Game Theoretical Approaches to Modelling and Simulation

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Overview

- Context
- A Classic: Hawks vs. Doves
- Pay-off matrices
- Nash-equilibria and Evolutionary Stable Strategies
- Pure vs. Mixed Strategies
- Population Games
- Replicator dynamics
- Asymmetric games: Bimatrix models
- Adaptive dynamics
- Evolution of cooperation

Context

- Game theoretical models are commonly used in three contexts:
  1. Evolutionary theory
  2. Economics
  3. Sociology
- Less-known fields are e.g. in A.I. (deciding optimal strategies) and in computer games.
- The games involved are usually very simple: two players, only a few types of moves.

Hawks vs. Doves I

- The Hawks vs. Doves game tries to explain ritualized “fighting” behaviour in many animals.
- Many animal species rarely cause real damage during conflicts:
  - King-cobra males try to push each-other’s heads to the ground rather than biting
  - Stags have roaring matches, walk parallel, and sometimes lock horns and push rather than trying to kill each-other
  - When cats fight they scream more than they actually fight
- The more dangerous the species, the more ritual the fight becomes.
- Naive reasoning using “survival of the fittest” suggest that killing rivals when an opportunity arises would be common

Hawks vs. Doves II

- Suppose two rivals of the same species meet at the location of some resource with value G.
- Suppose there are two “pure” strategies:

1. Hawk: you always escalate the conflict until either the other withdraws, or you are badly hurt.
2. Dove: you posture until the other withdraws, but you withdraw if the other escalates or seems too strong.

- If two hawks meet, one wins G but the other loses cost C, with $G < C$.
- If two doves meet, one withdraws and loses nothing, and the other wins G.
- If a hawk meets a dove, the dove loses nothing and the hawk wins G.

Hawks vs. Doves III

- We can formalize this in a *payoff matrix*:

	if it meets a hawk	if it meets a dove
a hawk receives	$\frac{G-C}{2}$	G
a dove receives	0	$\frac{G}{2}$

- In each cell the average payoff is listed for a given pair of strategies.
- We can now study the optimal behaviour, which depends on the population structure:
  - If doves dominate, it pays to be a hawk, because you almost always win G, vs. $\frac{G}{2}$ for doves.
  - If hawks dominate, it pays to be a dove, because you almost always win $0 > \frac{G-C}{2}$.

Hawks vs. Doves IV

- For a given population with fraction p of hawks and $1 - p$ of doves, the payoff for hawks P_h becomes:

$$P_h = \frac{G - C}{2} p + (1 - p) G \tag{1}$$

- Under the same assumptions, the payoff for doves P_d is

$$P_d = (1 - p) \frac{G}{2} \tag{2}$$

- The population reaches stability when $P_h = P_d$ or

$$(1 - p) \frac{G}{2} = \frac{G - C}{2} p + (1 - p) G \quad \Rightarrow p = \frac{G}{C} \tag{3}$$

- So if the cost C is high (the animals are dangerous) hawks becomes rare.

Hawks vs. Doves V

- Let us introduce two new strategies:
  1. bullies: Simulate escalation but run away if the other fights back.
  2. retaliators: behaves like dove *unless* the other escalates, then it fights back.
- The payoff matrix becomes:

if it meets	a hawk	a dove	a bully	a retaliator
a hawk receives	$\frac{G-C}{2}$	G	G	$\frac{G-C}{2}$
a dove receives	0	$\frac{G}{2}$	0	$\frac{G}{2}$
a bully receives	0	G	$\frac{G}{2}$	0
a retaliator receives	$\frac{G-C}{2}$	$\frac{G}{2}$	G	$\frac{G}{2}$

Hawks vs. Doves VI

- Let x_b, x_d, x_h, x_r, denote the numbers of bullies, doves, hawks and retaliators, and P_b, P_d, P_h, P_r their payoffs.
- The payoffs become:

$$P_h = (x_r + x_h)\frac{G - C}{2} + (x_b + x_d)G \tag{4}$$

$$P_d = (x_d + x_r)\frac{G}{2} \tag{5}$$

$$P_b = x_d G + x_b \frac{G}{2} \tag{6}$$

$$P_r = x_h \frac{G - C}{2} + (x_d + x_r)\frac{G}{2} + x_b G \tag{7}$$

Hawks vs. Doves VII

- Demanding that $P_b = P_d$ at stability means that

$$x_r = x_b + x_d \tag{8}$$

- If we insert this into (4) and (7), and then require that $P_h = P_r$ we arrive at

$$x_r(G - C) = x_b G \tag{9}$$

- This can only be the case if the signs of x_r and x_b are different which is meaningless in this context.
- Therefore, no 4-strategy equilibrium exists in this case.

Hawks vs. Doves VIII

- If $x_b = 0$, equating P_h and P_r yields

$$x_r = \frac{G}{C} x_d \tag{10}$$

- inserting this into (4) and (5) yields $x_h = 0$
- If $x_r = 0$, equating P_b and P_d yields that $x_b = -x_d$ which is again meaningless.
- The two-strategy options:
  - Doves and retaliators can live together in any ratio (why?)
  - Retaliators drive hawks to extinction (why?)
  - Retaliators drive bullies to extinction (why?)
  - Bullies drive doves into extinction
  - Bullies and hawks achieve the same equilibrium as hawks and doves.

Evolutionary Stable Strategies I

- A strategy S is an *Evolutionary Stable Strategy* (ESS) if a pure population of S-strategists is stable against invasion by *any* other strategy.
- Let the fitness of an individual with strategy S_i is a population P be denoted $F(S_i, P)$.
- A population consisting of a mixture of individuals with N strategies P may be denoted as $\sum_{j=1}^{N} p_j S_j$ with p_j the fraction of individuals with strategy S_j, and $\sum_{j=1}^{N} p_j = 1$.
- for a pure population of S_i-strategists we have $P = S_i$.
- S_i is an ESS if

$$F(S_i, S_i) > F(S_j, S_i) \quad \forall j \neq i \tag{11}$$

Evolutionary Stable Strategies II

- In the case above, neither hawks, doves, bullies nor retaliators are an ESS.
- ESSs are NOT unique: there are cases in which

$$F(S_1, S_1) > F(S_2, S_1) \quad \text{and} \quad F(S_2, S_2) > F(S_1, S_2) \tag{12}$$

- An example is that of toxin producing bacteria and their susceptible counterparts
- The payoffs $F(S, P)$ and $F(T, P)$ of the susceptible strategy S and toxin producing strategy T in a population $P = xS + (1 - x)T$ are

$$F(S, P) = \mu - (1 - x)\kappa \quad \text{and} \quad F(S, P) = \mu - \epsilon \tag{13}$$

with μ the relative growth rate, κ the toxin kill rate and ϵ the cost of toxin production.

- If $\epsilon < \kappa$, $F(S, S) = \mu > \mu - \epsilon = F(T, S)$ and $F(T, T) = \mu - \epsilon > \mu - \kappa = F(S, T)$.

Normal Form Games I

- In general, if N pure strategies are available we can allow mixed strategies S by assigning each strategy S_i a given probability p_i to be used by an individual.
- each strategy is now a point in the simplex

$$R_N = \left\{ \mathbf{p} = (p_1, p_2, \ldots, p_N), \sum_{i=1}^{N} p_i = 1 \right\} \tag{14}$$

- The corners of the simplex are given by the unit vectors along the coordinate axes: the pure strategies.
- The interior of the simplex consist of completely mixed strategies (all $p_i > 0$)
- On the boundary of the simplex are those strategies for which one or more $p_i = 0$

Normal Form Games II

- Each pure-strategy player using S_i extracts a payoff u_{ij} against a pure-strategy player using S_j.
- The $N \times N$ matrix $\mathbf{U} = (u_{ij})$ is the payoff matrix.
- An S_i-strategist extracts a payoff P_i against a $\mathbf{p}$-strategist of

$$P_i = (\mathbf{Up})_i = \sum_{j=1}^{N} u_{ij} p_j \tag{15}$$

- For a mixed strategy $\mathbf{q}$-player the payoff $P_{\mathbf{q}}$ against a $\mathbf{p}$-strategist becomes

$$P_{\mathbf{q}} = \mathbf{q} \cdot \mathbf{Up} = \sum_{j=1}^{N} \sum_{i=1}^{N} u_{ij} q_i p_j \tag{16}$$

Nash Equilibria

- We denote the *best replies* to strategy $\mathbf{p}$ as $\beta(\mathbf{p})$
- This is the set of strategies $\mathbf{q}$ for which the map $\mathbf{q} \to \mathbf{q} \cdot \mathbf{U}\mathbf{p}$ attains the maximum value.
- If

$$\mathbf{p} \in \beta(\mathbf{p}), \tag{17}$$

  p is a *Nash equilibrium*.
- Every normal form game has at least one Nash equilibrium.
- If

$$\mathbf{q} \cdot \mathbf{U}\mathbf{p} < \mathbf{p} \cdot \mathbf{U}\mathbf{p}, \forall \mathbf{q} \neq \mathbf{p}, \tag{18}$$

  $\mathbf{p}$ is a *strict* Nash equilibrium.
- All strict Nash equilibria are pure strategies

Nash Equilibria and ESS

- If a Nash equilibrium $\mathbf{p}$ is strict, any population consisting only of $\mathbf{p}$-strategists cannot be invaded by other strategies.
- Such strict Nash equilibria are ESS!
- Non-strict Nash equilibria are not necessarily ESS.
- ESS requires TWO conditions:

1. the *equilibrium condition*

$$\mathbf{q} \cdot \mathbf{U}\mathbf{p} \leq \mathbf{p} \cdot \mathbf{U}\mathbf{p}, \quad \forall \mathbf{q} \in R_N \tag{19}$$

2. the *stability condition*

$$\mathbf{q} \cdot \mathbf{U}\mathbf{q} < \mathbf{p} \cdot \mathbf{U}\mathbf{q}, \quad \text{if } \mathbf{q} \neq \mathbf{p}, \text{ and } \mathbf{q} \cdot \mathbf{U}\mathbf{p} = \mathbf{p} \cdot \mathbf{U}\mathbf{p} \tag{20}$$

Population Games: The Sex-Ratio Game

- Why do animals (including us) produce offspring of both sexes in equal numbers?
- Why is this a problem? Because populations with more females than males could reproduce faster (ask any farmer).
- However, if there are mostly females, then producing more males as offspring would be the better strategy:
  - some females might never find males
  - all males are likely to find females
- If males dominate producing females is better.
- Thus, the payoff of a strategy depends on the population, rather than on a single individual's strategy.
- This means, the Sex-Ratio Game is a Population Game.

The Sex-Ratio Game II

- Consider the payoff in terms of the number of *grandchildren*
- Let p be the sex-ratio of a given individual, and m the average sex ratio in the population
- Let the number of children of the current population be $N1$, of which mN_1 are males, and $(1-m)N_1$ females.
- Let the number of grandchildren of the current generation be N_2.
- Each male child of the current generation produces an average of N_2/mN_1 of these grandchildren
- Each female child $N_2/(1-m)N_1$.
- The number of grandchildren $N_{2,p}$ of an individual with sex ratio p is

$$N_{2,p} = p\frac{N_2}{mN_1} + (1-p)\frac{N_2}{(1-m)N_1} \tag{21}$$

The Sex-Ratio Game III

- The fitness is proportional to

$$w(p,m) = \frac{p}{m} + \frac{1-p}{1-m} \tag{22}$$

- If the population consists of a individuals with a sex ratio of q, is this evolutionary stable?
- Suppose a small fraction ϵ of the population mutates into some other value p for the sex ratio. The population mean m is then

$$m = \epsilon p + (1-\epsilon)q \tag{23}$$

- The q-strategy is only evolutionary stable if $w(p,m) < w(q,m)$.
- If $q = 0.5$ it can easily be shown that this holds true for all $p \neq q$.

Population Games

- This notion can be generalized by stating that the payoff P_i for a pure strategy player S_i depends on the probabilities of meeting a particular strategy in the population.
- More formally: P_i is a function of the frequencies m_j with which the *pure strategies* S_j are used *in the population*.
- Note that $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ is just a point in the simplex R_N.
- The payoff $P_{\mathbf{p}}$ for a $\mathbf{p}$-strategist is

$$P_{\mathbf{p}} = \sum_{i=1}^{N} p_i P_i(\mathbf{m}) = \mathbf{p} \cdot \mathbf{P}(\mathbf{m}). \tag{24}$$

Replicator Dynamics I

- These describe the evolution of the frequencies with which different strategies occur in the population
- Assume we have n types in the population with frequencies x_1 to x_n.
- As above, the fitness f_i will be a function of the state $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
- We assume that the population is large enough to model the rate of change of $\mathbf{x}$ as a function of time by differential equations.
- We assume the change in frequency is linked to its fitness compared to the average population fitness

$$\frac{1}{x_i}\frac{dx_i}{dt} = f_i(\mathbf{x}) - \bar{f}(\mathbf{x}) \quad \text{or} \quad \frac{dx_i}{dt} = (f_i(\mathbf{x}) - \bar{f}(\mathbf{x}))x_i \tag{25}$$

Replicator Dynamics II

- Assume that the replicator dynamics derive from a normal form game with N pure strategies $S_1, S_2, \ldots, S_N$ (note $N \neq n$), and payoff matrix $\mathbf{U}$.
- The n types in the population correspond to (possibly mixed) strategies $\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^n$.
- The state of the system is given by $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, which is a point in simplex R_n (*not* R_N).
- Let matrix $\mathbf{A} = (a_{ij}) = (\mathbf{p}^i \cdot \mathbf{U}\mathbf{p}^j)$.
- The fitness f_i now becomes

$$f_i(\mathbf{x}) = \sum_{j=1}^{n} a_{ij} x_j = (\mathbf{A}\mathbf{x})_i \tag{26}$$

Replicator Dynamics III

- The replicator equation (25) becomes

$$\frac{dx_i}{dt} = (((Ax)_i - \mathbf{x} \cdot \mathbf{A}\mathbf{x})x_i \qquad (27)$$

- We now say that a point $\mathbf{y} \in R_n$ is a *Nash equilibrium* if

$$\mathbf{x} \cdot \mathbf{A}\mathbf{y} \leq \mathbf{y} \cdot \mathbf{A}\mathbf{y} \quad \forall \mathbf{x} \in R_n \qquad (28)$$

- It is an *evolutionary stable state* (not strategy!) if

$$\mathbf{x} \cdot \mathbf{A}\mathbf{x} < \mathbf{y} \cdot \mathbf{A}\mathbf{x} \quad \forall \mathbf{x} \neq \mathbf{y} \qquad (29)$$

in a neighbourhood of $\mathbf{y}$.

Replicator Dynamics IV

- A simple example of replicator dynamics is in the Hawk-Dove game
- For two strategies we can simplify the replicator equation by replacing x_1 by x and x_2 by $1 - x$. We obtain

$$\frac{dx}{dt} = x(1-x)((\mathbf{Ax})_1 - (\mathbf{Ax})_2) \tag{30}$$

- In the Hawk-Dove case we obtain

$$\frac{dx}{dt} = x(1-x)(G - Cx) \tag{31}$$

- This has a point attractor G/C for the domain $]0,1[$.

Replicator Dynamics V

- More complex dynamics are obtained with the so called Rock-Scissors-Paper game.
- We now have a *zero sum* payoff matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{32}$$

- This yields replicator equations

$$\frac{dx_1}{dt} = x_1(x_2 - x_3) \tag{33}$$

$$\frac{dx_2}{dt} = x_2(x_3 - x_1) \tag{34}$$

$$\frac{dx_3}{dt} = x_3(x_1 - x_2) \tag{35}$$

Replicator Dynamics V

- This yields equilibria at

$$\mathbf{x} = (1/3, 1/3, 1/3) \quad (36)$$

$$\mathbf{x} = (1, 0, 0) \quad (37)$$

$$\mathbf{x} = (0, 1, 0) \quad (38)$$

$$\mathbf{x} = (0, 0, 1) \quad (39)$$

  (how do we find these?)

- The system will typically oscillate in orbits in which $x_1 x_2 x_3 = c$, centred on $(1/3, 1/3, 1/3)$.

Asymmetric games: Bimatrix models

- In the previous cases we have looked at symmetric games:
  - If moves are interchanged from player to player, so are the payoffs
  - Modelling is done using a single payoff matrix
- In practice, games between players may be asymmetric
- The goals may be different to players
- The values of resources may be different to different players: why is a hare faster than a fox? A hare runs for his life, a fox for his meal!
- The roles may be different (e.g. parent – child)
- To model such games we use *two* payoff matrices, or a *bimatrix*.

Bimatrix models: Battle of the Sexes I

- A classic example for a bimatrix game is the battle of the sexes, which concerns parental investment in the offspring
- For males who abandon the females after mating can go on to mate with other females.
- Females could prevent this by being "coy", demanding an investment E from the male before mating, during a so-called "engagement" period.
- After such an investment, it would pay more for a male to help raise his young (because he is now relatively sure they are his), rather than find another mate (by which time the mating season may be over).
- However, once all males have been selected for faithfulness, a "fast" female, who will mate without engagement cost will gain
- This in turn leads to the appearance of "philandering" males, who will mate and desert the females to mate with another

Bimatrix models: Battle of the Sexes II

- We can formalize this:
  - If a coy female mates with a faithful male, they both win gain G and share the cost of upbringing C, and both pay E, so each wins $G - C/2 - E$.
  - If a fast female mates with a faithful male, they both win gain G and share the cost of upbringing C, without the cost E, so each wins $G - C/2$.
  - If a fast female meets a philandering male, she gets $G - C$, whereas he gets G.
  - If a coy female encounters a philandering male, she refuses to mate, so both receive 0
- In terms of payoff matrices we have

$$\mathbf{A} = \begin{bmatrix} 0 & G \\ G - \frac{C}{2} - E & G - \frac{C}{2} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 & G - \frac{C}{2} - E \\ G - C & G - \frac{C}{2} \end{bmatrix} \tag{40}$$

  with $\mathbf{A}$ the matrix for males and $\mathbf{B}$ the matrix for females
- It turns out there is no stable equilibrium for this case.

Adaptive dynamics

- In the previous models we only look at single encounters
- Strategies change in a population through competition or copying
- Each strategy consists of a fixed set of probabilities for each move
- Unless mutations are allowed, no new strategies are developed
- In reality, players may change their strategy, depending on previous experience.
- Adaptive strategies are formulated differently;
  - The set of probabilities for moves is a function of the state of the player
  - The state of a player depends on previous games, either of the player himself, or those of others.
- The resulting adaptive dynamics can be highly complex

Evolution of Cooperation

- One field in which adaptive dynamics are important is that of the emergence of cooperation
- This emergence might concerns the evolution of cooperative behaviour in animals (not just the lack of aggression as in the Hawk-Dove game)
- It might also be the cooperation in economics and sociology: formation of coalitions, companies, etc.
- The core question is always: Why, when faced with an easy quick win at the expense of another, do many people or animals take a lower profit which does not harm the other.
- Another way of looking at the problem might be: why do we have such a strong feeling of fairness? Why do we get angry seeing someone cheat another when he should have shared?
- It turns out that single encounter games cannot solve this problem

Iterated Prisoner's Dilemma (IPD) I

- Iterated prisoner's dilemma is the classic example for adaptive strategies *and* the evolution of cooperation
- Prisoner's dilemma is a simple two player game in which there are two possible moves: cooperate (C) or defect (D)
- If both players cooperate, they receive a reward R
- If both players defect, they receive a punishment P
- If a player defects, but the other cooperates, he receives a temptation T
- If a player cooperates and the opponent defects, he receives the sucker's reward S
- In all cases we assume

$$T > R > P > S \quad \text{and} \quad 2R > T + S \tag{41}$$

IPD II

- The payoff matrix is

	C	D
C	R	S
D	T	P

- If the game is played once: the best strategy is always defect (*AllD*):
  - If the other cooperates, cooperating gets you R.
  - If the other cooperates, defecting gets you $T > R$.
  - If the other defects, cooperating gets you S.
  - If the other defects, defecting yourself gets you $P > S$.

  Therefore, you are always better off defecting
- This basically formalizes the selfishness problem

IPD III

- Now consider the case when you will encounter the same player again, with a probability w.
- Let A_n denote the payoff in the n-th round, the total expected payoff is given by

$$A = \sum_{n=0}^{\infty} A_n w^n \qquad (42)$$

- In the limiting case of $w = 1$, this diverges, so instead we take the limit of the mean

$$A = \lim_{n \leftarrow \infty} \frac{\sum_{n=0}^{N} A_n w^n}{N+1} \qquad (43)$$

- Obviously, if w is very small, each player should still just defect, since the possibilities for revenge are small.

IPD IV: Classifications of Strategies

- Strategies in IPD are programs which tell you which move to make in each round.
- Strategies are sometimes classified as:

  Nice: Does not defect first
  Retaliatory: Punishes defection
  Forgiving: Returns to cooperation following cooperation of opponent
  Suspicious: Does not cooperate until the other cooperates
  Generous: Does not always retaliate at a first defection

- No best strategy exist, it all depends on the opponent

IPD V

- If the opponent is an *AllC* player, *AllD* is best, because its payoff will be

$$A = \sum_{n=0}^{\infty} T w^n = \frac{T}{1-w} \tag{44}$$

- However, if the opponent is *Grim*, who is nice, retaliatory and totally unforgiving, the payoff after your first defection will be

$$A = \sum_{n=0}^{\infty} P w^n = \frac{P}{1-w} \tag{45}$$

  at best!

- This means *AllC* would perform better ($A = R/(1-w)$), provided

$$w > \frac{T-R}{T-P} \tag{46}$$

IPD VI: Tit-For-Tat

- A simple strategy which does very well in round-robin tournaments (each player competes in turn with each other player) is *Tit-For-Tat* (TFT).
- Curiously, TFT never gets *more* points per game than its opponent.
- It starts of with C, so it is nice
- It then copies the opponents last move
- This behaviour makes it retaliatory, because a defection will be repayed by a defection
- It is also forgiving, because it will return to playing C if the opponent returns to C
- It can outcompete a population of *AllD* and gain dominance if

$$w \geq \max\left(\frac{T-R}{T-P}, \frac{T-R}{R-S}\right) \tag{47}$$

IPD VII: Pavlov

- TFT has two weaknesses:
  1. It is sensitive to noise: if there is a small probability of a message (C or D) being misinterpreted, two TFT players enter into a round of mutual retaliations
  2. It is sensitive to invasion by other strategies such as *Pavlov*, or any nice strategy.
- *Pavlov* takes both his own and the opponents last move into account to compute the next
- This can be formalized as a function of the last *reward*
  - If the last reward is R, play C
  - If the last reward is P, play C
  - If the last reward is S, play D
  - If the last reward is T, play D
- In effect, *Pavlov* retains his strategy after high payoff (T or R) and changes strategy after low payoff (S or P).
- It can correct for occasional mistakes
- Strict cooperators cannot invade

IPD VIII: TFT variants

- The success of TFT resulted in the development of some variants
- *Tit-For-Two-Tats* (TF2T), which retaliates only after two Ds.
  - More generous
  - More tolerant for errors
  - Co-exists with TFT
- *Suspicious-Tit-For-Tat* (STFT) which starts with D instead of C, so it is not nice (gets on with TFT like a house on fire).
- *Observer Tit-For-Tat* Uses observations of potential opponents in other games to decide whether to start with D or C.
  - Requires the possibility of observations
  - Suppresses “Roving” strategies (*AllD* strategies which try to reduce w by selecting new opponents)

IPD IX: Stochastic Strategies

- Rather than using strict strategies, we can define probabilities with which strategies are used.
- This models:
  - Noise in the communications process
  - Faulty memory
- It also has the advantage that the importance of the initial move is lost after a sufficient number of moves.
- One way to define stochastic strategies is by defining 2-tuples (p, q) which denote the probabilities of a C after an opponent's C or D (respectively).
- Nowak and Sigmund (1992) found that *generous* TFT with $p = 1$ and

$$q = \min\left(1 - \frac{T-R}{R-S}, \frac{R-P}{T-P}\right) \tag{48}$$

was the optimum in the case of $w = 1$.

IPD X: Stochastic Strategies

- Note that *Grim* cannot be modelled using the 2-tuple approach
- The above stochastic model can be extend to include more strategies.
- By setting the probability of cooperation after receiving a reward $R, S, T,$ or P we can devise a large space of possible strategies, *including*
  - TFT: $(1, 0, 1, 0)$
  - *Pavlov*: $(1, 0, 0, 1)$
  - *Grim*: $(1, 0, 0, 0)$
  - *AllD*: $(0, 0, 0, 0)$
  - *AllC*: $(1, 1, 1, 1)$
- Strictly speaking, we should also add a fifth probability, i.e. the probability for C on the first move.

IPD XI: Experiments

- Using 100 random starting points in the 2-tuples-models, and a genetic algorithm using payoff as fitness function was implemented.
- Initially *AllD*-like strategies increased rapidly and *AllC*-like “suckers” were removed.
- Then, if sufficient TFT-like strategies were in the initial population, they eradicated the *AllD*-like strategies
- After this GTFT appeared and started to dominate.
- Similar results were obtained using the 4-tuple approach, but here Pavlov could appear, and did so (it was discovered this way).