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Morphological Image Analysis

Michael H. F. Wilkinson

Johan Bernoulli Institute of Mathematics and Computer Science
University of Groningen
The Netherlands

Overview

- What is mathematical morphology?
- Basic operators: dilation, erosion, opening, and closing
- Comparison to linear filtering
- Lattice theory: what is it and why do we need it?
- Extensions to vector images
- Basic multi-scale operators

Morphological Image Processing

- Started out as a **set-theoretical** approach to image analysis
- Simple **geometrical** interpretation
- Image is probed by small subsets B, **structuring elements**
- Extended to a **lattice-theoretical** approach to image analysis
- Includes very efficient **adaptive** filters

Erosions and Dilations

- The simplest operations in mathematical morphology are dilation and erosion.
- In the binary case the dilation is given by

$$\delta_B(X) = X \oplus B = \bigcup_{b \in B} X_b \tag{1}$$

in which B is the *structuring element* (S.E.),

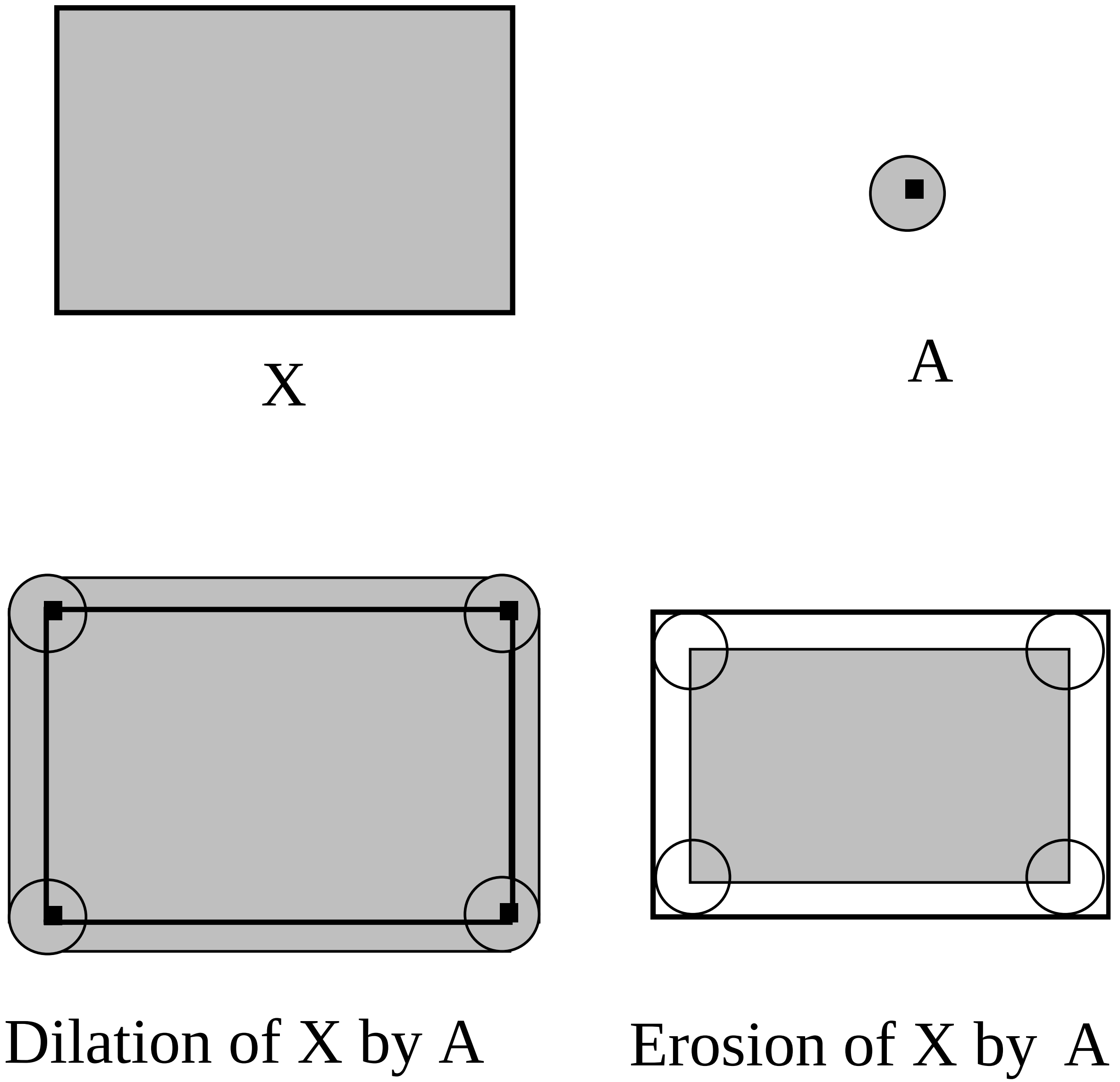
- X_b denotes the translation of X by b, i.e.

$$X_b = \{x + b | x \in X\}. \tag{2}$$

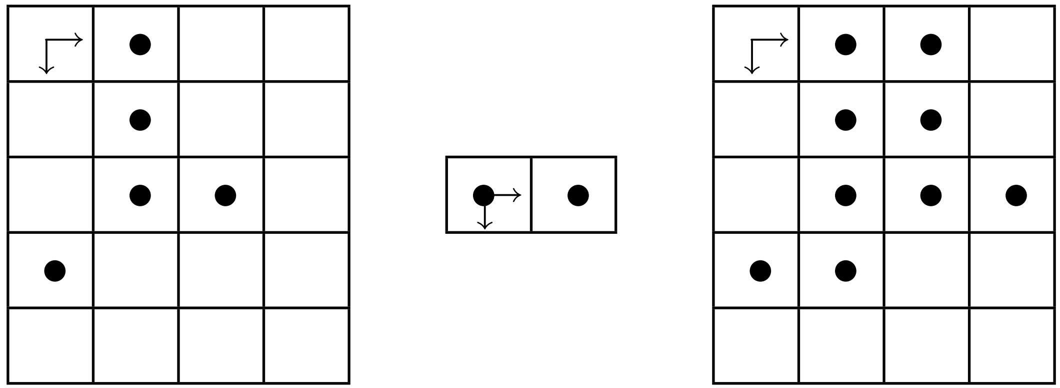
- The erosion is given by

$$\varepsilon_B(X) = X \ominus B = \bigcap_{b \in B} X_{-b} \tag{3}$$

Dilation/erosion: continuous case

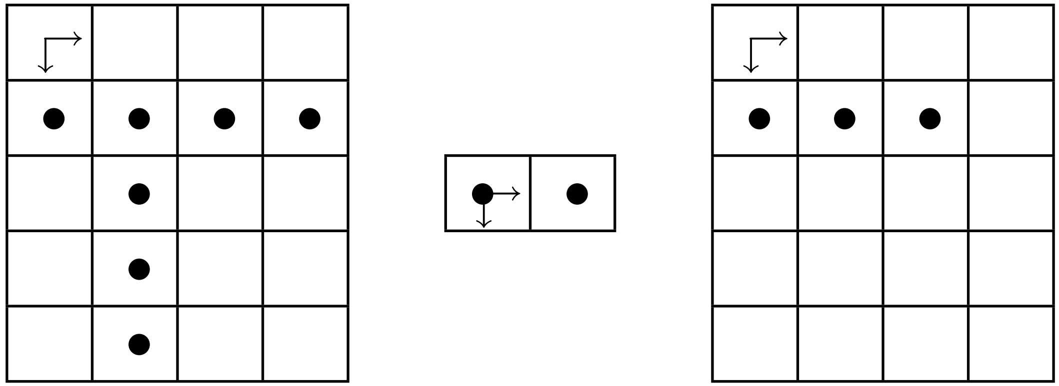


Dilation: discrete case



Left: binary image X. Middle: S.E. A. Right: dilation of X by A.

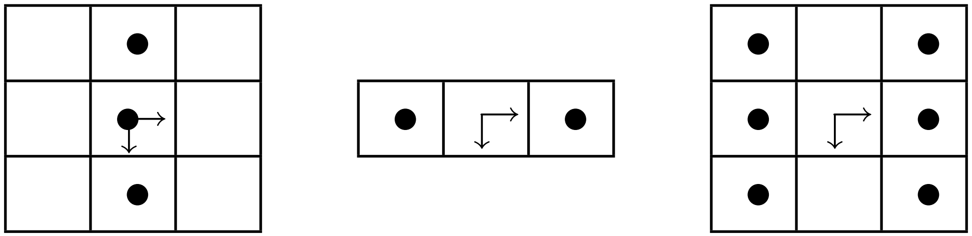
Erosion: discrete case



Left: binary image X. Middle: S.E. A. Right: erosion of X by A.

Dilation: discrete case

The S.E. does not need to contain the origin, so that $X \oplus A$ may have zero intersection with X.



Left: binary image X. Middle: S.E. A. Right: dilation of X by A.

Geometrical interpretation

$$\begin{aligned} X \oplus A &= \{h \in E : \check{A}_h \cap X \neq \emptyset\}, \\ X \ominus A &= \{h \in E : A_h \subseteq X\} \end{aligned}$$

where

$$\check{A} = \{-a : a \in A\}$$

is the *reflection* of A.

Characteristic properties

Distributivity : $(\bigcup_{i \in I} X_i) \oplus A = \bigcup_{i \in I} (X_i \oplus A)$

Translation invariance : $(X \oplus A)_h = X_h \oplus A.$

Increasing : $X \subseteq Y \implies X \oplus A \subseteq Y \oplus A$

Similarly for the **erosion** with **intersection** instead of union.

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Duality w.r.t. set-complementation

Let X^c denotes the complement of the set X. Then:

$$X \oplus A = (X^c \ominus \check{A})^c$$

In words: **dilating** an image by A gives the same result as **eroding** the **background** by $\check{A}$.

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Algebraic properties

$X \oplus A = A \oplus X$	**commutativity**
$(X \oplus A) \oplus B = X \oplus (A \oplus B)$	**associativity**
$(X \ominus A) \ominus B = X \ominus (A \oplus B)$	**iteration**
$(X \cup Y) \oplus A = (X \oplus A) \cup (Y \oplus A)$	**distributivity**
$(X \cap Y) \ominus A = (X \ominus A) \cap (Y \ominus A)$	**distributivity**
$X \oplus (A \cup B) = (X \oplus A) \cup (X \oplus B)$	
$X \ominus (A \cup B) = (X \ominus A) \cap (X \ominus B)$	

Structural Openings and Closings

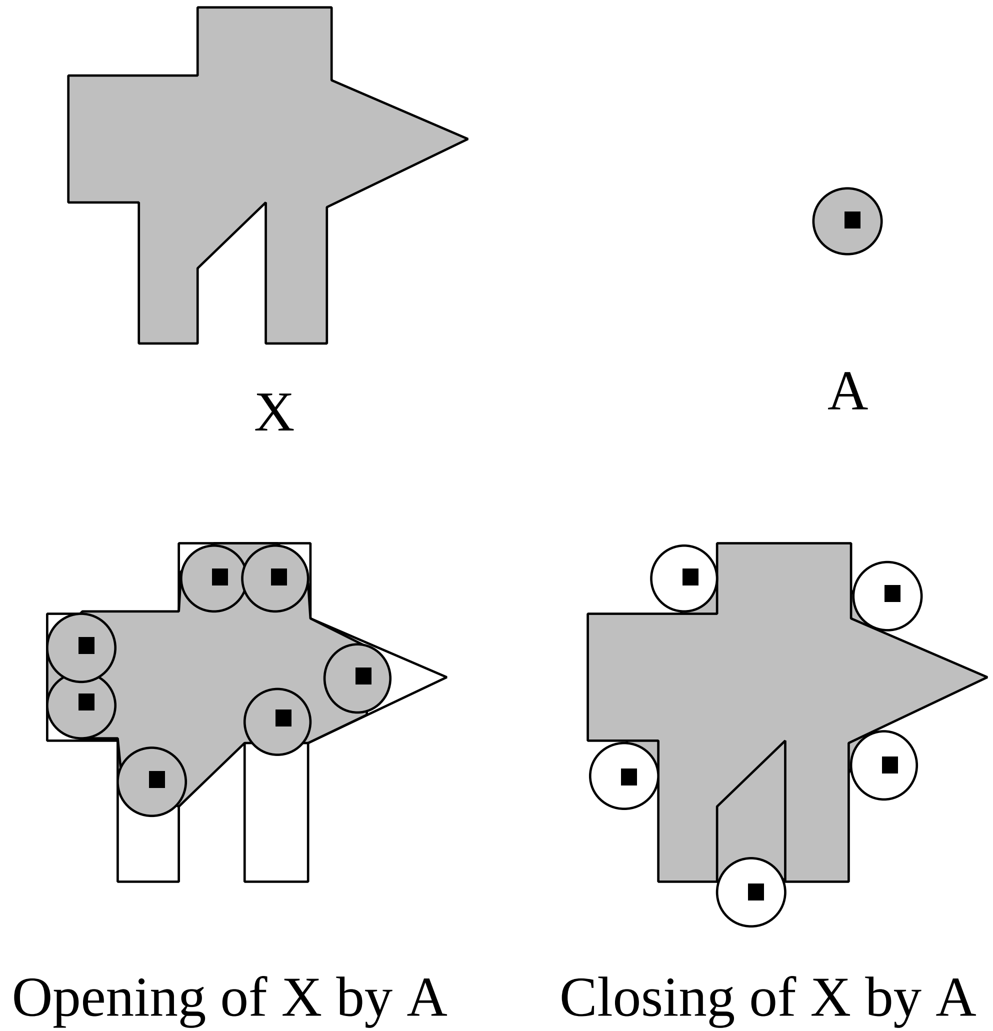
A *structural opening* γ_B by S.E. B is obtained by first applying an erosion, followed by a dilation with the same SE, i.e.,

$$\gamma_B(X) = \delta_B(\varepsilon_B(X)) \tag{4}$$

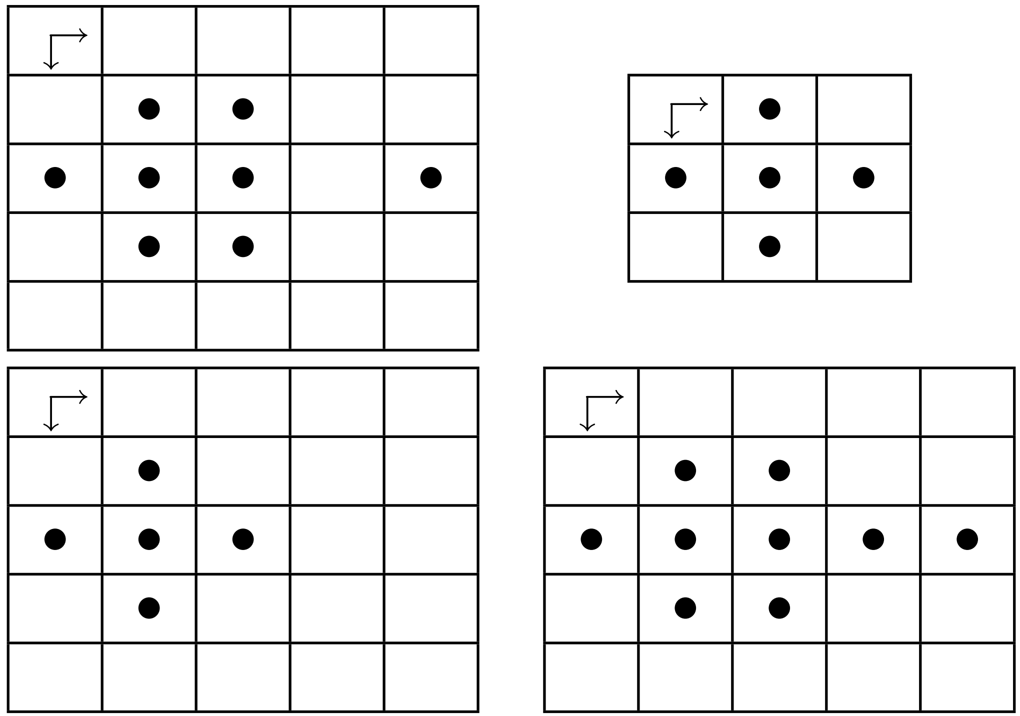
whereas the structural closing ϕ_B is defined as

$$\phi_B(X) = \varepsilon_B(\delta_B(X)) \tag{5}$$

Opening & Closing: continous case



Opening & Closing: discrete case



Upper left: binary image X. Upper right: S.E. A. Lower left: **opening** of X by A. Lower right: **closing** of X by A.

Properties of Openings and Closings

A mapping ψ is called:

1. **idempotent**, if $\psi(\psi(X)) = \psi(X)$
2. **increasing**, if $X \subseteq Y \implies \psi(X) \subseteq \psi(Y)$
3. **extensive**, if for every $X,\ \psi(X) \supseteq X$
4. **anti-extensive**, if for every $X,\ \psi(X) \subseteq X$

Theorem . *The **opening** is increasing, idempotent and **anti**-extensive. The **closing** is increasing, idempotent and extensive.*

Duality: $(X^c \circ A)^c = X \bullet \check{A}$.

Grey Scale Erosion and Dilation

In the grey scale case dilation and erosion become maximum and minimum filters respectively

$$(\delta_h(f))(x) = \bigvee_{k \in B} f(x - k) \tag{6}$$

and

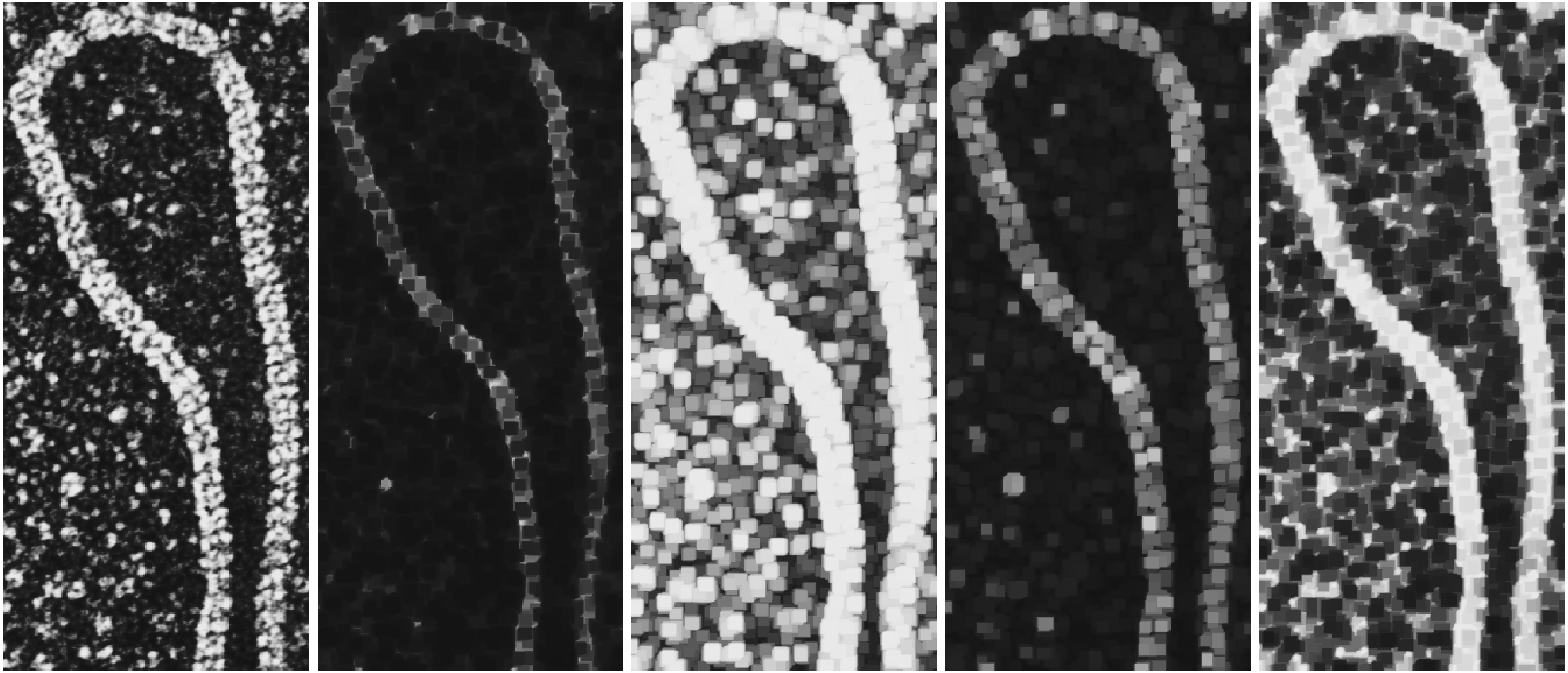
$$(\epsilon_h(f))(x) = \bigwedge_{k \in B} f(x + k) \tag{7}$$

A slightly more general form uses a function b with support B:

$$(\delta_h(f))(x) = \bigvee_{k \in B} (b(k) + f(x - k)) \tag{8}$$

and

$$(\epsilon_h(f))(x) = \bigwedge_{k \in B} (f(x + k) - b(k)) \tag{9}$$



(a) f (b) $\varepsilon_9 f$ (c) $\delta_9 f$ (d) $\delta_9\varepsilon_9 f$ (e) $\varepsilon_9\delta_9 f$

Characteristic properties

Distributivity : $\delta_A\left(\bigvee_{i\in I} f_i\right) = \bigvee_{i\in I}(\delta_A(f_i)$

Translation invariance : $(\delta_A(f))_h = \delta_A(f_h)$.

Increasing : $f \leq g \implies \delta_A(f) \leq \delta_A(g)$

Similarly for the **erosion** with **intersection** instead of union.

Relationship to Linear Filters

Let us recall linear filters, which are convolutions by some kernel h with support H, we have

$$(h * f)[n] = \sum_{k \in H} (h[k] f[n-k]) \tag{10}$$

Compare this to grey-scale dilation:

$$(\delta_h(f))(x) = \bigvee_{k \in B} (b(k) + f(x-k)) \tag{11}$$

Complete Lattices

- A complete lattice is defined as follows

  Definition 1. *A* ***complete lattice*** $\mathcal{L}$ *is a set with a partial order* $\leq$*, in which each subset* $A \subseteq \mathcal{L}$ *has an infimum* $\bigwedge A$ *and a supremum* $\bigvee A$ *contained in* $\mathcal{L}$

- Complete lattices have a least element denoted as $\mathbf{0}$ and a largest element denoted as $\mathbf{1}$

Examples of Complete Lattices

- The powerset $\mathcal{P}(E)$ of some universal set E with $\subseteq$ as the order, $\bigcap$ as infimum, and $\bigcup$ as supremum.
- The family of all functions $f : E \to \mathcal{T}$, with $\mathcal{T}$ some *totally ordered*, complete lattice (or chain), with

$$f \leq g \quad \equiv \quad f(x) \leq_{\mathcal{T}} g(x)\ \forall x \in E$$

with $\leq_{\mathcal{T}}$ the total order on $\mathcal{T}$

Adjunctions

- Given a lattice $\mathcal{L}$ with a partial order $\leq$
- A **dilation** δ is any operator which commutes with supremum $\bigvee$ and preserves the least element $\mathbf{0}$, or

$$\left(\bigvee_{a\in A} \delta(a)\right) = \delta\left(\bigvee_{a\in A} a\right), \quad \forall A \subseteq \mathcal{L} \tag{12}$$

and

$$\delta(\mathbf{0}) = \mathbf{0} \tag{13}$$

Adjunctions II

- An **erosion** ε is any operator which commutes with supremum $\bigwedge$ and preserves the greatest element $\mathbf{1}$, or

$$\left(\bigwedge_{a \in A} \varepsilon(a)\right) = \varepsilon\left(\bigwedge_{a \in A} a\right), \quad \forall A \subseteq \mathcal{L} \tag{14}$$

  and

$$\varepsilon(\mathbf{1}) = \mathbf{1} \tag{15}$$

- An **adjunction** is any pair (δ, ε) such that δ is a dilation and ε is an erosion, for which

$$a \leq \varepsilon(b) \Leftrightarrow \delta(a) \leq b. \tag{16}$$

Adjunctions III

- For every dilation δ there is precisely one erosion ε for which (16) holds.
- Likewise, for every erosion ε there is precisely one dilation δ for which (16) holds.
- If (δ, ε) is an adjunction

$$\gamma = \delta\varepsilon \tag{17}$$

is an algebraic opening, and

$$\phi = \varepsilon\delta \tag{18}$$

is an algebraic closing

Vector and Matrix Images

- Vector and matrix images can be modelled as mappings $f : E \rightarrow \mathcal{T}^N$, and $f : E \rightarrow \mathcal{T}^{N \times M}$ respectively.
- There is no natural order on either $\mathcal{T}^N$ or $\mathcal{T}^{N \times M}$.
- The most common solution: marginal processing
- This leads to **false colours**
- The alternative are lexicographic ordering, total preordering, or partial ordering.

Example

Lenna with noise

open-close

Example II


gaussian blur


area open-close

Size Distributions

A size distribution or *granulometry* is a set of openings $\{\alpha_r\}$ with r from some totally ordered set Λ with the following three properties:

$$\alpha_r(X) \subseteq X, \tag{19}$$

$$X \subseteq Y \Rightarrow \alpha_r(X) \subseteq \alpha_r(Y), \tag{20}$$

$$\alpha_r(\alpha_s(X)) = \alpha_{\max(r,s)}(X), \tag{21}$$

in the binary case, and in the grey scale case:

$$\alpha_r(f) \leq f, \tag{22}$$

$$f \leq g \Rightarrow \alpha_r(f) \leq \alpha_r(g), \tag{23}$$

$$\alpha_r(\alpha_s(f)) = \alpha_{\max(r,s)}(f), \tag{24}$$

Anti-Size Distributions

An anti-size distribution is a set of *closings* $\{\alpha_r\}$ with r from some totally ordered set Λ with the following three properties:

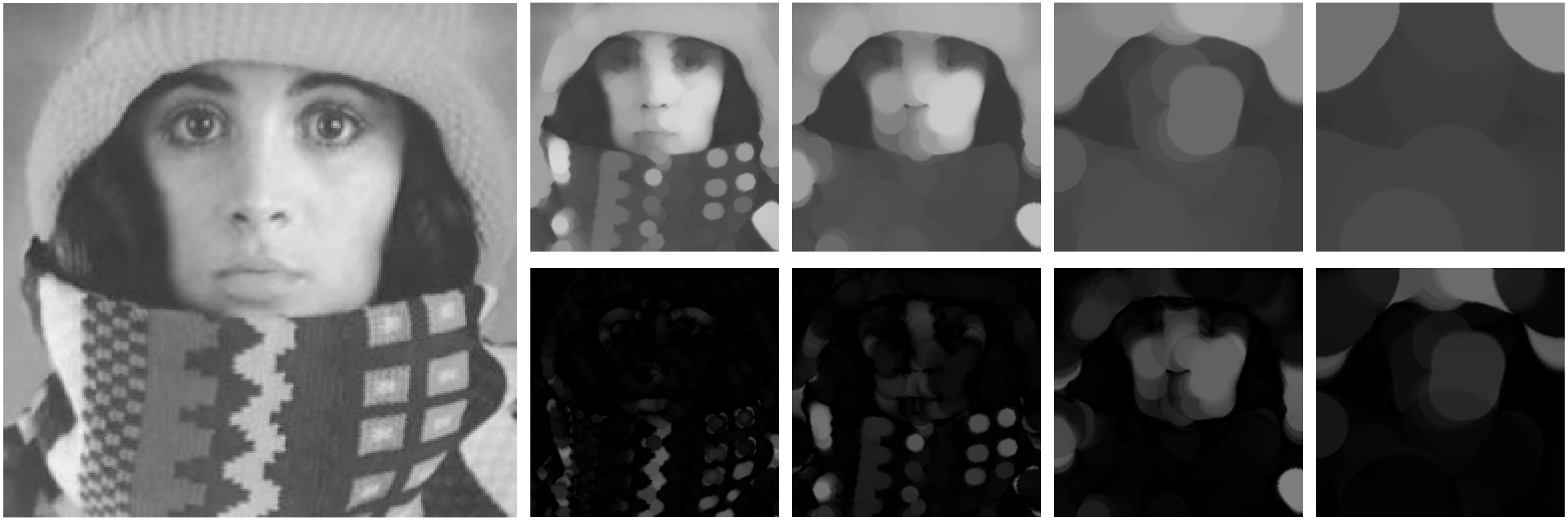
$$
\begin{aligned}
X &\subseteq \alpha_r(X), && (25)\\
X \subseteq Y &\Rightarrow \alpha_r(X) \subseteq \alpha_r(Y), && (26)\\
\alpha_r(\alpha_s(X)) &= \alpha_{\max(r,s)}(X), && (27)
\end{aligned}
$$

in the binary case, and in the grey scale case:

$$
\begin{aligned}
f) &\leq \alpha_r(f), && (28)\\
f \leq g &\Rightarrow \alpha_r(f) \leq \alpha_r(g), && (29)\\
\alpha_r(\alpha_s(f)) &= \alpha_{\min(r,s)}(f), && (30)
\end{aligned}
$$

Note that scale parameter r is usually held to be *negative*.

Example



- A grey-level image with the result of a size and anti-size distribution
- A sequence of approximations of circular S.E. are used.

Questions

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