Exercises for
Chapter 7: Tensor Visualization

1 EXERCISE 1

In data visualization, tensor attributes are some of the most challenging data types, due to their high dimensionality and abstract nature. In Chapter 7 (and also in Section 3.6), we introduced tensor fields by giving a simple example: the curvature tensor for a 3D surface. Give another example of a tensor field defined on 2D or 3D domains. For your example

- Explain why the quantity you are defining is a tensor
- Explain how the quantity you are defining varies both as a function of position but also of direction
- Explain what are the intuitive meanings of the minimal, respective maximal, values of your quantity in the directions of the respective eigenvectors of your tensor field.

2 EXERCISE 2

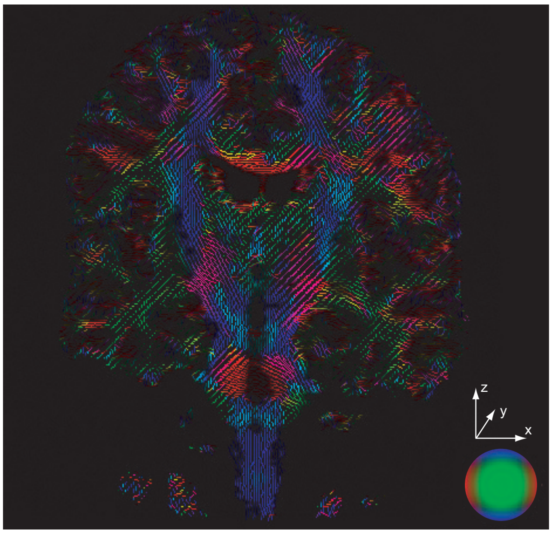
Consider a 2-by-2 symmetric matrix A with real-valued entries, such as the Hessian matrix of partial derivatives of some function of two variables. Now, consider the two eigenvectors $\mathbf{x}$ and $\mathbf{y}$ of the matrix A, and their two corresponding eigenvalues λ and μ. We next assume that these eigenvalues are different. Prove that the two eigenvectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Hints: There are several ways to prove this. One way is to use that the matrix is symmetric, hence $A = A^T$. Next, use the algebraic identity $< A\mathbf{x}, \mathbf{y} >=< \mathbf{x}, A^T\mathbf{y} >$, where $< \mathbf{a}, \mathbf{b} >$ denotes the dot product of two vectors **a** and **b**. To prove that **x** is orthogonal to **y**, prove that $<\mathbf{x},\mathbf{y}>= 0$.

3 EXERCISE 3

One basic solution for visualizing eigenvectors of a 3-by-3 tensor, such as the one generated from a diffusion-tensor MRI scan, is to color-code its (major) eigenvector using a directional colormap. Figure 7.6 (also shown below) shows such a colormap, where we modulate the basic color components R, G, and B to indicate the orientation of the eigenvector with respect to the x, y, and z axes respectively. For the same task of directional color-coding of a tensor field, imagine a different colormap, which, in your opinion, may be more intuitive than the red-green-blue colormap proposed here.

Hints: Think of directional hue-based color-coding schemes used to visualize 2D vector fields.

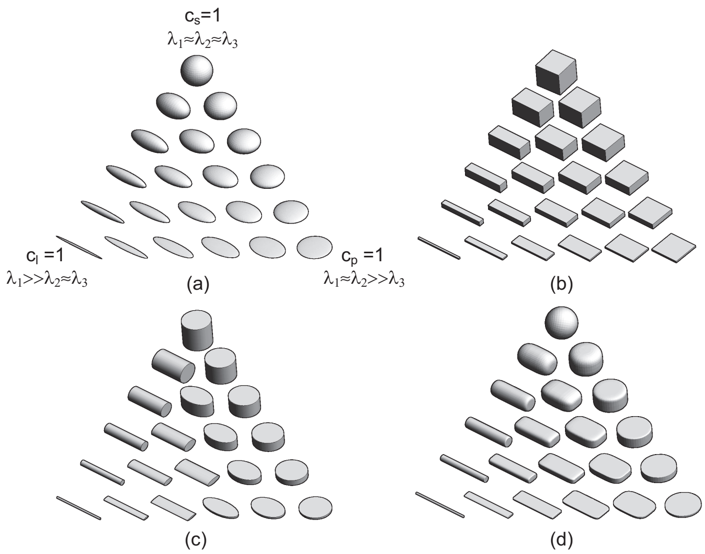


Major eigenvector of tensor field visualized with line glyphs color-coded by direction (Chapter 7).

4 EXERCISE 4

Tensor glyphs are a generalization of vector glyphs which attempt to convey three vectors (the eigenvectors of the tensor-field to be explored) at a given point over its domain. In Section 7.5 (Figure 7.8, also shown below), four kinds of tensor glyphs are proposed: ellipsoids, cuboids, cylinders, and superquadrics. Propose a different kind of tensor glyph. Sketch the glyph. For your proposed glyph, explain

- How the glyph's properties (shape, shading, color) convey the directions and magnitudes of the three eigenvectors
- How it is possible, by looking at the shape, to understand which is the direction of the major eigenvector, medium eigenvector, and minor eigenvector
- Which are, in your opinion, the advantages and/or disadvantages of your proposal as compared to the ellipsoid, cuboid, cylinder, and superquadric glyphs.



Different types of glyph shapes for visualizing tensor fields (Chapter 7).

5 EXERCISE 5

One way to visualize a symmetric 3D tensor field is to reduce it, by principal component analysis (PCA), to a set of three eigenvectors ($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$), whose corresponding lengths are given by three eigenvalues ($\lambda_1, \lambda_2, \lambda_3$). Such eigenvectors can be visualized, among other methods, by using vector glyphs. In this context, answer the two questions below

- If we use vector glyps, and since we have *three* eigenvectors, and all of them encode relevant information for the tensor field, why do we usually choose to visualize just the major-eigenvector field, rather than drawing a single image containing vector glyphs for all three eigenvector fields?

- *Oriented* glyphs such as arrows are typically preferred against unoriented ones (*e.g.* lines) when visualizing vector fields. Why do not we use such oriented glyphs, but prefer unoriented glyphs, when visualizing eigenvector fields?

6 EXERCISE 6

Apart from directional color coding and tensor glyphs, computing and visualizing the *anisotropy* is a third method that shows the difference between the principal components, or eigenvalues, of a tensor field, as a scalar field. Describe two anisotropy metrics for 3D symmetric tensor fields, and explain the meaning of the minimal, respective maximal, values of the metric.

7 EXERCISE 7

Consider that we have a (dense) point cloud $P = \{\mathbf{p}_i\}$ of N 3D points, which are the samples of a 3D smooth and non-intersecting surface. Many methods exist for the reconstruction of a meshed surface from such an unorganized point cloud. However, several such methods require to know the *orientation* of the surface normal $\mathbf{n}_i$ at each sample point $\mathbf{p}_i$. Describe in detail a method to compute this normal orientation based on principal component analysis applied to P.

8 EXERCISE 8

Given a 2D shape, represented as a binary image, or alternatively as a (densely sampled) 2D polyline, an important tool in graphics and visualization is finding the so-called *oriented bounding box* (OBB) of this shape. In 2D, the OBB is a (possibly not axis-aligned) rectangle which encloses the shape as tightly as possible. Present a way of computing an OBB, given an unordered set of 2D points $S = \{\mathbf{p}_i\}$ which densely sample the boundary of such a 2D shape, based on principal component analysis (PCA).

9 EXERCISE 9

(Hyper)streamline tracing, or tractography, is one of the best known methods for visualizing a 3D tensor field such as the ones produced by 3D diffusion tensor magnetic resonance imaging (DT-MRI). Both the seeding strategy and the streamline-tracing stop criterion have to be carefully set in function of the characteristics of the DT-MRI field to obtain useful visualizations. Describe one typical strategy for seeding and one for stopping the tracing, and explain how they are related to the DT-MRI field values.

10 EXERCISE 10

Hyperstreamlines visualize a tensor field by constructing streamlines in the vector field given by the major eigenvector of the tensor field. The medium and minor eigenvectors are encoded, at each point along a hyperstreamline, by using an ellipse whose half-axes are oriented along the medium and minor eigenvectors, and respectively scaled to reflect the sizes of the medium and minor eigenvalues. Propose a different hyperstreamline construction, whose cross-section would not be an ellipse, but a different shape.

Hints: Think about other tensor glyph shapes. Discuss the advantages and/or disadvantages of your proposal as compared to hyperstreamlines that use an elliptic cross-section.

11 EXERCISE 11

Fiber clustering is a method that, given a set of 3D curves computed *e.g.* by tracing streamlines along the major eigenvector of a tensor field, partitions (or clusters) this fiber-set into subsets of fibers that are very similar in terms of spatial location and curvature. Fiber clustering is useful into highlighting sets of similar fibers and thereby potentially simplifying the resulting visualization. However, using just geometric attributes to compare fibers ignores other information, such as encoded by the medium and minor eigenvectors and the corresponding eigenvalues. Propose an alternative similarity function for fibers that, apart from the geometric information, would also consider similarity of the medium and minor eigenvectors and eigenvalues. Describe your similarity function in (mathematical) detail, and discuss why it would produce a different (and potentially more insightful) clustering of tensor fibers.

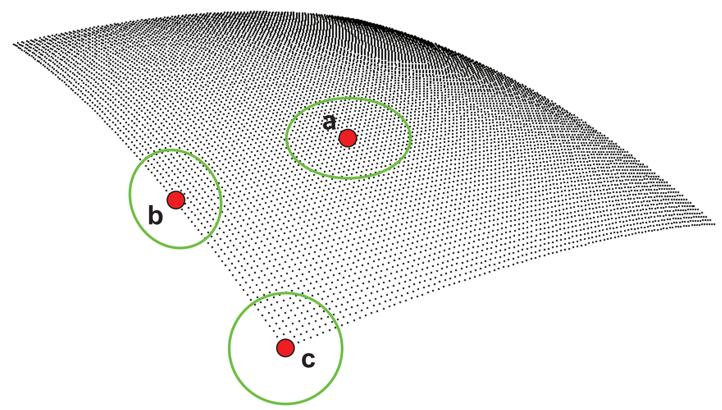
12 Exercise 12

Image-based flow visualization (or IBFV) is a method that depicts a vector field by means of an animated luminance texture which gives the impression to 'flow' along the vector field (see Section 6.6.1). Imagine an extension of IBFV that would be used to visualize 2D tensor fields. The idea is to use the major eigenvector field to construct the IBFV animation, and, additionally, encode the minor eigenvector and/or eigenvalue in other attributes of the resulting visualization, such as color, luminance, or shading. How would you modify IBFV to encode such additional attributes?

Hints: Take care that modifying luminance may adversely affect the result of IBFV, e.g., destroy the apparent flow patterns that convey the direction of the major eigenvector field.

13 Exercise 13

Consider a point cloud that densely samples a part of the surface of a sphere of radius R, defined in polar coordinates α, ϕ by the ranges $[\alpha_{min}, \alpha_{max}]$ and $[\beta_{min}, \beta_{max}]$. The 'patch' created by this sampling is shown in the figure below. Given the three points a, b, c indicated in the same figure, describe what are the three eigenvectors of the principal component analysis (PCA) applied to the points' covariance matrix for small neighborhoods of each of these three points. The neighborhood sizes are indicated by the circles in the figure. For this, indicate which are the directions of these eigenvectors, and (if possible from the provided information), which are their relative magnitudes.



Point cloud sampling a sphere patch with three points of interest.

End of Exercises for Chapter 7: Tensor Visualization