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Connected Filters

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Outline

- Filters by Reconstruction
- Area Filters
- Attribute Filters
- Algorithms
- Vector-Attribute Filters

What Are Connected Filters?

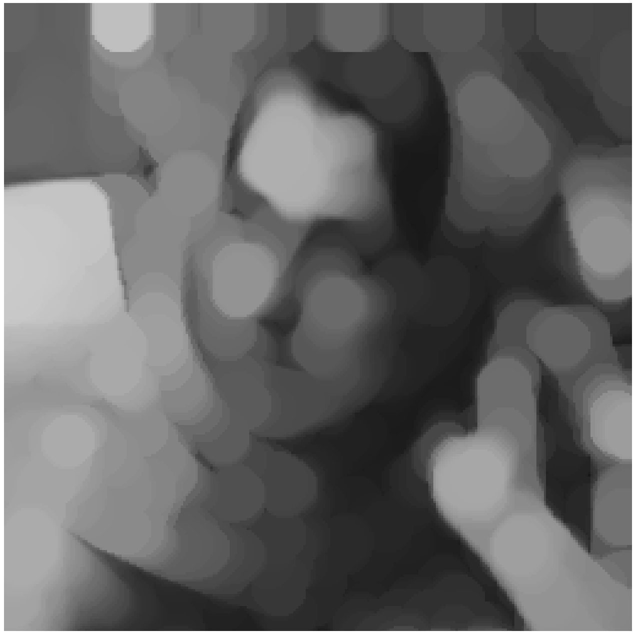
- Connectivity preserving morphological filters
- The image domain $\mathbf{M}$ can be partitioned into disjoint sets based on
  - connected components in the binary case
  - connected zones of constant grey/colour level in the grey-scale/colour case
- A connected filter works by
  - merging disjoint sets in the partition
  - assigning new grey levels or colours to them
- This means that no new edges are introduced by connected filters.
- Connected filtering therefore works on image structures rather than pixels
- Note that these filters depend on a definition of *connectivity*

Key Properties

- Connected filters differ from other morphological filters in that they:
  - work on the connected components of images, rather than on single pixels or rigidly defined neighbourhoods,
  - are strictly edge preserving,
  - can be used to create strictly causal scale-spaces,
  - can perform both low level and intermediate to high level processing tasks
  - can be given many useful invariance properties such as scale invariance.
- Problems with classical connected filters include:
  - noise along edges of objects cannot be removed
  - objects linked by noise pixels cannot be separated
  - objects broken up by imaging artefacts cannot be joined
- These problems can partly be solved by *second-order* connectivities.

Reconstruction

original f



marker $g = \gamma_{21} f$

reconstruction of f by g

The edge preserving effect of openings-by-reconstruction compared to structural openings

Reconstruction

- The basis of an opening by reconstruction is the reconstruction of image f from an arbitrary marker g.
- This is usually defined using geodesic dilations $\bar{\delta}_f$ defined as

$$\bar{\delta}_f^1(g) = f \wedge \delta(g). \tag{1}$$

- This operator is used iteratively until stability, to perform the reconstruction ρ i.e.

$$\rho(f|g) = \lim_{n\to\infty} \bar{\delta}_f^n g = \underbrace{\bar{\delta}_f^1 \ldots \bar{\delta}_f^1 \bar{\delta}_f^1}_{\text{until stability}}(g). \tag{2}$$

- In practice we apply $\bar{\delta}_f^n$ with n the smallest integer such that

$$\bar{\delta}_f^n g = \bar{\delta}_f^{n-1} g. \tag{3}$$

Openings-by-Reconstruction

- What this process does in the binary case is reconstruct any connected component in f which intersects some part of g.
- An opening-by-reconstruction $\bar{\gamma}_X$ with structuring element (S.E) X is computed as
$$\bar{\gamma}_X(f) = \rho(f|\gamma_X(f)), \quad (4)$$
in which γ_X denotes an opening of f by X.
- Reconstructing from this marker preserves any connected component in which X fits at at least one position.
- Closing-by-reconstruction $\bar{\phi}_X$ can be defined by duality, i.e.
$$\bar{\phi}_X(f) = -\bar{\gamma}_X(-f) \quad (5)$$

- Openings-by-reconstructions are anti-extensive, and closings-by-reconstructions are extensive, removing bright or dark image details respectively.
- Meyer (*J.Math. Imag. Vis.* 2004) proposed levelings as an auto-dual extension of reconstruction filters.
- In this case a marker is used which may lie partly above and partly below the image.
- We can compute a leveling of $\lambda(f|g)$ of f from marker g as

$$(\lambda(f|g))(x) = \begin{cases} (\rho(f|g))(x) & \text{if } f(x) \geq g(x) \\ -(\rho(-f|-g))(x) & \text{if } f(x) < g(x), \end{cases} \tag{6}$$

- Levelings allow edge-preserving simplification of images, by simultaneously removing bright and dark details.

Example: Levelings

original

blurred by Gaussian

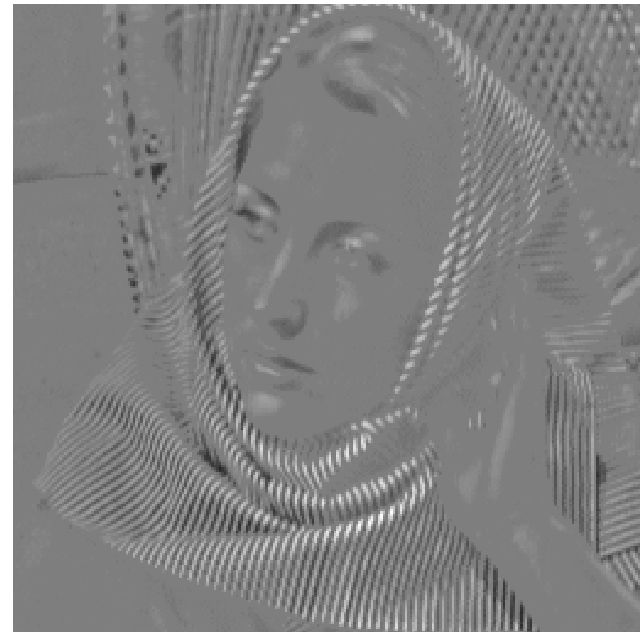
leveling

Leveling using a Gaussian filter to simplify the image in an auto-dual manner.

Example: Leveling Cartoons

original

cartoon



texture channel

Leveling cartoons for texture/cartoon decomposition.

Area openings

- The opening by reconstruction can also be defined as

$$\bar{\gamma}_X(f) = \bigvee_{B \in \mathcal{B}_X} \gamma_B(f) \tag{7}$$

with $\mathcal{B}_X$ the family of all connected S.E. B such that $X \subseteq B$.

- The area opening γ_λ^a can be defined as

$$\gamma_\lambda^a(f|\gamma_B(f)) = \bigvee_{B \in \mathcal{B}_\lambda} \gamma_B(f) \tag{8}$$

with $\mathcal{B}_\lambda$ the family of all connected S.E. B such that its area $A(B) \geq \lambda$.

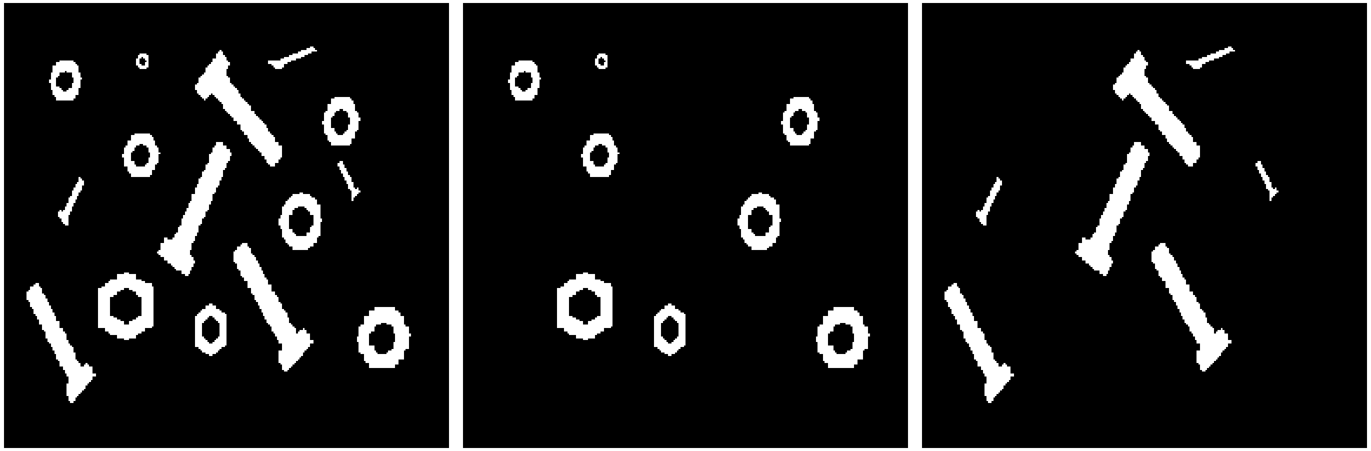
- These were the first two **attribute openings**

Attribute Filters I

- Introduced by Breen and Jones in 1996.
- Examples: area openings/closings, attribute openings, shape filters
- How do they work?

Binary image :

1. compute attribute for each connected component
2. keep components of which attribute value exceeds some threshold λ



Attribute Openings: Formally

- Let $T : \mathcal{P}(E) \rightarrow \{false, true\}$ be an increasing criterion, i.e. $C \subseteq D$ implies that $T(C) \Rightarrow T(D)$.
- A binary *trivial opening* $\Gamma_T : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ using T as defined above is defined as

$$\Gamma_T(C) = \begin{cases} C & \text{if } T(C), \\ \emptyset & \text{otherwise.} \end{cases} \tag{9}$$

- A typical form of T is

$$T(C) = (\mu(C) \geq \lambda) \tag{10}$$

  in which μ is some increasing scalar attribute value (i.e. $C \subseteq D \Rightarrow \mu(C) \leq \mu(D)$), and λ is the attribute threshold.
- The binary *attribute opening* Γ^T is defined as

$$\Gamma^T(X) = \bigcup_{x \in X} \Gamma_T(\Gamma_x(X)), \tag{11}$$

  in other words it is the union of all connected foreground components of X which meet the criterion T.

Attribute Openings: Examples

X

$T = A(C) \geq 11^2$

$T = I(C) \geq 11^4/6$

- An *area opening* is obtained if the criterion $T = A(C) \geq \lambda$, with A the area of the connected set C.
- A *moment-of-inertia opening* is obtained if the criterium is of the form $T = I(C) \geq \lambda$, with I the moment of inertia.

Other Attribute Filters

- If criterion T is non-increasing in (9), Γ_T becomes a *trivial thinning*, or *trivial, anti-extensive grain filter* Φ_T:

$$\Phi_T(C) = \begin{cases} C & \text{if } T(C), \\ \emptyset & \text{otherwise.} \end{cases} \tag{12}$$

- Using a trivial thinning rather than a trivial opening in (11), Γ_T becomes an *attribute thinning* or *anti-extensive grain filter* Φ^T:

$$\Phi^T(X) = \bigcup_{x \in X} \Phi_T(\Gamma_x(X)), \tag{13}$$

- The extensive dual of the atribute opening Γ_T is the *attribute closing* Ψ_T, which is defined as

$$\Psi_T(X) = (\Gamma_T(X^c))^c. \tag{14}$$

- The extensive dual of the attribute thinning is the *attribute thickening*, which is defined as above, but with a non-increasing criterion.

Non-Increasing Attributes

- Attribute thinnings can be defined using the usual form $T(C) = (\mu(C) \geq \lambda)$ if μ is non-increasing, e.g.:
  - Perimeter length P
  - Circularity (or boundary complexity) P^2/A
  - Concavity: $(H - A)/A$, with H the convex hull area
  - Elongation (non-compactness): I/A^2
  - Any of Hu's moment invariants
- Alternatively, increasing attributes (i.e. $C \subseteq D \Rightarrow \mu(C) \leq \mu(D)$) can be used if the form of T is changed:
  - $T = (\mu(C) = \lambda)$
  - $T = (\mu(C) \leq \lambda)$
  - etc.

The Grey-scale Case

- In the case of attribute openings, generalization to grey scale is achieved through *threshold decomposition*.
- A threshold set X_h of grey level image (function) f is defined as

$$X_h(f) = \{x \in E | f(x) \geq h\}. \tag{15}$$

- The grey scale attribute opening γ^T based on binary counterpart Γ^T is given by

$$(\gamma^T(f))(x) = \sup\{h \leq f(x) | x \in \Gamma^T(X_h(f))\} \tag{16}$$

- Closings ψ^T are defined by duality:

$$\psi^T(f) = -\gamma^T(-f). \tag{17}$$

- The non-increasing case will be dealt with after discussing the algorithms.

(Semi) Auto-dual filtering

- A filter is auto-dual (or self-dual) if it is invariant to inversion:

$$\psi(f) = -\psi(-f) \tag{18}$$

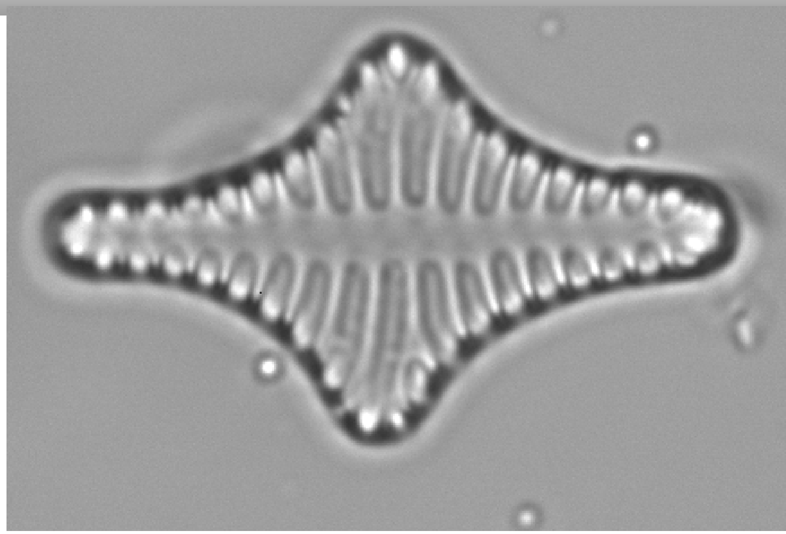
- An approximation is offered by *alternating sequential filters* (ASFs), which consist of an alternating sequence of openings and closings of increasing scale (e.g. radius of structuring element).
- Let γ_λ^a be a area opening of attribute threshold λ, and ϕ_λ^a the corresponding area closing.
- The *area* N-Sieve ψ_λ^N is given by

$$\psi_\lambda^N(f) = \phi_\lambda^a(\gamma_\lambda^a(\ldots(\phi_2^a(\gamma_2^a(\phi_1^a(\gamma_1^a(f)))))\ldots)) \tag{19}$$

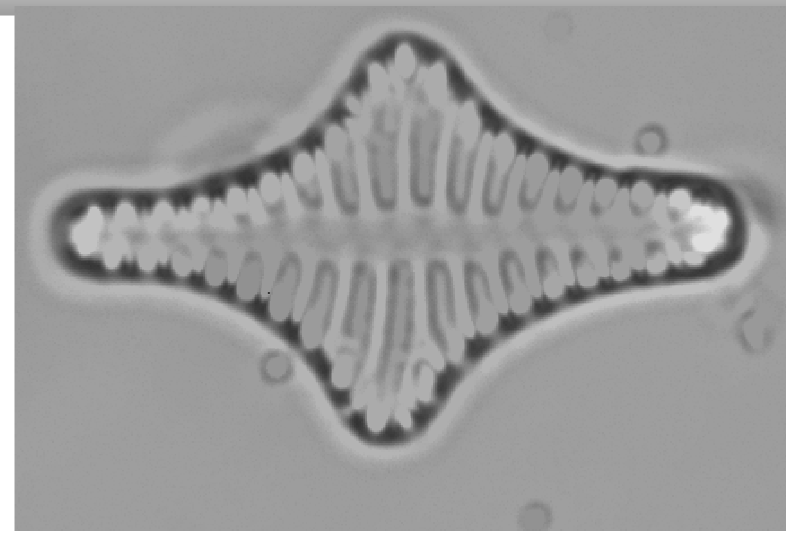
  and is an alternating sequential filter.
- The corresponding M-Sieve ψ_λ^M is just

$$\psi_\lambda^M(f) = \gamma_\lambda^a(\phi_\lambda^a(\ldots(\gamma_2^a(\phi_2^a(\gamma_1^a(\phi_1^a(f)))))\ldots)) \tag{20}$$

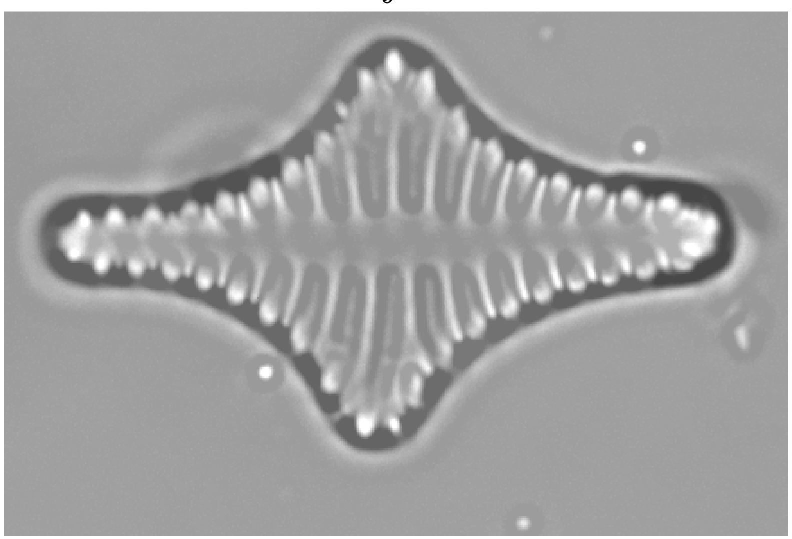
Grey Scale Example



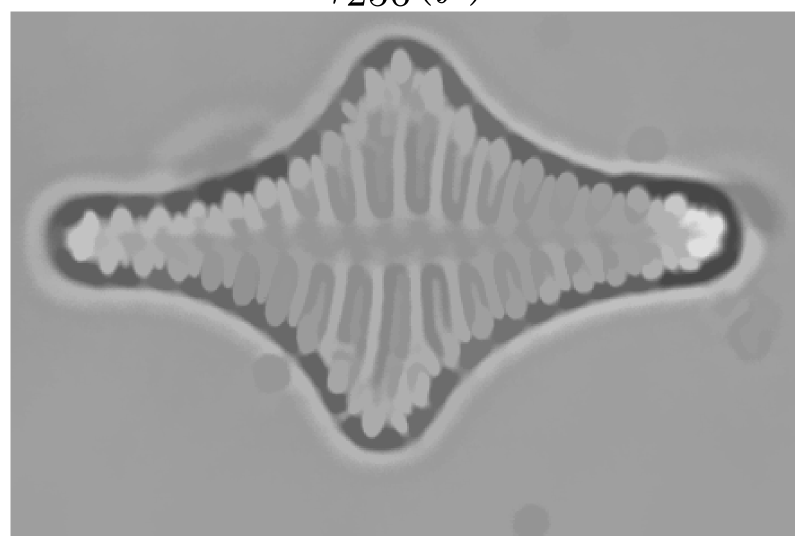
f



$\gamma^{a}_{256}(f)$



$\phi^{a}_{256}(f)$



$\psi^{N}_{256}(f)$

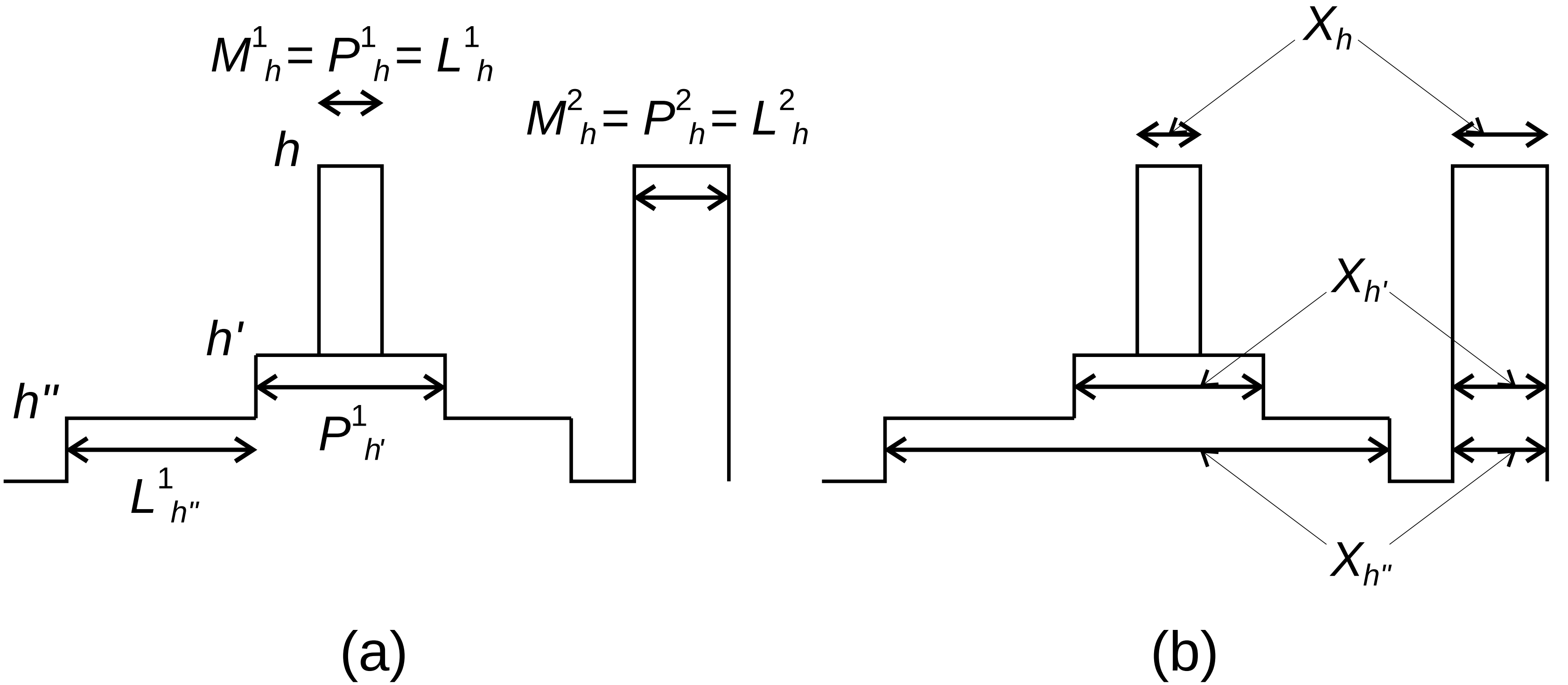
Definitions for Grey Scale

- A *level set* $\mathcal{L}_h$ of image f is defined as

$$\mathcal{L}_h(f) = \{x \in E | f(x) = h\} \tag{21}$$

- A *flat zone* or *level component* L_h at level h of a grey scale image f is a connected component of the level set $\mathcal{L}_h(f)$.
- *peak component* P_h at level h is a connected component of the thresholded set $X_h(f)$.
- A *regional maximum* M_h at level h is a level component no members of which have neighbors larger than h. A
- At each level h there may be several such components, which will be indexed as L_h^i, P_h^j and M_h^k, respectively.
- Any regional maximum M_h^k is also a peak component, but the reverse is not true.

Definitions for Grey Scale



One-dimensional example of level components, peak components and regional maxima.

Algorithms for the Grey-scale Case

- Naive computation of these filters in the grey-scale case can be done by threshold decomposition. This is SLOW!
- Three faster algorithms have been proposed
  - A priority-queue based approach (Vincent, 1993; Breen & Jones, 1996): low memory cost, time complexity $O(N^2 \log N)$.
  - A union-find approach (Meijster & Wilkinson 2002): low memory cost, time complexity $O(N \log N)$, fastest in practice, only for increasing filters.
  - The Max-tree based approach (Salembier *et al.*, 1998): high memory cost, time complexity $O(N)$, most flexible.

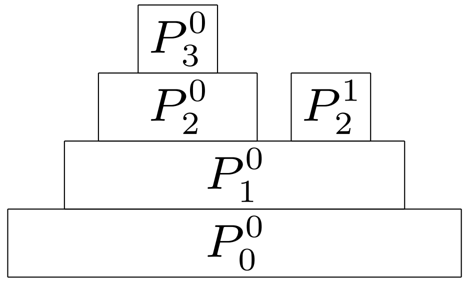
Tree Structures for Connected Filtering

- Because peak components at different grey levels are nested within eachother, it is possible to represent the entire component structure as a tree.
- In *Max-trees* (Salembier et al., 1998) the nodes represent peak components.
- In *Min-trees* the nodes represent *valley components* (peak components of the inverted image).
- *Level-line trees* are built by computing a Min-tree and a Max-tree and merging these in such a way that the leaves of the tree are both minima and maxima in the image.
- Removing nodes in the Max-tree is leads to anti-extensive filtering
- Removing nodes in the Min-tree is leads to extensive filtering
- Removing nodes in the Level-line tree leads to auto-dual filtering.

Max-Tree representation

0	1	2	3	2	1	2	1	0

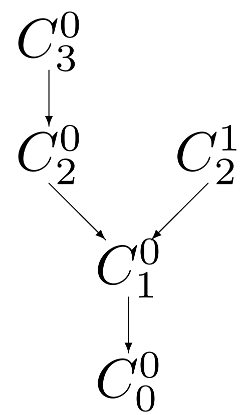
input signal



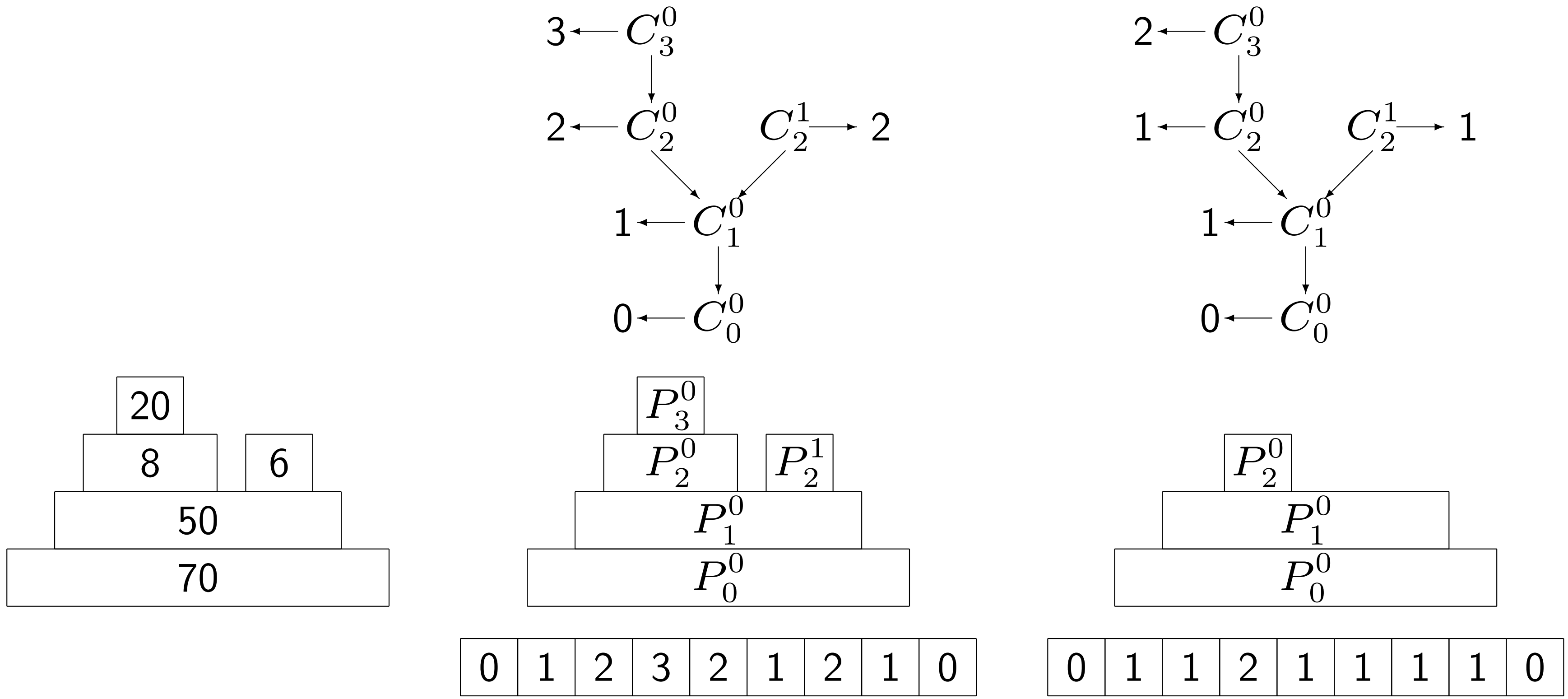
peak components

C_0^0	C_1^0	C_2^0	C_3^0	C_2^0	C_1^0	C_2^1	C_1^0	C_0^0

labelling



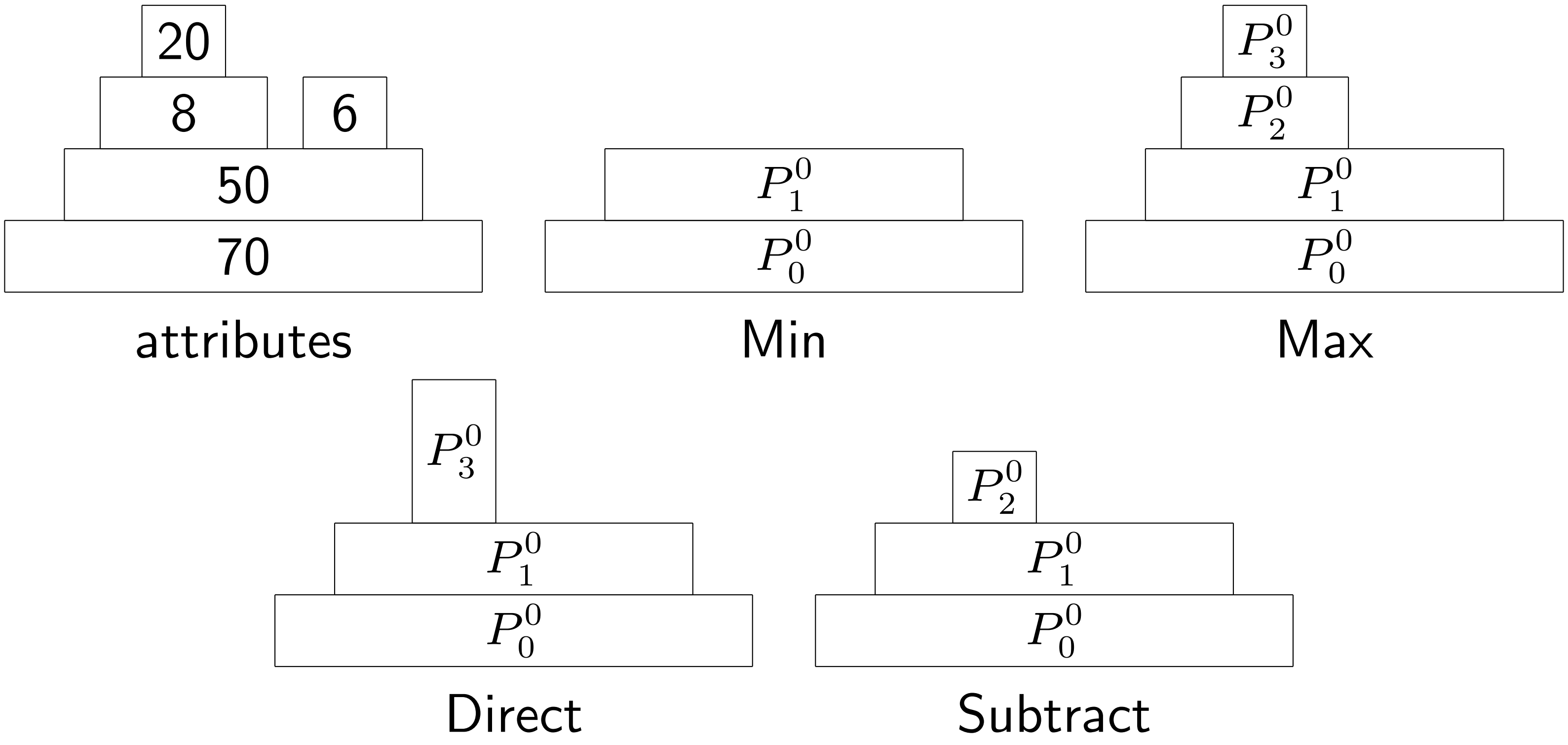
Max-Tree



attribute values **original** **filtered** $(\lambda = 10)$

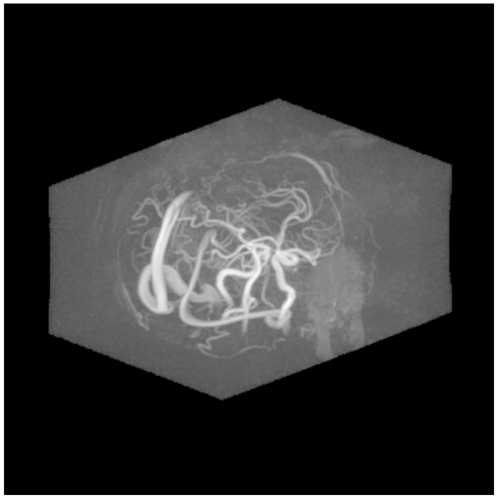
Filtering Rules

Different rules exist for removal of nodes:

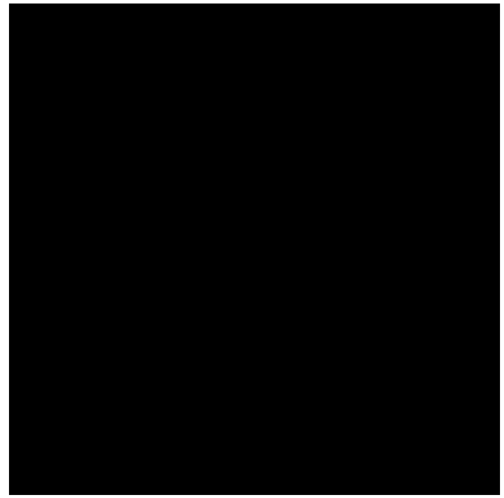


The first two are "pruning" rules, the second two "non-pruning". These different rules have an impact on the way "top-hat" equivalents of grey-scale shape filters work.

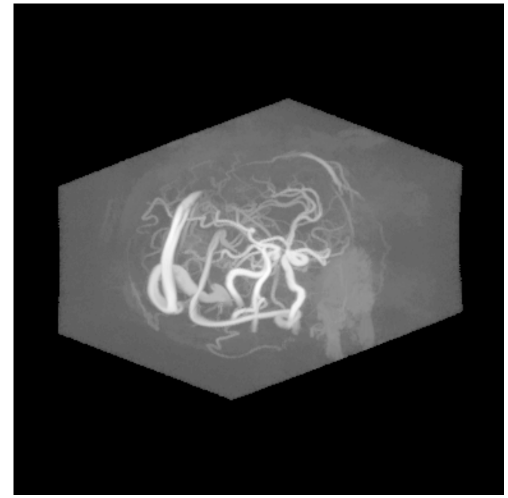
The Difference between Filtering Rules



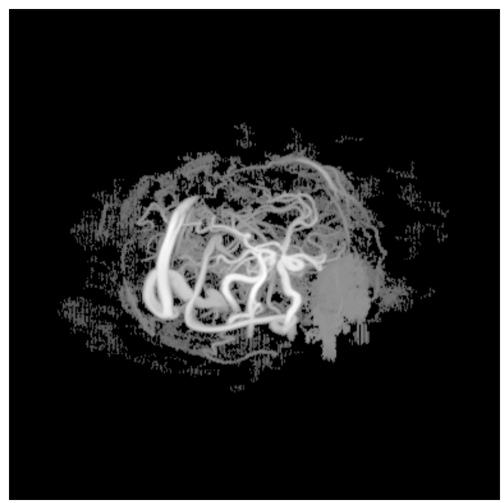
original



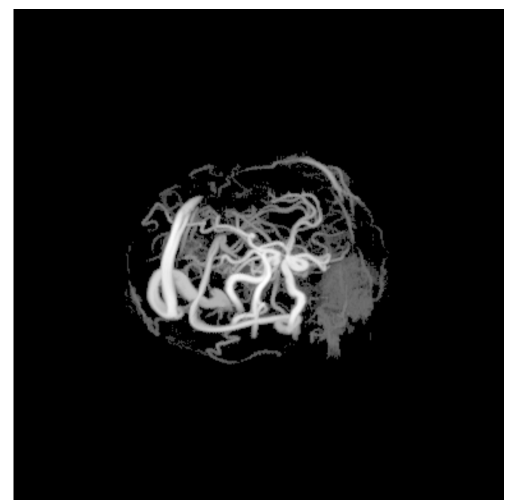
min



max



direct



subtractive

Invariances in Attribute Filters

- Very often in image analysis, we want our methods to be invariant to certain transforms.
- Most, if not all filters are shift invariant
- Rotation invariance can be obtained in structural filtering by:
  - Using a rotation invariant structuring element (SE), or
  - Using a non-rotation invariant SE at all possible rotations.
- In attribute filtering *invariance properties of the attribute carry over in the filter* if the connectivity is also invariant.
- Example: area is a rotation invariant attribute. and so is the area opening.
- Scale invariance is easily achieved in attribute filtering: use scale-invariant attributes: I/A^2.
- This leads to so-called *shape-filters*.

Why shape filters?

- Shape extraction is required whenever the objects of interest are characterized by shape, rather than scale.
- The common approach to this problem is by using multi-scale processing techniques.
- One example is finding elongated structures (vessels) is by using successive top-hat filters to obtain features of different width, followed by selection of sufficiently long features at each width scale by area openings.
- Multi-scale operators usually require multiple applications of filters to a single image.
- It may be more economical to design filters select for shape directly, in a single filter step.

Grey-Scale Image Decomposition by Shape

If we filter a grey-scale image f using shape criteria, we want the following properties to hold:

- *All* connected components of any threshold set of the filtered image $\phi_r^T(f)$ satisfy the shape criterion used.
- *None* of the connected components of any threshold set of the *difference* between the filtered image and original image $\phi_r^T(f) - f$ satisfy the shape criterion used

More formally we have

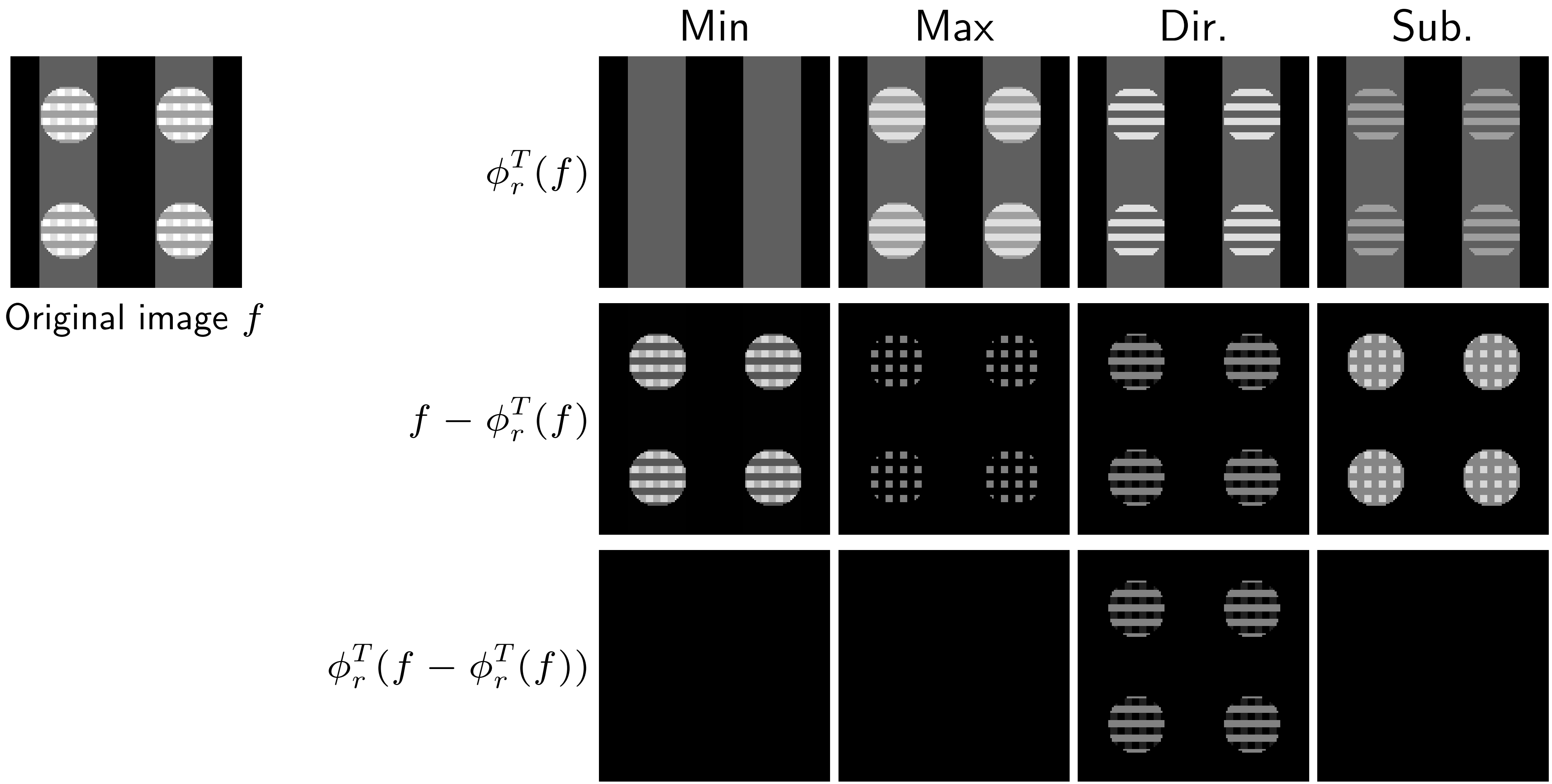
$$\Phi_r^T(X_h(\phi_r^T(f))) = X_h(\phi_r^T(f)) \tag{22}$$

and

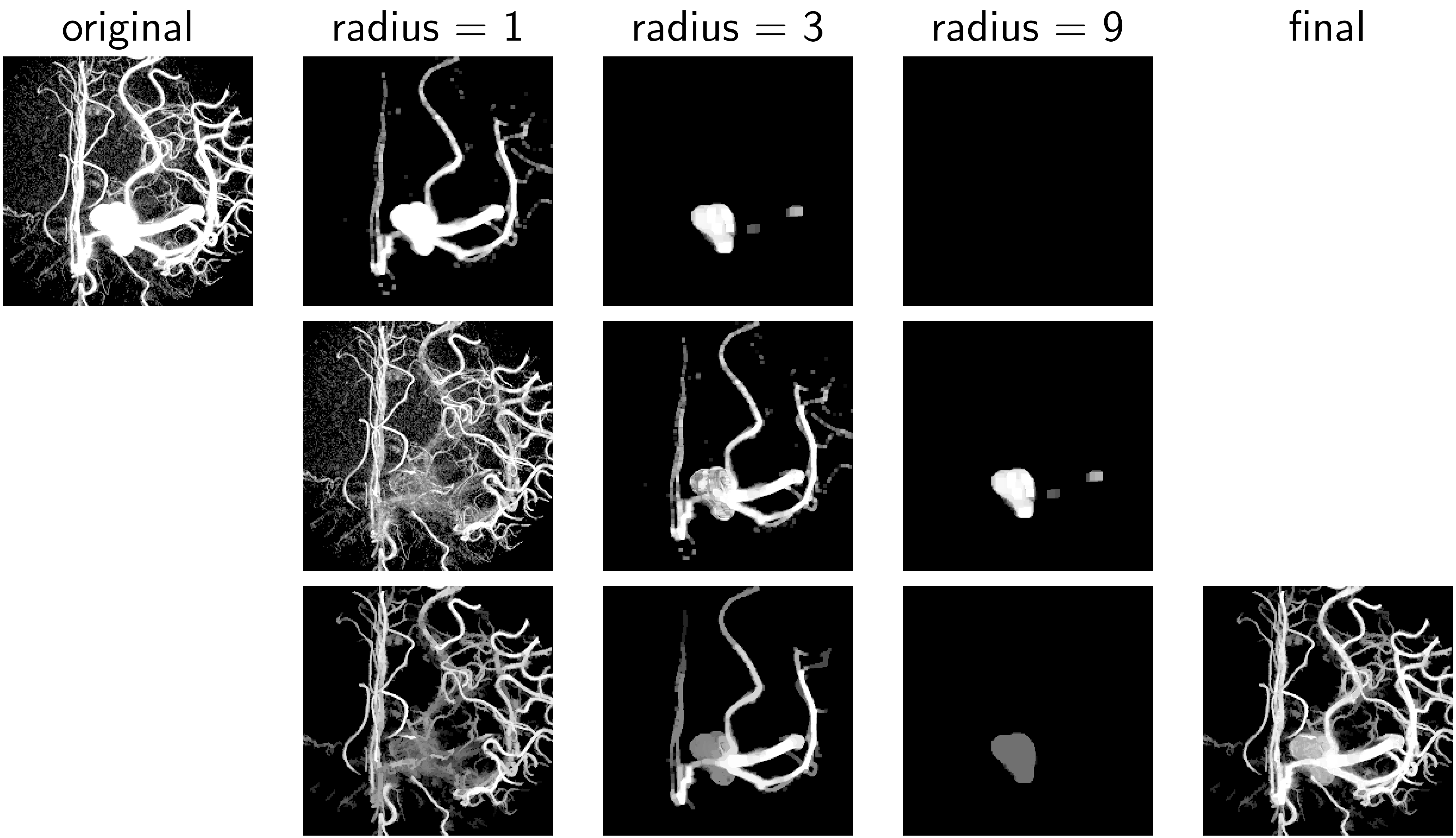
$$\Phi_r^T(X_h(f - \phi_r^T(f))) = \emptyset \tag{23}$$

for all h.

Grey-Scale Image Decomposition by Shape



Explicit Multiscale Approach



Shape filters: formal description

- Let us define a scaling X_λ of set X by a scalar factor $\lambda \in \mathbb{R}$ as

$$X_\lambda = \{x \in \mathbb{R}^n | \lambda^{-1} x \in X\}, \tag{24}$$

- An operator ϕ is said to be *scale invariant* if

$$\phi(X_\lambda) = (\phi(X))_\lambda \tag{25}$$

  for all $\lambda > 0$.

- If an operator is scale, rotation and translation invariant, we call it a *shape operator*.

An instance of a shape filter

- The binary connected opening Γ_x extracts the connected component to which x belongs, discarding all others.
- The trivial thinning Φ_T of a connected set C with criterion T is just the set C *if* C satisfies T, and is empty otherwise. Furthermore, $\Phi_T(\emptyset) = \emptyset$.
- The binary attribute thinning Φ^T of set X with criterion T is given by

$$\Phi^T(X) = \bigcup_{x \in X} \Phi_T(\Gamma_x(X)) \tag{26}$$

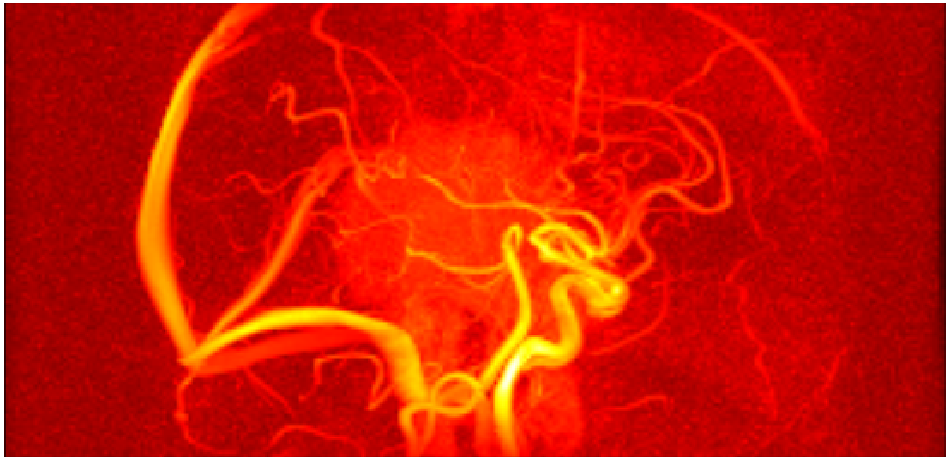
- If T is scale, rotation and translation invariant, Φ^T is a shape filter. An example would be:

$$T(C) = \left(\frac{I(C)}{A^2(C)} \geq \lambda \right). \tag{27}$$

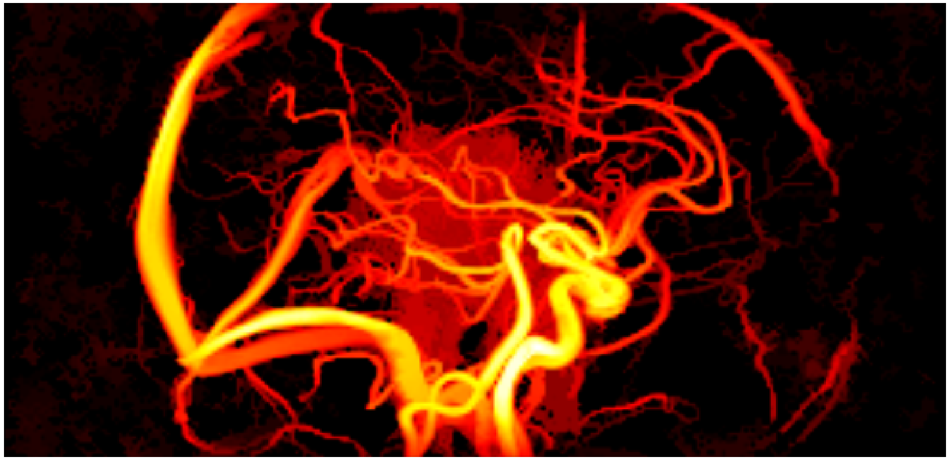
An Application

- In angiography it is often necessary to enhance curvilinear detail before segmentation.
- Standard multi-scale techniques require filtering at multiple scales *and* orientations, and may require > 1 hr CPU-time.
- Shape filtering using $I/V^{5/3} > \lambda$ as 3D shape criterion can be used instead.
- The result can be computed in 12 s on a Pentium 4 at 1.9 GHz for a 256^3 volume.

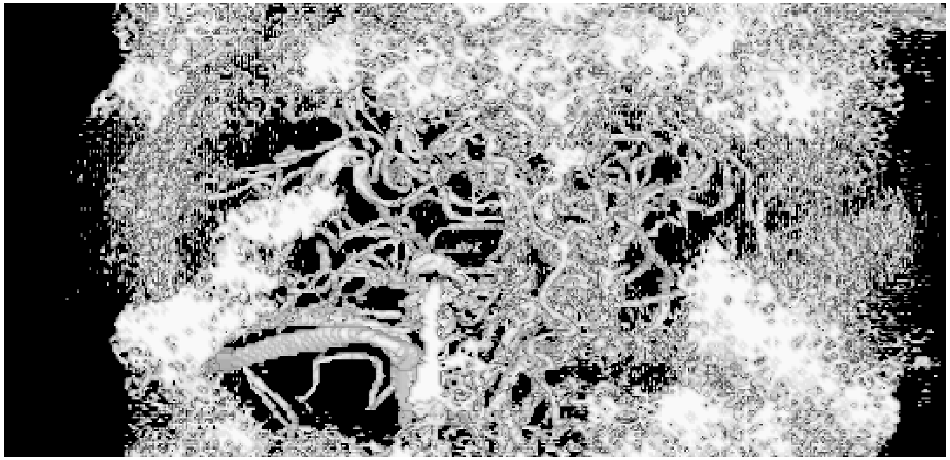
Blood-Vessel Enhancement



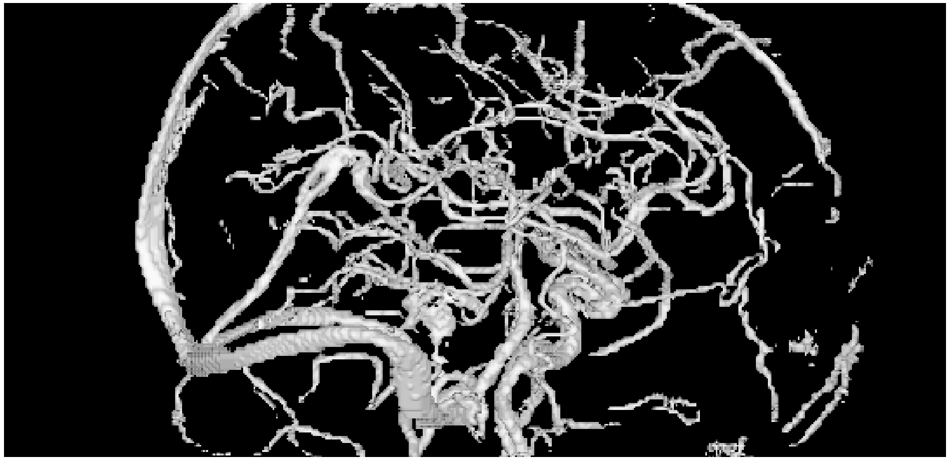
angiogram



filtered $\lambda = 2.0$



segmentation of original



segmentation of filtered set

Size Distributions

A size distribution or *granulometry* is a set of openings $\{\alpha_r\}$ with r from some totally ordered set Λ with the following three properties:

$$\alpha_r(X) \subseteq X, \tag{28}$$

$$X \subseteq Y \Rightarrow \alpha_r(X) \subseteq \alpha_r(Y), \tag{29}$$

$$\alpha_r(\alpha_s(X)) = \alpha_{\max(r,s)}(X), \tag{30}$$

in the binary case, and in the grey scale case:

$$\alpha_r(f) \leq f, \tag{31}$$

$$f \leq g \Rightarrow \alpha_r(f) \leq \alpha_r(g), \tag{32}$$

$$\alpha_r(\alpha_s(f)) = \alpha_{\max(r,s)}(f), \tag{33}$$

Anti-Size Distributions

An anti-size distribution is a set of *closings* $\{\alpha_r\}$ with r from some totally ordered set Λ with the following three properties:

$$X \subseteq \alpha_r(X), \tag{34}$$

$$X \subseteq Y \Rightarrow \alpha_r(X) \subseteq \alpha_r(Y), \tag{35}$$

$$\alpha_r(\alpha_s(X)) = \alpha_{\max(r,s)}(X), \tag{36}$$

in the binary case, and in the grey scale case:

$$f) \leq \alpha_r(f), \tag{37}$$

$$f \leq g \Rightarrow \alpha_r(f) \leq \alpha_r(g), \tag{38}$$

$$\alpha_r(\alpha_s(f)) = \alpha_{\min(r,s)}(f), \tag{39}$$

Note that scale parameter r is usually held to be *negative*.

Pattern Spectrum

- The pattern spectrum $s_\alpha(X)$ obtained by applying granulometry $\{\alpha_r\}$ to a binary image X is defined as

$$(s_\alpha(X))(u) = -\frac{\partial A(\alpha_r(X))}{\partial r}\bigg|_{r=u} \tag{40}$$

in which $A(X)$ is a function denoting the Lebesgue measure in $\mathbb{R}^n$.

- In the case of discrete images, and with $r \in \Lambda \subset \mathbb{Z}$, this differentiation reduces to

$$(s_\alpha(X))(r) = \#(\alpha_r(X) \setminus \alpha_{r^+}(X)) \tag{41}$$

$$= \#(\alpha_r(X)) - \#(\alpha_{r^+}(X)), \tag{42}$$

with $r^+ = \min\{r' \in \Lambda | r' > r\}$, and $\#(X)$ the number of elements of X.

Computing Pattern Spectra in Grey Scale

- For structural openings, we generally use a set of structuring elements $\{B_r\}$ (e.g. discs) of increasing size.
- From this we construct a granulometry $\{\alpha_r\}$ for which

$$\alpha_r(f) = f \circ B_r \tag{43}$$

- In this case the pattern spectrum is generally computed by naive implementation of the equation for the patter spectrum $s_f(r)$

$$s_f(r) = \sum_x ((f \circ B_{r-1})(x) - (f \circ B_r)(x)) \tag{44}$$

- This requires one structural opening per bin of the spectrum.

Using Connected Filters

- The nesting property of peak components makes computation of patterns spectra in the case of connected filters very simple.
- Any of the algorithms for attribute openings can be adapted to computation of pattern spectra with *any number of bins* in *just one* application of the algorithm.
- As each peak component is processed, simply add its grey-level sum to the appropriate bin based on the attribute.
- The method also works for *shape spectra* using attribute thinnings rather than openings.

Shape distributions I

A shape distribution is a set of operators $\{\beta_r\}$ with r from some totally ordered set Λ, with the following three properties

$$\beta_r(X) \subset X \tag{45}$$

$$\beta_r(X_\lambda) = (\beta_r(X))_\lambda \tag{46}$$

$$\beta_r(\beta_s(X)) = \beta_{\max(r,s)}(X), \tag{47}$$

for all $r, s \in \Lambda$ and $\lambda > 0$ in the binary case, and in the grey-scale case:

$$(\beta_r(f))(x) \leq f(x) \tag{48}$$

$$\beta_r(f_\lambda) = (\beta_r(f))_\lambda \tag{49}$$

$$\beta_r(\beta_s(f)) = \beta_{\max(r,s)}(f), \tag{50}$$

Shape distributions II: Example

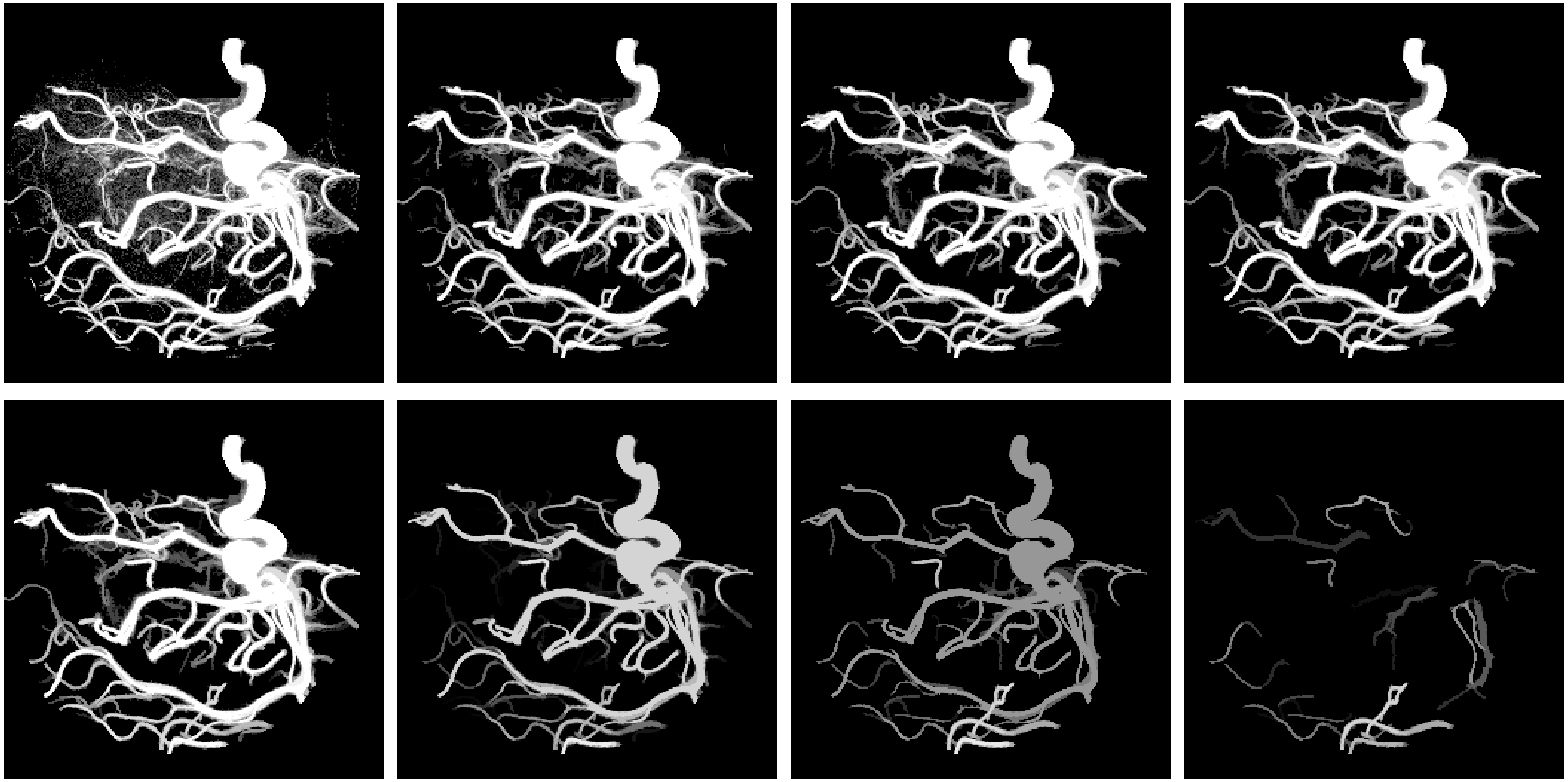
- Shape distributions can be implemented using families of attribute thinnings.
- Care must be taken that the third (absorption) property holds.
- If $\tau(C)$ is scale, rotation, and translation-invariant attribute of connected set C, the family of shape filters $\{\Phi^{T_\lambda}\}$ is a shape distribution, if T has the form:

$$T(C) = (\tau(C) > \lambda). \tag{51}$$

An example would be:

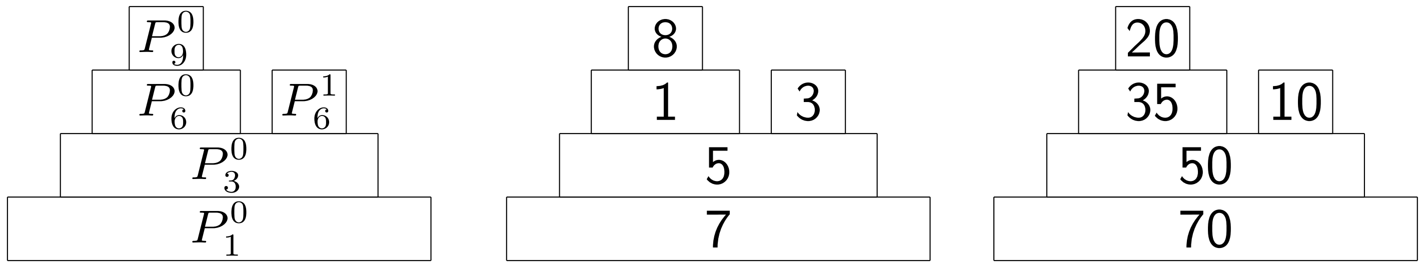
$$T(C) = \left(\frac{I(C)}{A^2(C)} > \lambda\right). \tag{52}$$

An Example



Applying the $I/V^{5/3}$-based shape distribution to an angiogram (top left) with $\lambda =$ 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, and 4.0.

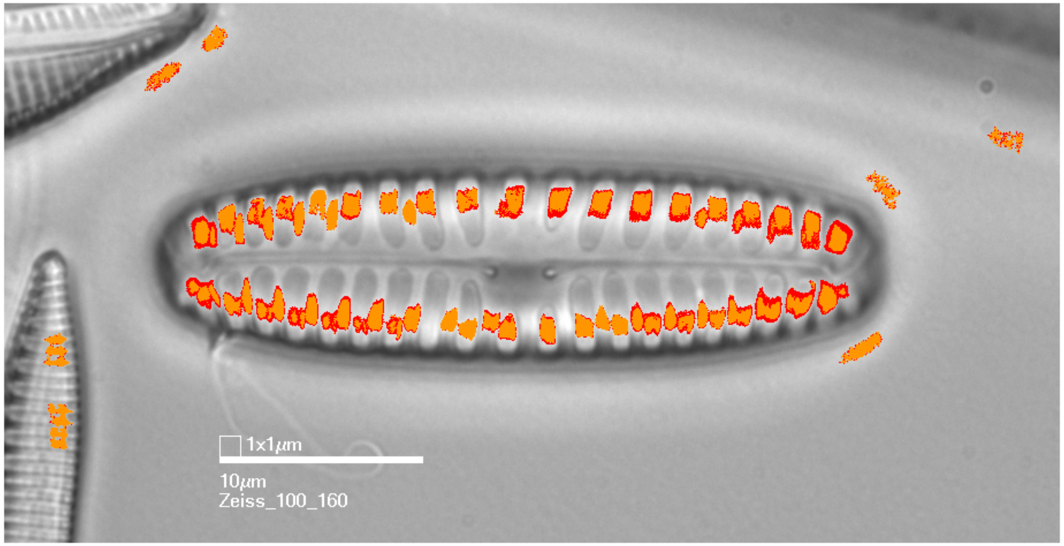
Computation of pattern spectrum using Max-Tree (Subtractive):

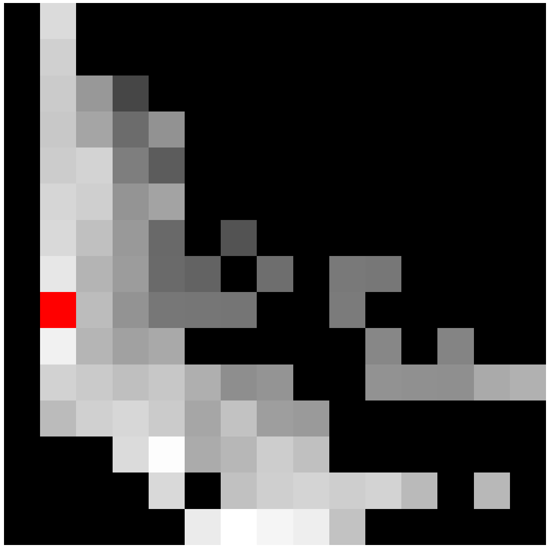


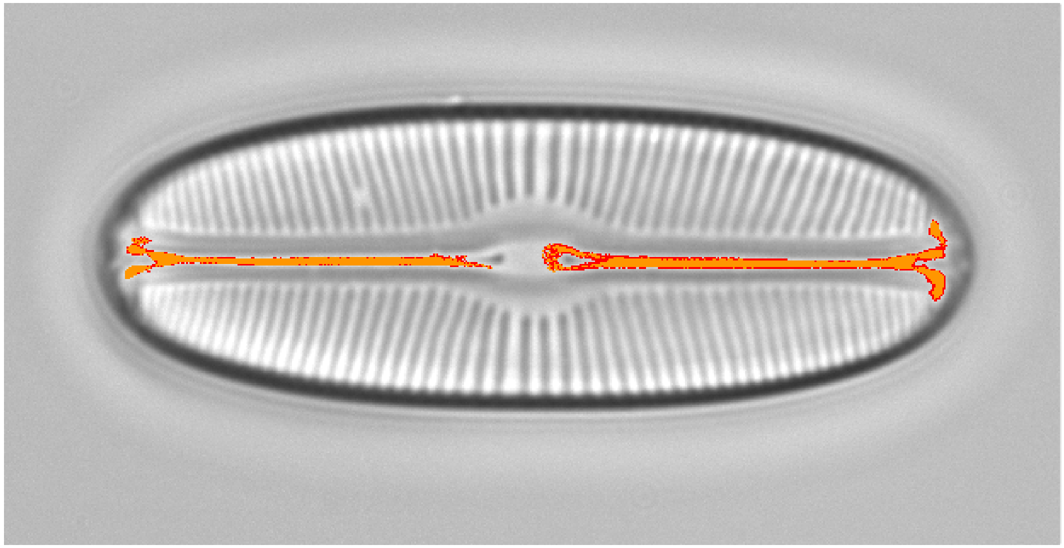
Peak components **Elongation** **Area**

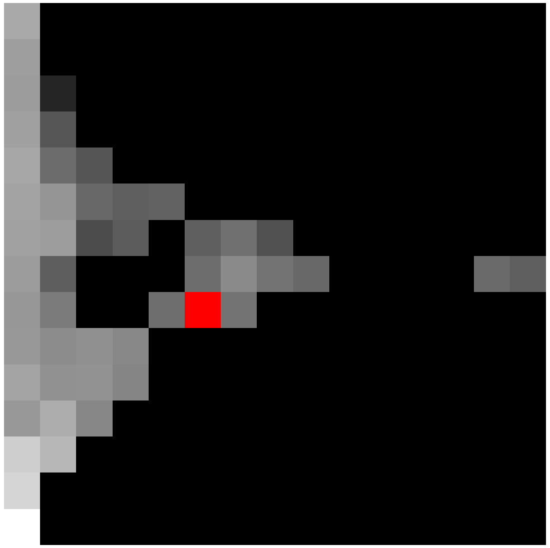
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
10	0	0	**30**	0	0	0	0	0	0
20	0	0	0	0	0	0	0	**60**	0
30	0	0	0	0	0	0	0	0	0
40	**105**	0	0	0	0	0	0	0	0
50	0	0	0	0	**100**	0	0	0	0
60	0	0	0	0	0	0	0	0	0
70	0	0	0	0	0	0	**70**	0	0
80	0	0	0	0	0	0	0	0	0

Application to Diatom Identification



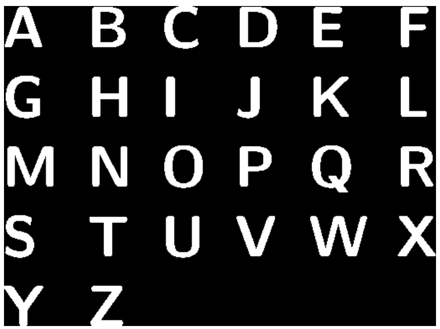




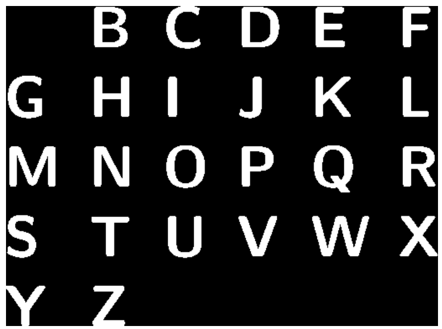


Vector-attribute filters

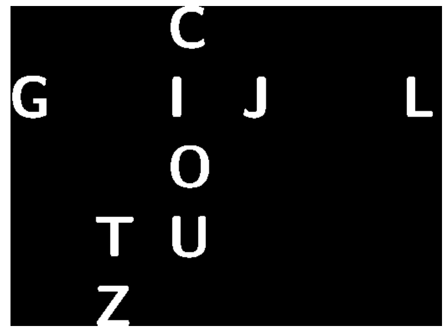
- Aim: Removing objects that are similar enough to a given shape.
- Example: removing objects that are similar enough (ϵ) to the reference shape (letter A).



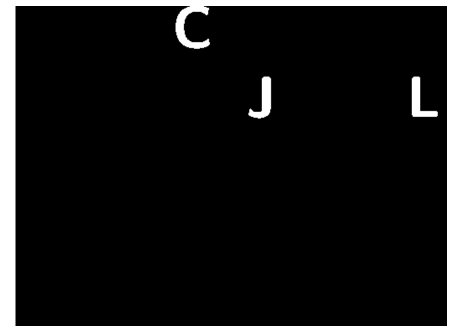
Original image X



$\epsilon = 0.01$



$\epsilon = 0.10$



$\epsilon = 0.15$

- A value of $\epsilon = 0$ means only those shapes are removed that are exactly the same as the reference shape.
- To gain more descriptive power we may use more than one attribute per node.

Binary Vector-Attribute Thinning

- A multi-variate attribute thinning $\Phi^{\{T_i\}}(X)$ with scalar attributes $\{\tau_i\}$ and their corresponding criteria $\{T_i\}$, with $1 \leq i \leq N$, preserves a component C if $\exists i : T_i$, $T_i = \tau_i(C) \geq r_i$:

$$\Phi^{\{T_i\}}(X) = \bigcup_{i=1}^{N} \Phi^{T_i}(X). \quad (53)$$

- An alternative is the *vector-attribute thinning*, in which C is preserved if $\vec{\tau}(C) \in \mathbb{R}^D$ satisfies criterion

$$T^{\vec{\tau}}_{\vec{r},\epsilon}(C) = d(\vec{\tau}(C), \vec{r}) \geq \epsilon \quad (54)$$

  in which dissimilarity measure $d : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{R}$ quantifies the difference between $\vec{\tau}(C)$ and $\vec{r}$.

- A binary vector-attribute thinning $\Phi^{\vec{\tau}}_{\vec{r},\epsilon}(X)$, with D-dimensional vectors, removes the connected components of a binary image X whose vector-attributes differ less than ϵ from a reference vector $\vec{r} \in \mathbb{R}^D$.

Definition 1. *The vector-attribute thinning $\Phi^{\vec{\tau}}_{\vec{r},\epsilon}$ of X with respect to a reference vector $\vec{r}$ and using vector-attribute $\vec{\tau}$ and scalar value ϵ is given by*

$$\Phi^{\vec{\tau}}_{\vec{r},\epsilon}(X) = \{x \in X |\ T^{\vec{\tau}}_{\vec{r},\epsilon}(\Gamma_x(X))\}. \quad (55)$$

Possible choices for d:

- Euclidean distance $d(\vec{u}, \vec{v}) = ||\vec{v} - \vec{u}||$.
- Manhattan distance $d(\vec{u}, \vec{v}) = \sum |v_i - u_i|$
- Any dissimilarity measure can be used (such as Mahalanobis distance).
- Since the triangle inequality $d(a, c) \leq d(a, b) + d(b, c)$ is not required, d need not be a distance.

Thinning with Respect to a Shape (Family)

- To select the appropriate vector $\vec{r}$ we can provide a shape in a binary image and compute its vector attributes.

Definition 2. *The vector-attribute thinning $\Phi^{\vec{\tau}}_{S,\epsilon}$ of X with respect to a reference* shape S *and using vector-attribute $\vec{\tau}$ and scalar value ϵ is given by*

$$\Phi^{\vec{\tau}}_{S,\epsilon}(X) = \Phi^{\vec{\tau}}_{\vec{\tau}(S),\epsilon}(X) \tag{56}$$

- More robustness can be obtained using a series of example shapes in a shape family $F = \{S_1, S_2, \ldots, S_n\}$:

Definition 3. *The vector-attribute thinning $\Phi^{\vec{\tau}}_{F,\epsilon}$ of X with respect to a reference* shape family F *and using vector-attribute $\vec{\tau}$ and scalar value ϵ is given by*

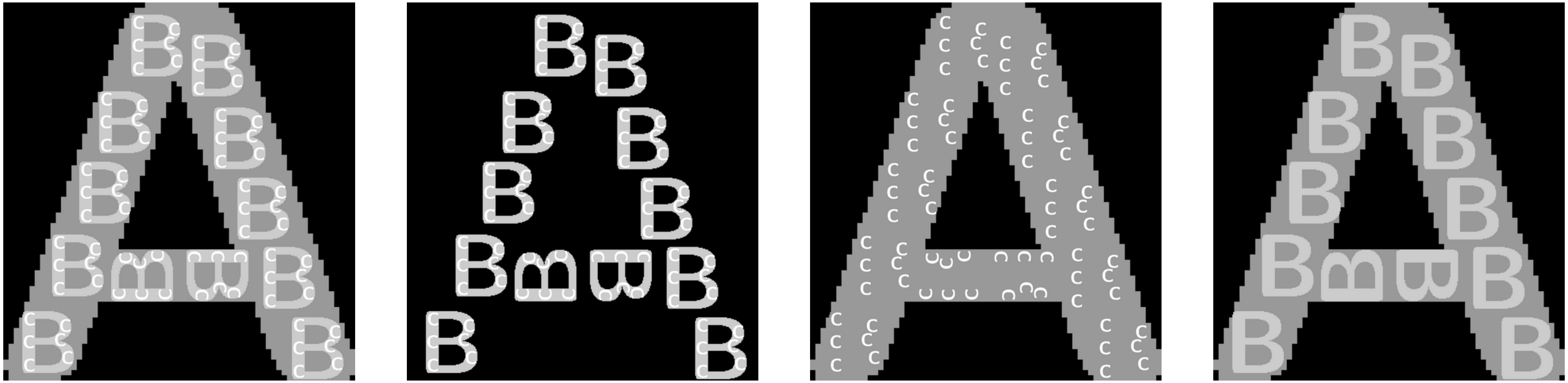
$$\Phi^{\vec{\tau}}_{F,\epsilon}(X) = \bigcap_{S \in F} \Phi^{\vec{\tau}}_{S,\epsilon}(X) \tag{57}$$

- This removes objects if they are similar enough to any of the example shapes

Extension to gray-scale using threshold decomposition:

$$\phi^{\vec{\tau}}_{\vec{r},\epsilon}(f) = \sup\{h|\ T^{\vec{\tau}}_{\vec{r},\epsilon}(\Gamma_x(X_h(f)))\}, \qquad (58)$$

where threshold set $X_h(f)$ is defined as: $X_h(f) = \{x \in \mathbf{M} | f(x) \geq h\}$. Example: removing letters from image f consisting of nested versions of the letters A, B, and C.



f $\phi^{\vec{\tau}}_{S_A,\epsilon}(f)$ $\phi^{\vec{\tau}}_{S_B,\epsilon}(f)$ $\phi^{\vec{\tau}}_{S_C,\epsilon}(f)$

(Central) moments up to some order $(p+q)$ are computed:

$$\text{Moments:} \qquad m_{pq} = \iint_{\mathbb{R}^2} x^p y^q f(x,y)\, dx\, dy \tag{59}$$

$$\text{Central moments:} \qquad \mu_{pq} = \iint_{\mathbb{R}^2} (x-\bar{x})^p (y-\bar{y})^q f(x,y)\, dx\, dy \tag{60}$$

$$\text{where } \bar{x} = \frac{m_{10}}{m_{00}} \quad \text{and} \quad \bar{y} = \frac{m_{01}}{m_{00}} \tag{61}$$

$$\text{Normalized central moments:} \qquad \eta_{pq} = \frac{\mu_{pq}}{\mu_{00}^{\gamma}} \tag{62}$$

$$\text{where } \gamma = \frac{p+q}{2} + 1 \tag{63}$$

(64)

Hu's Moment Invariants

Hu's set of seven moment invariants is defined as:

$$\phi_1 = \eta_{20} + \eta_{02} \tag{65}$$

$$\phi_2 = (\eta_{20} - \eta_{02})^2 + 4\eta_{11}^2 \tag{66}$$

$$\phi_3 = (\eta_{30} - 3\eta_{12})^2 + (3\eta_{21} - \eta_{03})^2 \tag{67}$$

$$\phi_4 = (\eta_{30} + \eta_{12})^2 + (\eta_{21} + \eta_{03})^2 \tag{68}$$

$$\begin{aligned}\phi_5 = {} & (\eta_{30} - 3\eta_{12})(\eta_{30} + \eta_{12})[(\eta_{30} + \eta_{12})^2 - 3(\eta_{21} + \eta_{03})^2] \\ & + (3\eta_{21} - \eta_{03})(\eta_{21} + \eta_{03})[3(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2]\end{aligned} \tag{69}$$

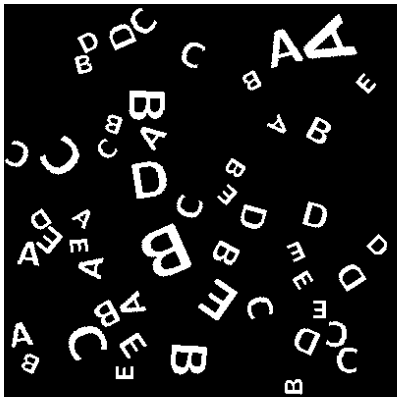
$$\phi_6 = (\eta_{20} - \eta_{02})[(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2] + 4\eta_{11}(\eta_{30} + \eta_{12})(\eta_{21} + \eta_{03}) \tag{70}$$

$$\begin{aligned}\phi_7 = {} & (3\eta_{21} - \eta_{03})(\eta_{30} + \eta_{12})[(\eta_{30} + \eta_{12})^2 - 3(\eta_{21} + \eta_{03})^2] \\ & + (3\eta_{12} - \eta_{30})(\eta_{21} + \eta_{03})[3(\eta_{30} + \eta_{12})^2 - (\eta_{21} + \eta_{03})^2]\end{aligned} \tag{71}$$

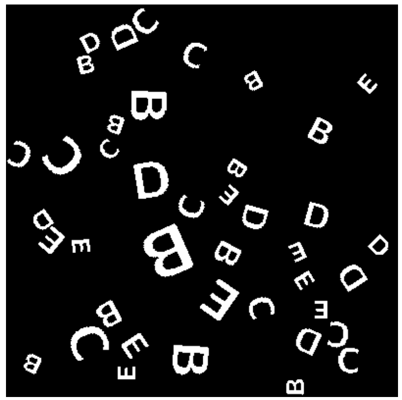
Note that these seven moment invariants are computed using central moments up-to(and including) order 3.

Filtering using Hu's Moment Invariants

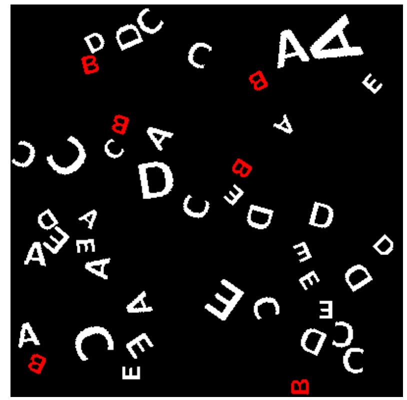
Using vector-attribute thinning with Hu's set of 7 moment invariants as vector-attribute to remove from image X the letters A, B, and C respectively.



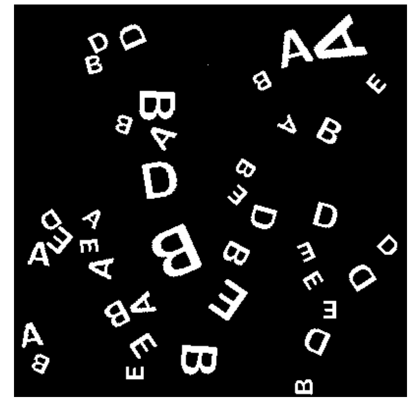
X



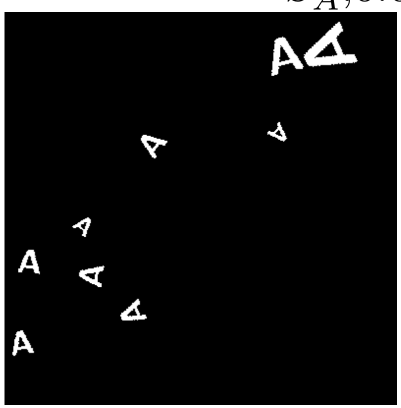
$\Phi^{\vec{\tau}}_{S_A,0.010}(X)$



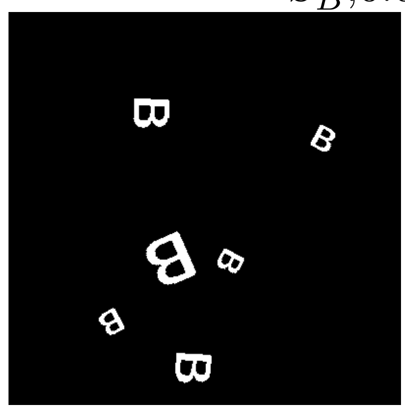
$\Phi^{\vec{\tau}}_{S_B,0.013}(X)$



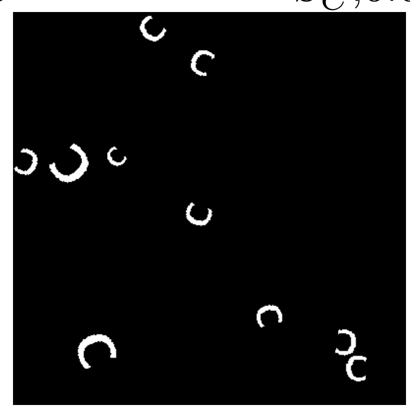
$\Phi^{\vec{\tau}}_{S_C,0.010}(X)$



$X - \Phi^{\vec{\tau}}_{S_A,\epsilon}(X)$



$X - \Phi^{\vec{\tau}}_{S_B,\epsilon}(X)$



$X - \Phi^{\vec{\tau}}_{S_C,\epsilon}(X)$

Vector Images

- All Max-Tree, Min-Tree, and level-line trees rely on a total order of the pixel values
- Max-Trees for vectorial images can be built using a total preorder

Definition 4. *A* **total preorder** *on $\mathcal{T}$ is any binary relation $\leq$ which is*

1. *reflexive:* $a \leq a$ *is true*
2. *transitive:* $a \leq b \wedge b \leq c \Rightarrow a \leq c$
3. *total:* $(a \leq b) \vee (b \leq a)$ *is true*

- To become a total order an extra property is needed

4. *antisymmetric:* $(a \leq b) \wedge (b \leq a) \Rightarrow a = b$

Marginal Processing Results

Lenna with noise

Marginal processing

Color Max-Tree Results

“All nodes mean”

“Only change filtered”

Binary Partition Trees

- Avoid the problem of total order by focusing on **differences**
- Start with homogenous regions as their leaves
- Simplest form: vector flat zones $F_{\vec{h}}$ containing point x can be defined as

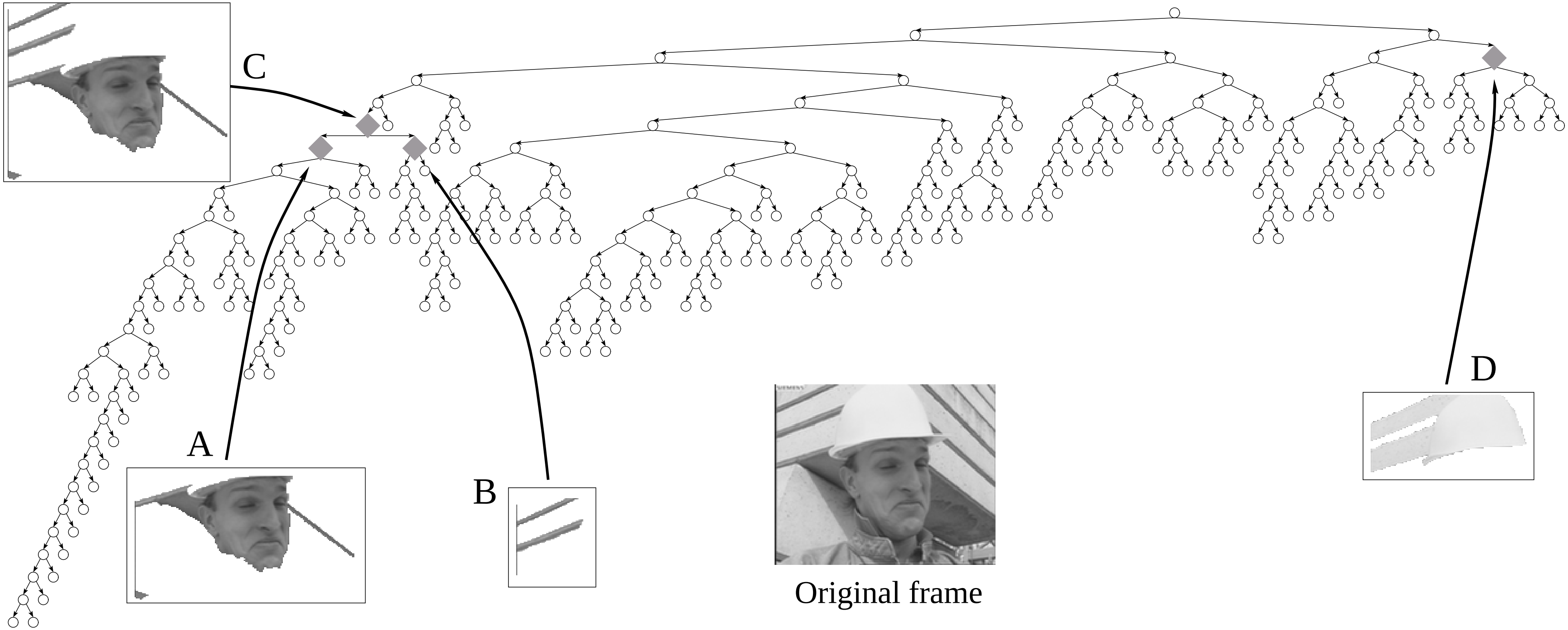
$$F_x(\vec{f}) = \Gamma_x(L_{\vec{f}(x)}) \tag{72}$$

with vector "level set" $L_{\vec{h}}$ defined as

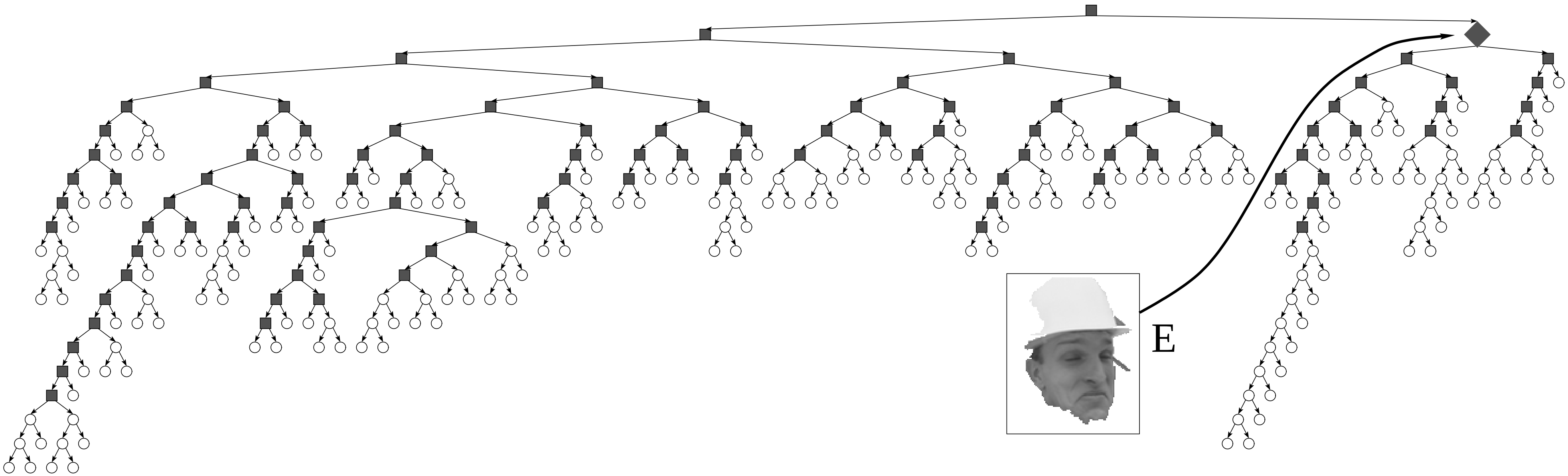
$$L_{\vec{h}} = \{x \in E | \vec{f}(x) = \vec{h}\}. \tag{73}$$

- Nodes at coarser levels are formed by hierarchical merger based on homogeneity criterion

Binary Partition Trees II



Merged based on color homogeneity criterion



Merged based on motion and color homogeneity criterion

Reconstruction Criteria

- One way to deal with leakage is to change the notion of connectivity
- An alternative to use reconstruction criteria (Terol-Villalobos and Vargas-Vázquez, *J. Electron. Imag.* 2005)
- This is implemented by performing an opening by a ball γ between each pair of conditional dilations, i.e.

$$\rho_\gamma(f|g) = \lim_{n\to\infty} \bar{\delta}_f^n g = \underbrace{\bar{\delta}_f^1 \gamma_B \ldots \bar{\delta}_f^1 \gamma_B \bar{\delta}_f^1 \gamma_B}_{\text{until stability}}(g). \tag{74}$$

- This means that at every step of the iteration, the growing region is restricted to a union of balls of the diameter of B.
- Denoted more compactly we have

$$\rho_\gamma(f|g) = (\bar{\delta}_f \gamma_B)^n g. \tag{75}$$

- This scheme has only been implemented in an iterative way.

Example: Cartoons with Criteria

original cartoon texture channel

Leveling cartoons for texture/cartoon decomposition are far better using criteria.

Complexity

- A key problem with the iterative approach is its computational complexity.
- Usually, very many iterations are required before stability.
- It is possible to construct an image in which 50% of the N pixels must be reconstructed, *and*
- in which at each iteration *only one* pixel is assigned its final value.
- Each iteration is $O(N)$, so if $N/2$ iterations are needed, the algorithm is $O(N^2)$.
- For regular reconstruction, fast algorithms which are $O(N)$ in practice exist (Vincent, *IEEE Trans. Image Proc.* 1993).
- When using reconstruction criteria these algorithms do not work.

Viscous Reconstruction

- We can consider the reconstruction criteria as using a viscous fluid modelled by balls B.
- We can study the centroids of these balls by rewriting (75) as

$$\rho_\gamma(f|g) = (\bar{\delta}_f \gamma_B)^n g = (\bar{\delta}_f \delta_B \epsilon_B)^n g. \tag{76}$$

- At each step, the centroids of these balls are obtained just after the erosion by B.
- Because the erosion is applied to a subset of f at each step, these can only lie in the region defined by the erosion $\epsilon_B f$.
- So as an alternative, we might reconstruct $\epsilon_B f$ from an appropriate marker.
- We follow this by a dilation by B and a conditional dilation $\bar{\delta}_f$ as post-processing.
- This change in processing order does not guarantee an identical result.

Viscous Reconstruction II

- We poceed as follows:
  - First erode both the marker g and the image f by B.
  - Then reconstruct the erosion of f using the erosion of g as marker.
  - The process reconstructs any connected component of $\epsilon_B f$ intersected by $\epsilon_B g$.
  - We now have the collection of centroids reached by the flooding process.
  - Dilate the reconstructed region to obtain the balls themselves,
  - Follow by a last conditional dilation.
- We can therefore define this approximate operator ρ'_γ as

$$\rho'_\gamma(f|g) = \bar{\delta}_f \delta_B \rho(\epsilon_B f|\epsilon_B g). \tag{77}$$

- The cost of this operator is that of two erosions, one dilation, one conditional dilation (all $O(N)$) and an ordinary reconstruction (also $O(N)$ in practice).

But is it the same?

$\rho_\gamma(f|g)$ $\rho'_\gamma(f|g)$ difference

A comparison. Note that the difference image is contrast-stretched $(32\times)$.

An Experiment

- To measure the speed difference two test images were chosen: a street scene and an image of a comet (both 3 megapixel).
- For each image, markers were created by performing openings with Euclidean discs of diameters ranging from 11 to 161 pixels
- Reconstruction with reconstruction criteria were computed using balls ranging from 3 pixels to 45 pixels diameter.
- For all S.E. opperations the algorithms of Urbach and Wilkinson (*IEEE Trans. Image Proc.* 2008) were used.
- A Max-tree based reconstruction method was used for $\rho'_\gamma(f|g)$
- Timings were performed on a Core 2 Quad machine running at 2.4 GHz with 2 GB of RAM.

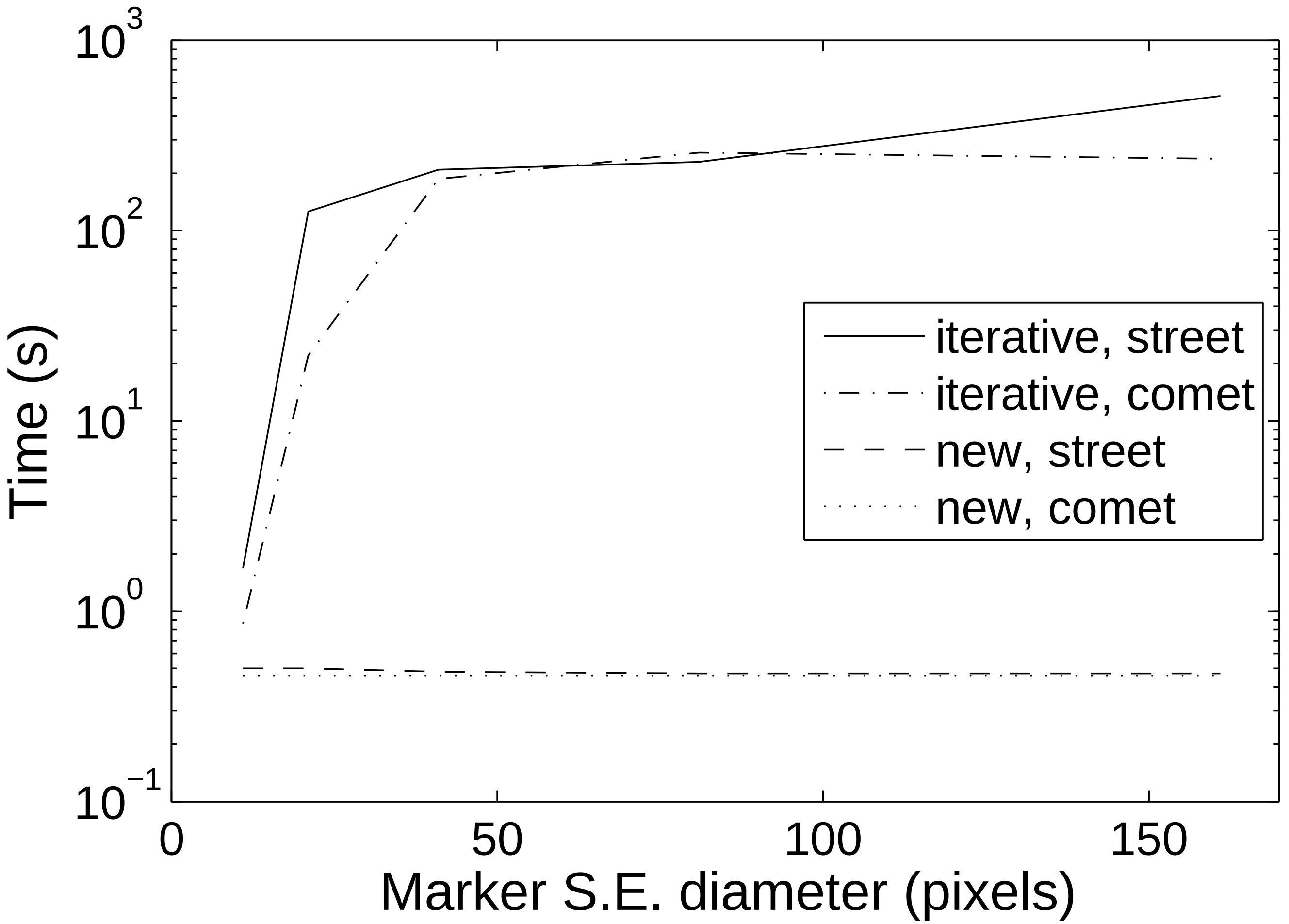
Test Images

street

comet

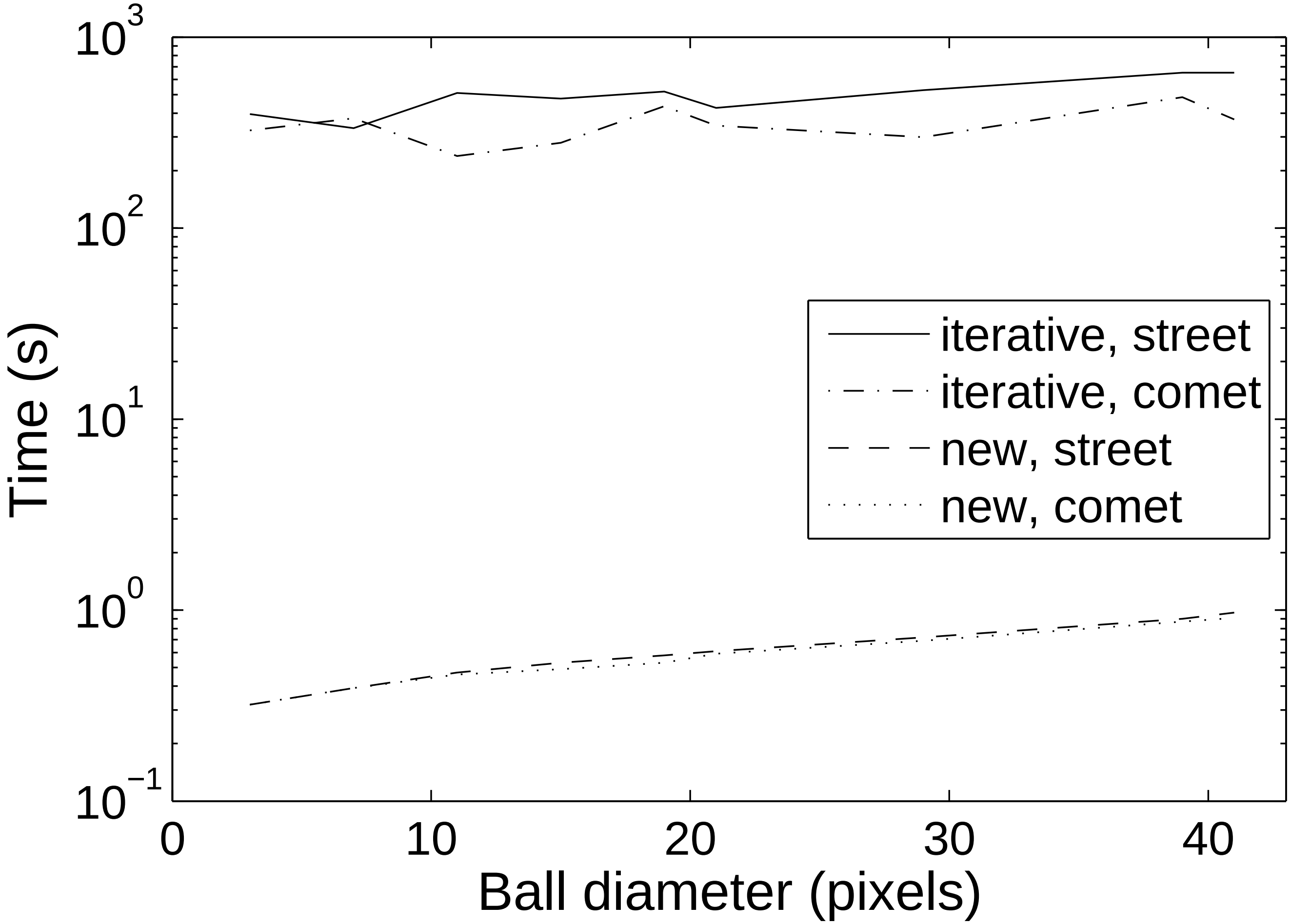
The two 3-megapixel test images

Timings: $\rho_\gamma(f|g)$ vs $\rho'_\gamma(f|g)$



- The computing times for a criterion using a ball of diameter 11 as a function of the S.E. diameter used to obtain the marker.

Timings: $\rho_\gamma(f|g)$ vs $\rho'_\gamma(f|g)$



- The computing times for a fixed marker (obtained with and S.E. of 161 pixels diameter) as a function of the diameter of the ball B used for the criterion.

Other work

- Morphological connected hat scale-spaces based on Max-trees have been constructed for contour and texture analysis.
- The C-trees for multi-scale connectivity analysis of binary images as suggested by Tzafestas & Maragos (2003) can be implemented rapidly as Max-trees of opening transforms.
- Derived connectivities (i.e. using openings or closings) can be incorporated into the Max-tree by constructing the tree not from one, but from two images. The second image encodes the altered connectivity.
- Extending the attributes for shape filtering.
- Making shape filters trainable by examples.
- Other vector approaches are being developed
- Parallel algorithms for the Max-tree have been developed

Questions

?