not need to be refined since their union with the other cells in $\mathfrak{M} \setminus \mathfrak{M}^1_+$ (shown in white) does.

4.5. Hierarchical Butterfly subdivision

Butterfly subdivision is an interpolatory algorithm for surface design based on triangular meshes. More precisely, the limit surface interpolates the vertices of the given mesh. We consider the modified Butterfly scheme (further referred to simply as the Butterfly scheme), introduced in [41] as an improved version of the original scheme presented in [9]. A non-stationary variant of Butterfly subdivision has appeared recently [25].

Domain and generating systems. The domain $\mathfrak{M}$ of Butterfly subdivision splines is a two–dimensional topological manifold $\mathfrak{M}/\sim$, which is constructed in an analogous fashion as in the previous section. We consider four different types of cells A, B, C and R, see Fig. 10 (left). Extraordinary cells (type A) contain an EV. Regular cells of types B and C share an edge or only a vertex with an extraordinary cell, respectively. The remaining regular cells have type R. We restrict ourselves to domains where only these four types of cells are

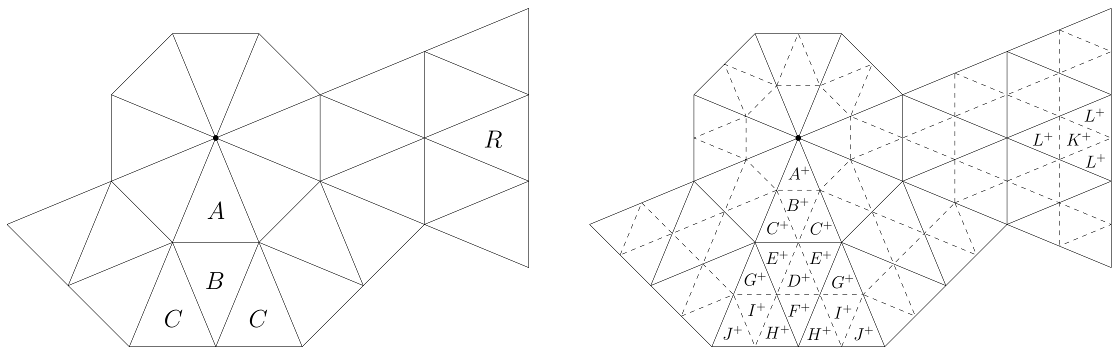


Figure 10: Types A, B, C, R of cells of level ℓ (left) and $A^+, \ldots, L^+$ of level $\ell+1$ (right) for Butterfly subdivision.

present. Consequently, the shortest paths connecting any two EVs contain at least three edges.

The generating systems G^ℓ consist of Butterfly blending functions, which are determined by the refinement matrices. The refinement matrices preserve the coefficients that are associated with existing vertices. Consequently, each Butterfly blending function of level ℓ takes the value 1 at the vertex it corresponds to, and the value 0 at all other vertices.

The support of Butterfly blending functions has a fractal structure. In order to define the hierarchical generating system, one needs to decide whether the support of a Butterfly blending functions is contained in a subdomain or not. We shall assume that the subdomains $\mathfrak{M}^\ell$ consist of cells (triangles) of level ℓ. Consequently, it suffices to identify the triangles of level $\ell+1$ that contribute to the support of a level ℓ Butterfly blending function, see Fig. 11 for an example. Analyzing the Butterfly refinement rules confirms that SFR is satisfied.

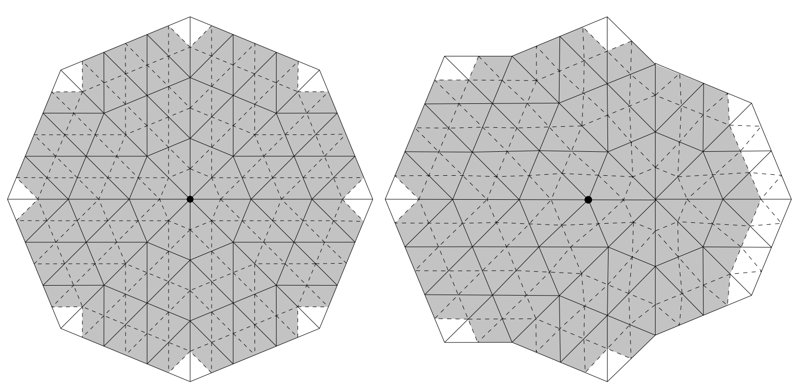


Figure 11: Cells of level $\ell+1$ that contribute to the supports of two Butterfly blending functions of level ℓ in the vicinity of an EV of valency 8. The vertices that correspond to the blending functions are marked by a black dot.

Linear independence. The Butterfly blending functions of level ℓ are linearly independent on any subset of the vertex set of level ℓ, due to the interpolatory nature of this subdivision

valence	ℓ			$\ell+1$									
ν	A	B	C	A^+	B^+	C^+	D^+	E^+	F^+	G^+	H^+	I^+	J^+
3	✓	✓	✓	×	✓	✓	✓	✓	✓	✓	✓	✓	✓
4	✓	✓	✓	×	×	✓	✓	✓	✓	✓	✓	✓	✓
5	✓	✓	✓	×	×	✓	✓	✓	✓	✓	✓	✓	✓
6 (regular)		R : ✓						K^+: ✓	L^+: ✓				
7	✓	✓	✓	×	×	✓	✓	✓	✓	✓	✓	×	×
8	✓	✓	×	×	×	✓	✓	✓	×	×	×	×	×
9	✓	✓	×	×	×	×	×	✓	×	×	×	×	×
10	✓	✓	×	×	×	×	×	×	×	×	×	×	×
11	✓	×	×	×	×	×	×	×	×	×	×	×	×
12	✓	×	×	×	×	×	×	×	×	×	×	×	×

Figure 12: Linear independence of Butterfly blending functions of level ℓ on cells of levels ℓ (types A, B, C, R) and $\ell+1$ (types A^+ to L^+) for various values of the valence ν. The case of regular cells is covered by $\nu = 6$.

scheme. Similar to the previous sections we use Lemmas 6 and 7 for generating the catalogs $\mathbf{C}^\ell$. More precisely, we analyze the linear independence of level ℓ Butterfly blending functions on cells of levels ℓ and $\ell+1$.

Besides irregular cells (type A), there are three types B, C and R of regular ones, see Fig. 10 (left). Splitting them into cells of level $\ell+1$ gives 12 different types $A^+, \ldots, L^+$. The two types K^+ and L^+ are created when splitting a cell of type R, see Fig. 10 (right).

We use Lemma 7 and the second part of Lemma 6 to prove that some of the cells are contained in $\mathbf{C}^\ell$. More precisely, for a suitable $k > \ell$, we choose the set $\mathfrak{S}$ to be the subset of the level k vertex set that is contained in the cell under consideration. We then analyze the row rank of the corresponding matrix in Lemma 7, whose entries are simply the values of the subdivision blending functions of level ℓ at the vertices forming $\mathfrak{S}$. If this matrix is of full row rank, then the second part of Lemma 6 allows us to conclude that G^ℓ is linearly independent on the cell.

In addition, we rely on Lemma 7 for identifying cells where G^ℓ is linearly dependent. In this case, the set $\mathfrak{S}$ is the entire cell under consideration. The matrix entries are the values of the subdivision blending functions at vertices from the level k vertex set for a suitable $k > \ell$ within the cell and located on the two layers (with respect to the grid of level k) surrounding it.

Our results are summarized in Table 12. The results for valences other than 7, 9 and 11 were obtained using symbolic computations. The results for the remaining high odd valencies were obtained by numerical computations in MATLAB as the coefficients [41] involved for these valences do not admit representations by radicals. This complicates the symbolic manipulation of the resulting expressions considerably.

Example. We reconsider the example from Fig. 9 (left) and use the catalog presented in Fig. 12 to perform adaptive refinement with the help of ARLI. We consider a subdomain hierarchy with two levels $\mathfrak{M} = \mathfrak{M}^0 \supseteq \mathfrak{M}^1 = \emptyset$, where $\mathfrak{M}$ is a domain manifold consisting of triangular cells. After selecting a region $\mathfrak{R}$ (left, gray) for refinement, the algorithm generates a new subdomain hierarchy $\mathfrak{M} = \mathfrak{M}^0_+ \supseteq \mathfrak{M}^1_+ \supseteq \mathfrak{M}^2_+ = \emptyset$, see Fig. 9 (right). We need to add 6 cells of level 1. Although some cells (right, dark gray) are not contained in the catalog $\mathbf{C}^0$, they are not included in $\mathfrak{M}^1_+$ since each Butterfly blending function acting on them takes non-zero values on another white cell in $\mathfrak{M} \setminus \mathfrak{M}^1_+$ as well.

5. Conclusion

We have extended the construction of hierarchical generating systems and the use of truncation to restore the partition of unity property, which were presented in [39] for generating systems containing non-negative functions defined on open subsets of $\mathbb{R}^d$, to more general generating systems defined on domain manifolds. We also proposed a refinement algorithm that maintains linear independence based on catalogs containing subdomains with linearly independent restricted generating systems.

Based on these abstract results, we studied doubly hierarchical B-splines and hierarchical Zwart-Powell elements as well as subdivision splines generated by the Catmull-Clark, Loop and Butterfly subdivision schemes. In particular, we provided catalogs of subdomains with linearly independent generating systems, which are required by our refinement algorithm. For Catmull-Clark and Loop subdivision, these catalogs are covered by the results in [27], which we summarized and re-confirmed using our framework. In addition, we were able to obtain similar catalogs also for other hierarchical constructions, which include the modified Butterfly subdivision introduced in [41]. It should be noted that the resulting conditions for linearly independent (truncated) hierarchical generating systems based on the Butterfly scheme are far more restrictive than those for Catmull-Clark and Loop subdivision, due to the larger number of cells where the functions are linearly dependent.

Potential applications include adaptive subdivision surface fitting and numerical simulation using Isogeometric Analysis. Truncated hierarchical Catmull-Clark subdivision is explored in the recent articles [37, 38]. Clearly, besides using Catmull-Clark subdivision [2, 24], which operates on quadrangular meshes, subdivision schemes for triangular meshes (such as Loop subdivision) are of great interest for analysis as well, cf. [4]. Moreover, we feel that the use of *interpolatory* subdivision schemes in Isogeometric Analysis might be appealing to practitioners in the finite element community due to the simplicity of enforcing Dirichlet boundary conditions.

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