THE GENERALIZED TAILOR PROBLEM *

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Abstract. The so-called 'Tailor Problem' concerns putting a number of sets within another set by translation, such that the translated sets do not overlap. In this paper we consider a generalization of this problem in which also rotations of the sets are allowed.

Key words: Tailor problem, Minkowski operations, group morphology.

1. Introduction

The goal of this paper is to give a solution by morphological operators to the following *Generalized Tailor Problem*:

Problem *Given a set X and a collection of sets $A_1, A_2, \ldots, A_n$, is it possible to put $A_1, A_2, \ldots, A_n$ within X using translations and rotations such that no two of the translated and/or rotated sets intersect? If so, what are the possible solutions?*

The problem where only translations are allowed (the *Tailor Problem*) was posed by Serra [5], see also [2]. He obtained an elegant solution in terms of Minkowski operations.

Our solution of the Generalized Tailor Problem involves a general construction of morphological operators on spaces with transitive transformation groups [4]. In the original case, the group is given by the translation group, which acts on the plane. When also rotations are allowed, the group becomes the Euclidean motion group. The methods of this paper can also be used for spaces with a symmetry group different from the Euclidean motion group.

Using methods from computational geometry, Li and Milenkovic [1] study the related problem of constructing the smallest rectangle that will contain a given set of parts, with applications to making cutting plans for clothing manufacture.

2. The Tailor Problem

In this section we summarize the solution of the Tailor Problem as obtained by Serra [5] for the case of one, two or three sets — called 'pieces' — to be put into a given set. All sets are subsets of $E = \mathbb{R}^n$ or $E = \mathbb{Z}^n$.

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2.1. One piece

There is one set A which is to be put inside a set X. The solubility of the problem depends on the non-emptiness of the following set:

$$R_1(X; A) := X \ominus A, \tag{1}$$

which is simply the erosion of X by A. The set $R_1(X; A)$ is called the *residue* of X w.r.t. A.

2.2. Two pieces

Now there are two sets $A_1 = A, A_2 = B$ which are to be put inside a set X. That is, we are looking for $a, b \in E$ such that

$$A_a \subseteq X; \qquad B_b \subseteq X \setminus A_a. \tag{2}$$

The solubility of the problem now depends on the non-emptiness of the following residue:

$$R_2(X; A, B) := (X \ominus B) \cap [(X \ominus A) \oplus (A^c \ominus B)]. \tag{3}$$

After a choice $b \in R_2(X; A, B)$ to put B into position, the translation vector a can be chosen from the set $R_1(X \setminus B_b; A) = (X \setminus B_b) \ominus A$.

2.3. Three pieces

In this case there are three sets $A_1 = A, A_2 = B, A_3 = C$ which are to be put inside a set X. That is, we are looking for $a, b, c \in E$ such that

$$A_a \subseteq X; \qquad B_b \subseteq X \setminus A_a; \qquad C_c \subseteq (X \setminus A_a) \setminus B_b. \tag{4}$$

The solubility of the problem depends on the non-emptiness of the following residue:

$$R_3(X; A, B, C) := \bigcup_{a \in X \ominus A} \; \bigcup_{b \in X \ominus B} \delta_{A,B,C}(x, y), \tag{5}$$

where

$$\delta_{A,B,C}(a, b) = \begin{cases} (X \ominus C) \cap (A^c \ominus C)_a \cap (B^c \ominus C)_b, & A_a \cap B_b = \emptyset \\ \emptyset & else \end{cases} \tag{6}$$

It can be shown that the following recursive expression holds:

$$R_3(X; A, B, C) = \bigcup_{a \in X \ominus A} R_2(X \setminus A_a; B, C). \tag{7}$$

After a choice $c \in R_3(X; A, B, C)$ to put C into position, the translation vector b can be chosen from the set $R_2(X \setminus C_c; A, B)$; finally, a can be chosen from $R_1(X \setminus (B_b \cup C_c); A)$. The generalization to n pieces is straightforward, cf. [5].

3. Group morphology

3.1. Generalized Minkowski operators

On any group $\mathbf{\Gamma}$ one can define generalizations of the Minkowski operations [4]. For any subsets G, H of $\mathbf{\Gamma}$ define the $\mathbf{\Gamma}$-dilation and $\mathbf{\Gamma}$-erosion by

$$\delta(G) := G \overset{\Gamma}{\oplus} H := \bigcup_{h \in H} Gh = \bigcup_{g \in G} gH, \tag{8}$$

$$\epsilon(G) := G \overset{\lambda}{\ominus} H := \bigcap_{h \in H} Gh^{-1}. \tag{9}$$

Here

$$gH := \{gh : h \in H\}, \quad Gh := \{gh : g \in G\}, \tag{10}$$

with gh the group product of g and h, and h^{-1} is the group inverse of h. Both mappings are *left-invariant*, e.g. $\delta(gG) = g\delta(G), \quad \forall g \in \mathbf{\Gamma}$. This is the reason for the superscript 'λ' on the '$\ominus$' symbol.

3.2. Group actions and morphological operations

Let E be a non-empty set, $\mathbf{\Gamma}$ a transformation group (or group action) on E [6]. Each $g \in \mathbf{\Gamma}$ maps a point $x \in E$ to a point $gx \in E$. The group $\mathbf{\Gamma}$ is called *transitive on* E if for each $x, y \in E$ there is a $g \in \mathbf{\Gamma}$ such that $gx = y$, and *simply transitive* when this element g is unique. The translate of a set $A \subseteq E$ by $g \in \mathbf{\Gamma}$ is defined by $gA := \{ga : a \in A\}$. If $\mathbf{\Gamma}$ acts on E, the *stabilizer* of $x \in E$ is the subgroup $\mathbf{\Gamma}_x := \{g \in \mathbf{\Gamma} : gx = x\}$. A mapping $\psi : E \to E$ is called $\mathbf{\Gamma}$-*invariant* if $\psi(gX) = g\psi(X), \quad \forall X \subseteq E, \forall g \in \mathbf{\Gamma}$.

In the following we present two examples, as we will need them in what follows. In each case $\mathbf{\Gamma}$ denotes the group and E the corresponding set.

Example 1 E = Euclidean space $\mathbb{R}^n$, $\mathbf{\Gamma}$ = the Euclidean translation group $\mathbf{T}$, which is abelian. Elements of $\mathbf{T}$ can be parameterized by vectors $h \in \mathbb{R}^n$, with τ_h the translation over the vector h:

$$\tau_h\, x = x + h, \quad h \in \mathbf{T}, x \in \mathbb{R}^n. \tag{11}$$

Example 2 E = Euclidean space $\mathbb{R}^n$ $(n \geq 2)$, $\mathbf{\Gamma}$ = the Euclidean motion group $\mathbf{M}$, i.e. the group generated by translations and rotations (see [3]). The subgroup leaving a point p fixed is the set of all rotations around that point. $\mathbf{M}$ is not abelian. The collection of translations forms the Euclidean translation group $\mathbf{T}$. The stabilizer of the origin, denoted by $\mathbf{R}$, equals the (commutative) group of rotations around the origin. Let τ_h denote the translation over the vector $h \in \mathbb{R}^2$ and ρ_ϕ^p the rotation over an angle ϕ around the point p. Let $\gamma_{h,\phi}$ denote a rotation around the origin followed by a translation:

$$\gamma_{h,\phi} = \tau_h\, \rho_\phi^0, \quad h \in \mathbb{R}^2, \; \phi \in [0, 2\pi). \tag{12}$$

Any element of $\mathbf{M}$ can be written in this form.

3.3. Morphological operations

One can construct morphological operations on a space E with a group $\mathbf{\Gamma}$ acting on it as follows. Let the 'origin' ω be an arbitrary point of E. To each subset X of E associate all elements of the group which map the origin ω to an element of X. To go back from the group $\mathbf{\Gamma}$ to the space E, associate to each subset G of $\mathbf{\Gamma}$ the collection of all points $g\omega$ where g ranges over G.

Definition 3 *The* lift $\vartheta : \mathcal{P}(E) \to \mathcal{P}(\mathbf{\Gamma})$ *and* projection $\pi : \mathcal{P}(\mathbf{\Gamma}) \to \mathcal{P}(E)$ *are defined by*

$$\begin{aligned} \vartheta(X) &= \{g \in \mathbf{\Gamma} : g\omega \in X\}, \quad X \subseteq E \\ \pi(G) &= \{g\omega : g \in G\}, \quad G \subseteq \mathbf{\Gamma}. \end{aligned}$$

For the case of the Euclidean motion group $\mathbf{M}$ the formula for the lift specializes to [3]:

$$\vartheta(X) = \bigcup_{x \in X} \tau_x \mathbf{R} = \tau(X) \overset{\mathrm{M}}{\oplus} \mathbf{R}, \tag{13}$$

where $\mathbf{R}$ denotes the group of rotations around the origin, and

$$\tau(X) := \{\tau_x : x \in X\}, \tag{14}$$

with τ_x the (unique) Euclidean translation which maps the origin to x.

In [3,4] a construction was performed of various morphological operators between the distinct lattices $\mathcal{P}(E)$ and $\mathcal{P}(\mathbf{\Gamma})$. Here we only need erosions from $\mathcal{P}(E)$ to $\mathcal{P}(\mathbf{\Gamma})$. That is, consider the mapping which associates to a subset X of E the set of group elements $g \in \mathbf{\Gamma}$ for which the translated set gA is included in X:

$$\vartheta(X) \overset{\lambda}{\ominus} \vartheta(A) := \{g \in \mathbf{\Gamma} : gA \subseteq X\}. \tag{15}$$

The mapping $X \mapsto \vartheta(X) \overset{\lambda}{\ominus} \vartheta(A)$ is an erosion $\mathcal{P}(E) \to \mathcal{P}(\mathbf{\Gamma})$ which is $\mathbf{\Gamma}$-invariant.

4. The Generalized Tailor Problem

The solution of the Generalized Tailor Problem can be obtained in a way which is completely analogous to that of the Tailor Problem, cf. Sect. 2. The basic observation is that formula (15) expresses the containment relation on which the method is based. We summarize the solution for the cases of one, two and three sets or 'pieces' to be put into a given set. The generalization to n pieces is straightforward, cf. [5].

4.1. One piece

The solubility of the problem depends on the non-emptiness of the following set, called the *residue* of X w.r.t. A:

$$R_1^{\mathbf{\Gamma}}(X; A) := \vartheta(X) \overset{\lambda}{\ominus} \vartheta(A), \tag{16}$$

which is simply the $\mathbf{\Gamma}$-erosion of $\vartheta(X)$ by $\vartheta(A)$. Notice that the residue $R_1^{\mathbf{\Gamma}}(X; A)$ is a subset of $\mathbf{\Gamma}$. It is easy to see that

$$R_1^{\mathbf{\Gamma}}(X; A) = \bigcup_{\phi} (X \ominus \rho_\phi A)\, \rho_\phi = \bigcup_{\phi} R_1(X; \rho_\phi A)\, \rho_\phi \tag{17}$$

where ρ_ϕ is short for ρ^0_ϕ, and we have written $X \ominus \rho_\phi A$ instead of $\tau(X \ominus \rho_\phi A)$, since the points of a set $X \subseteq E$ are in 1-1 correspondence to points of the set $\tau(X) \subseteq \mathbf{T}$. Therefore, $R_1(X; \rho_\phi A)$ is to be interpreted as a subset of the translation group $\mathbf{T}$, which can be multiplied from the right by a rotation ρ_ϕ according to the second equation in formula (10).

This equation expresses the obvious fact that $R_1^{\mathbf{\Gamma}}(X; A)$ can be obtained by considering all rotations of the structuring element A, and solving the ordinary Tailor Problem with structuring element $\rho_\phi A$.

4.2. Two pieces

Consider two sets $A_1 = A, A_2 = B$ which are to be put inside a set X. That is, we are looking for $a, b \in \mathbf{\Gamma}$ such that

$$aA \subseteq X; \qquad bB \subseteq X \setminus aA. \tag{18}$$

The solubility depends on the non-emptiness of the following residue:

$$R_2^{\mathbf{\Gamma}}(X; A, B) := [\vartheta(X) \overset{\lambda}{\ominus} \vartheta(B)] \cap [(\vartheta(X) \overset{\lambda}{\ominus} \vartheta(A)) \oplus (\vartheta(A)^c \overset{\lambda}{\ominus} B)]. \tag{19}$$

After a choice $b \in R_2^{\mathbf{\Gamma}}(X; A, B)$ to put B into position, the group element a can be chosen from the set $R_1(X \setminus B_b; A) = \vartheta(X \setminus B_b) \overset{\lambda}{\ominus} \vartheta(A)$. Note the similarity of these expressions to those in Sect. 2.

Again we can express $R_2^{\mathbf{\Gamma}}(X; A, B)$ in terms of the residue of the ordinary Tailor Problem. The result is:

$$R_2^{\mathbf{\Gamma}}(X; A, B) = \bigcup_{\phi'} \left(\bigcup_{\phi} R_2(X; \rho_\phi A, \rho_{\phi'} B) \right) \rho_{\phi'} \tag{20}$$

4.3. Three pieces

Now there are three sets $A_1 = A, A_2 = B, A_3 = C$ which are to be put inside a set X. That is, we are looking for $a, b, c \in \mathbf{\Gamma}$ such that

$$aA \subseteq X; \qquad bB \subseteq X \setminus aA; \qquad cC \subseteq (X \setminus aA) \setminus bB. \tag{21}$$

The solubility of the problem depends on the non-emptiness of the following residue:

$$R_3^{\mathbf{\Gamma}}(X; A, B, C) := \bigcup_{a \in \vartheta(X) \overset{\lambda}{\ominus} \vartheta(A)} \; \bigcup_{b \in \vartheta(X) \overset{\lambda}{\ominus} \vartheta(B)} \delta_{A,B,C}(x, y), \tag{22}$$

where

$$\delta_{A,B,C}(a, b) = [\vartheta(X) \overset{\lambda}{\ominus} \vartheta(C)] \cap [\vartheta(aA)^c \overset{\lambda}{\ominus} \vartheta(C)] \cap [\vartheta(bB)^c \overset{\lambda}{\ominus} \vartheta(C)] \tag{23}$$

when $\vartheta(aA) \cap \vartheta(bB) = \emptyset$ and $\delta_{A,B,C}(a, b) = 0$ otherwise. It can be shown that the following recursive expression holds:

$$R_3^{\mathbf{\Gamma}}(X; A, B, C) = \bigcup_{a \in \vartheta(X) \overset{\lambda}{\ominus} \vartheta(A)} R_2^{\mathbf{\Gamma}}(X \setminus aA; B, C). \tag{24}$$

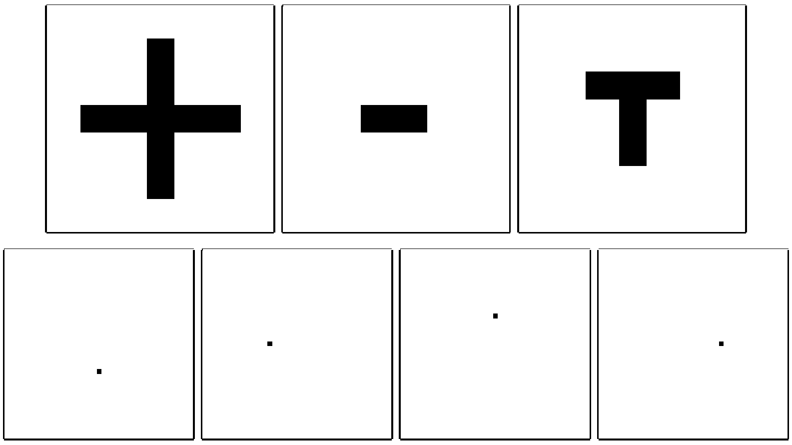

Fig. 1. The Generalized Tailor Problem for two pieces. Top row, from left to right: set X, set A, set B. Bottom row, from left to right: allowed translations of $\rho_{\phi'}B$ for $\phi' = 0, \pi/2, \pi, 3\pi/2$, respectively.

Also, we can express $R_3^{\Gamma}(X; A, B)$ in terms of the residue R_3 appearing in the ordinary Tailor Problem:

$$R_3^{\Gamma}(X; A, B, C) = \bigcup_{\phi''} \left(\bigcup_{\phi, \phi'} R_3(X; \rho_\phi A, \rho_{\phi'} B, \rho_{\phi''} C) \right) \rho_{\phi''} \qquad (25)$$

After a choice $c \in R_3^{\Gamma}(X; A, B, C)$ to put C into position, b can be chosen from the set $R_2^{\Gamma}(X \setminus cC; A, B)$; finally, a can be chosen from $R_1^{\Gamma}(X \setminus (bB \cup cC); A)$.

5. Experimental results

We have implemented the formulas above using dilations, erosions and set complementation for the case of one and two pieces. For the case $n = 3$ the formula (25) is used, where the sets $R_3(X; \rho_\phi A, \rho_{\phi'} B, \rho_{\phi''} C)$ are computed recursively using (7). The set of rotations is restricted here for simplicity to multiples of $\pi/2$.

As a first example consider the case of two pieces. The set X and the sets A and B to be fitted within X are shown in the top row of Fig. 1. In the second row of this figure we show the possible positions of the set $\rho_{\phi'}B$ for a given angle $\phi' = 0, \pi/2, \pi, 3\pi/2$, i.e. the set $\bigcup_\phi R_2(X; \rho_\phi A, \rho_{\phi'} B)$, cf. (20). In each of these four pictures, a black dot represents a single pixel. The results show that for each orientation of B there is only a single solution. This is obvious from the form of the sets involved.

The second example is for the case of three pieces, cf. Fig. 2. Again, the set X and the sets A, B and C to be fitted within X are shown in the top row. These sets have been constructed in such a way that there is only a single way to fit the sets, as shown in Fig. 2, second row, leftmost picture, in which the sets A, B, C have been

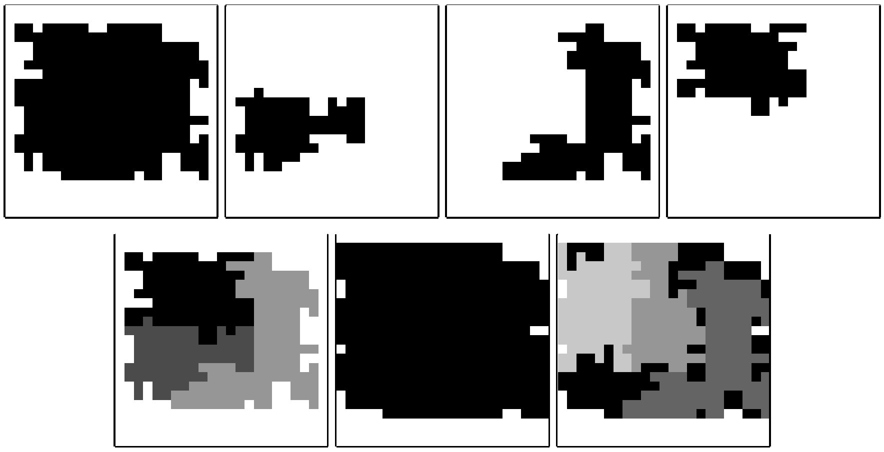

Fig. 2. The Generalized Tailor Problem for three pieces. Top row, from left to right: set X, sets A, B and C. Bottom row, left: the unique fit of A, B and C in X; middle: dilation X' of X; right: possible fit of A, B and C in X'.

given distinct grey values to show how they fit in. To find this complete solution, we first computed $R_3^{\mathbf{\Gamma}}(X; A, B, C)$ to find the allowed positions and orientations for C. Next we computed $R_2^{\mathbf{\Gamma}}(X \setminus C; A, B)$, yielding the allowed positions and orientations for B, and finally the allowed positions and orientations of A were obtained from $R_1^{\mathbf{\Gamma}}(X \setminus (B \cup C); A)$. The result in all three cases was that there is a single solution involving zero rotation and zero translation.

Next we perform a dilation of the set X with a 3×3 square structuring element, resulting in the set X', cf. Fig. 2. When we again apply the generalized tailor algorithm, we find 9 solutions (including the original solution) without rotation of A, B, C, but also a solution where A and C are rotated over $\pi/2$. This is shown as the last picture in Fig. 2, where the sets A, B, C (shaded) have been superimposed upon X'. This solution is certainly more difficult to guess, but the generalized tailor algorithm readily shows its existence.

6. Discussion

In this paper the solution of the Tailor Problem in terms of morphological operators [5] has been generalized to the case where rotations of the sets are allowed. By using the formalism of morphological operators on transformation groups, we have obtained a solution of the Generalized Tailor Problem which is completely similar in form to the case with translations only. When the group $\mathbf{\Gamma}$ equals the ordinary translation group, the formulas in this paper reduce to those found by Serra [5]. We presented some experimental results showing the possibilities of the method. As far as computational complexity is concerned, it may be remarked that the method for three pieces is already becoming time consuming. This may be improved by using a polygonal representation of the sets instead of a pixel representation, and applying methods from computational geometry, such as those of [1].

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