Sponges for Generalized Morphology

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Abstract. Mathematical morphology has traditionally been grounded in lattice theory. For non-scalar data lattices often prove too restrictive, however. In this paper we present a more general alternative, sponges, that still allows useful definitions of various properties and concepts from morphological theory. It turns out that some of the existing work on "pseudo-morphology" for non-scalar data can in fact be considered "proper" mathematical morphology in this new framework, while other work cannot, and that this correlates with how useful/intuitive some of the resulting operators are.

Keywords: mathematical morphology, pseudo-morphology, weakly associative lattices, sponges

1 Introduction

Lattice theory has brought mathematical morphology very far when it comes to processing binary and greyscale images. However, for vector- and tensor-valued images lattice theory appears to be overly restrictive [5, 12, 20, 27]: vectors and tensors simply do not seem to naturally fit a lattice structure. For example, we cannot have a lattice that is compatible with a vector space structure while also behaving in a rotationally invariant manner [20]. And having a lattice that can deal with periodic structures is equally impossible (due to it being based on an order relation).

Some attempts have been made to still apply mathematical morphology to vector- and tensor-valued images by letting go of the lattice structure while still having something resembling the infimum and supremum operations [1, 3, 6, 7, 10, 11, 17, 30]. However, over the years, mathematical morphology has developed a host of concepts that (in their usual formulation) rely on a lattice structure. Take away the lattice structure and all these concepts make very little sense any more. For example, a dilation is defined as an operator that commutes with the supremum of the lattice. And some pseudo-morphological operators can indeed lead to unintuitive (and undesired) behaviour [19].

In this paper we introduce a novel theoretical framework which generalizes lattices, while retaining some crucial properties of infima and suprema. This

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more flexible structure, which we will call a "sponge", is inspired by the vector levelings of Zanoguera and Meyer [30] and the tensor dilations/erosions by Burgeth et al. [6, 9, 11], and is effectively a variant of what is known as a "weakly associative lattice" or "trellis" [14, 15, 28]. Its relations to lattices and the earlier generalizations are discussed and examples are shown of existing and new methods that are not interpretable in a traditional lattice theoretic framework, but that do lead to sponges. The method proposed by Burgeth et al. [6], on the other hand, is shown not to lead to a sponge (consistent with some of the issues with this method). We hope this new framework will be useful in guiding future developments in non-scalar morphology, and that it will provide more insight into the properties of operators based on such schemes.

2   Related work

If we step away from lattices, what options do we have? Some attempts at developing specific methods that still behave much like a traditional lattice include non-separable vector levelings by Zanoguera and Meyer [30], morphology for hyperbolic-valued images by Angulo and Velasco-Forero [3], and the Loewner-order based operations by Burgeth et al. [6]. All these methods support the concepts of upper and lower bounds, as well as some sort of join and meet (infimum and supremum), but do not rely on a lattice structure. The framework presented in this work will be shown to encompass some of these methods, but not all (in the latter case this can be linked to some issues with the method).

Below we will present a generalization of a partial order that will be called an oriented set, as a starting point for our generalization of a lattice. An oriented set is so named because it can also be considered an oriented graph[1] and vice versa. Also, if all elements in some subset of an oriented set are comparable, this subset can be called a tournament (analogous to a chain). This structure was already used, under different names, as the basis for a subtly different generalization of a lattice: a weakly associative lattice (WAL), trellis, or T-lattice [14, 15, 28].

Based on oriented sets we will introduce a generalization of a lattice called a sponge, which supports (partial) join and meet operations on sets of elements. A sponge is a lattice if the orientation is transitive *and* the join and meet are defined for all pairs (as a consequence of being a lattice they must then also be defined for all finite sets). If the latter condition does not hold the result would still be a partially ordered set with a join and meet defined for all finite subsets that have an upper/lower bound (which is a bit more specific than the concept of a partial lattice used by [18, Def. 12]). On the other hand, a weakly associative lattice [14, 15, 28] is defined in *almost* the exact same way as a sponge. The difference is that a weakly associative lattice requires the join and meet of *every* pair of elements to be defined, while not guaranteeing that the existence of an upper/lower bound implies the existence of the join/meet of a (finite) set of

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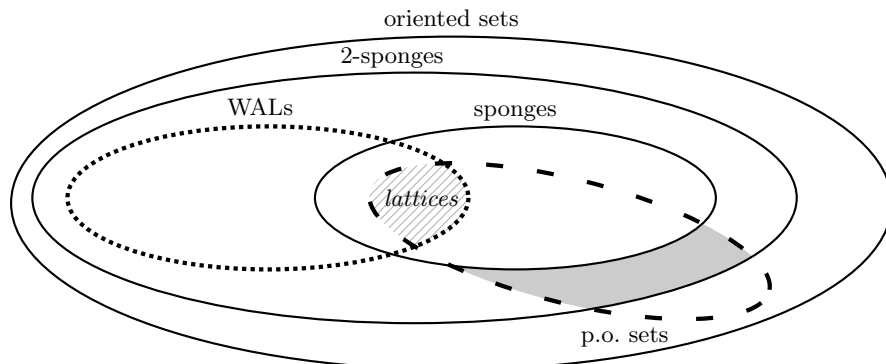[1] Fried and Grätzer [16] called an oriented graph a directed graph.

Fig. 1. An Euler diagram of the oriented and ordered structures discussed in the text. The shaded area is empty, but all other areas are not. The set of all lattices is the (hatched) intersection of WALs and p.o. sets.

elements [13]. The concept of a *partial* weakly associative lattice seems to be no more powerful than that of an oriented set [16, Lemma 1][2].

   Imagine a variant of a sponge where the join and meet only need to be defined for all pairs (rather than all finite sets) with upper/lower bounds. If (against our better judgement, since the concept is more general than a sponge) we call such structures 2-sponges, then (as captured in Fig. 1)

   – WALs, sponges and partially ordered sets generalize lattices,
   – there are WALs that are not sponges (and vice versa),
   – a WAL that is also a partial order is a lattice (and thus also a sponge),
   – a partially ordered set that is also a 2-sponge is a sponge,
   – there are partially ordered sets that are not 2-sponges (and vice versa),
   – 2-sponges are (strictly) more general than both WALs and sponges,
   – oriented sets are (strictly) more general than 2-sponges and partial orders.

   Since many morphological operators and concepts are based on joins and meets of sets, sponges provide a much more natural framework for generalized morphology than WALs. Also, WALs require the join and meet to be defined for *all* pairs of elements, and all our examples violate this property. Partial WALs on the other hand provide too few guarantees to really be useful. As a consequence, we believe sponges are the right choice in the current context.

3   Sponges

In order to define sponges, it is useful to first quickly recap the two main ways of defining lattices (we will provide analogous definitions for sponges):

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[2] The definition of a partial weakly associative lattice is a little vague, but it seems clear that at least any oriented set gives rise to a partial weakly associative lattice.

1. A set with a binary relation is considered a partial order if the relation is reflexive, antisymmetric and transitive. A partial order is a lattice if every pair of elements has a unique greatest lower bound (infimum) and a unique least upper bound (supremum).
2. A set with two (binary) operators called meet and join is a lattice if and only if the operators are commutative and associative, and satisfy the absorption property: the join of an element with the meet of that same element with any element is always equal to that first element, and similarly with the roles of the join and meet swapped.

Roughly speaking, sponges let go of transitivity of the partial order, or, equivalently, the associativity of the join and meet.

3.1 Sponges as oriented sets

We define a *(partially) oriented set*[3] S to be a set with a binary relation '$\preceq$' – a (partial) orientation[4] – that has the following two properties:

reflexivity: $a \preceq a$ for all $a \in S$, and
antisymmetry: $a \preceq b$ and $b \preceq a \implies a = b$.

We also write $P \preceq Q$ for subsets P and Q of S if and only if $[\forall a \in P, b \in Q : a \preceq b]$. For reasons of simplicity, we *will* say that a is less than or equal to b (or a lower bound of b) if $a \preceq b$, even though the relation need not be transitive. If the orientation relation is total (in the sense that all elements are comparable), the set is called totally oriented and the relation is a total orientation (or tournament).

We now define a sponge as an oriented set in which there exists a supremum/infimum for every *finite* subset of S which has at least one common upper/lower bound. Here a supremum a of a subset P of S is defined as an element in S such that $P \preceq \{a\}$ and $a \preceq b$ for all b such that $P \preceq \{b\}$; the infimum is defined analogously. Note that antisymmetry guarantees that if a supremum/infimum exists, it is unique.[5]

3.2 Algebraic definition of sponges

Analogous to the algebraic definition of a lattice, we now define a sponge as a set S with partial functions J (join) and M (meet) defined on *finite* subsets of the set S, satisfying the properties (with $a, b \in S$ and P a finite subset of S)

idempotence: $M(\{a\}) = a$,
absorption: if $M(P)$ is defined, then $[\forall a \in P : J(\{a, M(P)\}) = a]$,
part preservation: $[\forall a \in P : M(\{a, b\}) = b] \implies M(\{M(P), b\}) = b$,

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[3] Fried [13] called an oriented set a partial tournament.
[4] Rachůnek [26] called an orientation a semi-order, while Skala [28] and Fried and Grätzer [16] called it a pseudo-order.
[5] In fact, Fried [13] already showed that in any orientation the set of "least upper bounds" of a set is either empty or a set of just one element.

and the same properties with J substituted for M and vice versa. Since J and M are operators on (sub)sets, they preserve the commutativity of lattice-based joins and meets, but not necessarily their associativity. In a lattice $\mathcal{L}$, idempotence follows from absorption: $a \wedge a = a \wedge (a \vee (a \wedge b)) = a$ for all $a, b \in \mathcal{L}$. In a sponge, on the other hand, the join and meet need *not* be defined for all pairs of elements in the sponge, and this argument breaks down (but we still need it, so it is included as a separate property). Part preservation[6] is essentially "half" of associativity, in the sense that if the implication was replaced by a logical equivalence, J and M would be associative. In some cases we wish to write down $M(P)$ or $J(P)$ without worrying about whether or not it is actually defined for the set P. We then consider M or J to return a special value when the result is undefined. This value propagates much like a `NaN`: if it is part of the input of M or J, then the output takes on this "undefined" value as well.

It is important to note if a (finite) subset P of a sponge has a common lower (upper) bound b, the premise of part preservation is true, and P must then have a meet (join), or the left-hand side of the conclusion would be undefined.

From now on, we will omit braces around explicitly enumerated sets whenever this need not lead to any confusion (as this greatly enhances readability). So we will write $M(a, b)$ and $P \preceq a$ rather than $M(\{a, b\})$ and $P \preceq \{a\}$.

3.3 Equivalence of definitions

We now proceed to show that both definitions above are, in fact, equivalent. For example, part preservation can be interpreted as: $b \preceq P$ implies $b \preceq M(P)$.

Theorem 1. *An oriented set-based sponge gives rise to an algebraic sponge, in which the partial functions J and M recover precisely the suprema and infima in the oriented set.*

Proof. Since the supremum is unique whenever it is defined, we can construct a partial function J that gives the supremum of a (finite) set of elements; we construct M analogously. Due to reflexivity the resulting J and M must be idempotent. Part preservation also follows, as by definition any upper bound of a set of elements is an upper bound of the supremum of those elements (note that by definition, if there is an upper bound, there must also be a supremum).

To see that our candidate sponge also satisfies the absorption laws, suppose that the set P has a common lower bound, so its infimum $\inf(P)$ is defined. By definition, $a \preceq a$ as well as $\inf(P) \preceq a$ for any $a \in P$. Since the two elements share an upper bound (a), the supremum of a and $\inf(P)$ must be defined. Again by definition, we must have that $a \preceq \sup(a, \inf(P))$, but also that $\sup(a, \inf(P)) \preceq a$ (since a is an upper bound for all of the arguments). Due to the antisymmetry of the orientation $\sup(a, \inf(P))$ must thus equal a. Since the same can be done in the dual situation, the J and M induced by '$\preceq$' must give rise to an algebraic

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[6] The name of this property was taken from the analogous property on binary joins/meets given by Skala [28], and presumably refers to the meet (join) preserving all joint lower (upper) bounds ("parts").

sponge, in which J and M recover the suprema and infima in the oriented set.

$\square$

Theorem 2. *An algebraic sponge gives rise to an oriented-set-based sponge, such that a finite set has a supremum (infimum) if and only if J (M) of the set is defined, and if it is, J (M) gives the supremum (infimum).*

Proof. We define $a \preceq b$ if and only if $M(a,b) = a$. Note that the absorption laws guarantee that it does not matter whether we base the relation $\preceq$ on M or on J, as they imply that $M(a,b) = a \implies J(a,b) = J(M(a,b),b) = b$ (the dual statement follows analogously).

Since M is idempotent, the induced relation '$\preceq$' must be reflexive. Also, as M is a (partial) function, $a \preceq b$ and $a \neq b$ together imply $b \not\preceq a$ (a function cannot take on two values at the same time). In other words: the relation '$\preceq$' is antisymmetric, and we can thus conclude that '$\preceq$' is an orientation.

We will now show that every finite set with a common upper (lower) bound (according to '$\preceq$') has a supremum (infimum) if and only if J (M) is defined for that set, and that if it exists, the supremum (infimum) is given by J (M). Due to the absorption and part preservation properties, the join provides every finite set that has a common upper bound with a supremum. Thus, the relation '$\preceq$' induced by J and/or M is a sponge. Furthermore, we cannot have any finite subsets for which J (M) is not defined but a supremum (infimum) does exist, as J and M must be defined for *all* finite subsets with a common upper/lower bound (due to the part preservation property). This concludes our proof. $\square$

3.4 Further properties

Lemma 1. *For any finite set of finite subsets $P_1, P_2, \ldots$ of a sponge, we have $M(P) \preceq M(M(P_1), M(P_2), \ldots)$, with $P = P_1 \cup P_2 \cup \ldots$, assuming $M(P)$ exists (and similarly for joins).*

Proof. We have $M(P) \preceq P$ (absorption). Now, since $P_i \subseteq P$, $M(P)$ is a lower bound of all elements of P_i (for any i), and thus of $M(P_i)$ as well (part preservation). The lemma now follows from another application of part preservation (since we have established that $M(P)$ is a lower bound of all the P_i). $\square$

In a sponge the join of a finite set exists if (and only if) there is a common upper bound of that set. A conditionally complete sponge guarantees that this is true for *all* non-empty sets (finite or otherwise) that have at least one common upper bound, and similarly for the meet. A complete sponge would be a sponge for which all sets are guaranteed to have a join and a meet. All of the examples given in Section 4 are conditionally complete, and most of them have a smallest element (so all non-empty meets exist). We expect conditionally complete sponges with a least element to play a role analogous to that of complete lattices in traditional morphological theory. In such a sponge the meet of any set is well-defined, as is the join of any set with a common upper bound, making something similar to a structural opening well-defined (see Section 5).

Analogous to semilattices, we can define semisponges: a meet-semisponge is an oriented set such that any finite set with a lower bound has an infimum (a join-semisponge can be defined analogously). We can consider a meet-semisponge to have an operator M (the meet) that gives the infimum of a set. As the infimum is defined as the unique lower bound that is an upper bound of all lower bounds, M would still satisfy the part preservation property, as well as a modified form of the absorption property: if $M(P)$ is defined, then $[\forall a \in P : M(a, M(P)) = M(P)]$.

Theorem 3. *If S is a conditionally complete meet-semisponge it is a conditionally complete sponge (by duality the same holds for a conditionally complete join-semisponge).*

Proof. If S is a conditionally complete meet-semisponge, this means that the meet is defined for all (non-empty) sets that have a lower bound. We can now define J as giving the meet of the set of all upper bounds of a given set. For any non-empty set with a common upper bound, the set of all (common) upper bounds is again non-empty, and bounded from below by the original set, so its infimum is well-defined. As a consequence, if this construction turns S into a sponge, then this sponge is conditionally complete. Due to part preservation, the meet of the set of all upper bounds of a set is still an upper bound of the original set, and due to the absorption property it must also be a lower bound of all the upper bounds of the original set. In other words, J can indeed be interpreted as giving the supremum of any (non-empty) subset of S with an upper bound. We can thus conclude that S is a conditionally complete sponge. □

If a conditionally complete meet-semisponge has a least element we can give meaning to the meet of *all* non-empty subsets, as well as the join of the empty set (which would give the least element).

Another important property of a sponge is whether it is acyclic or not. In a lattice we cannot have any cycles whatsoever (due to the combination of transitivity and antisymmetry of the underlying partial order). In a sponge we can have cycles (an example will be given below), but not having cycles allows us to define a partial order or preorder on a sponge, making it easier to talk about the convergence of sequences. It may also be interesting to consider *locally* acyclic sponges, whose restriction to any set of lower/upper bounds is acyclic.
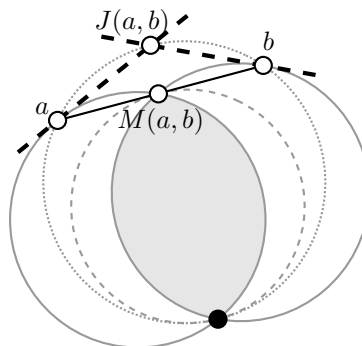
Yet another interesting property that sponges can have is that the meet of a set of lower bounds is still a lower bound (and dually for upper bounds/joins). It is possible to construct sponges that do not have this property, but the examples given in Section 4 do all have this property. An immediate consequence is that in these sponges the join and meet preserve *both* upper and lower bounds: $a \preceq P$ and $P \preceq b$ imply $a \preceq M(P)$ and $M(P) \preceq b$ (and similarly for the join).

4 Examples

4.1 Inner product and hyperbolic sponges

Inspired by the vector levelings developed by Zanoguera and Meyer [30], we can consider a vector $\mathbf{a}$ in some Hilbert space as "less" than (or equal to) another

Fig. 2. Computing the meet and join of the points a and b in the 2D (Euclidean) inner product sponge. The line segment between a and b is the convex hull of those two points. Each point has an associated circle enclosing all of its lower bounds (the dark dot represents the origin). The thick, dashed *lines* show the boundaries of the half-spaces of upper bounds for a and b. The shaded area below the meet is the intersection of the lower bounds of a and b, and is a subset of the set of lower bounds of the meet, consistent with part preservation.

vector $\mathbf{b}$ if and only if $\mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) \geq 0$. This does not give rise to a partial order, or even a preorder, as it is not transitive. However, it can be checked that it does give an orientation, and we will show that it even gives rise to a sponge (Fig. 2 illustrates the orientation and a meet and join in toy example).

The relation $\mathbf{a} \preceq \mathbf{b} \iff \mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) \geq 0$ implies that the set of upper bounds of some element $\mathbf{a}$ is the half-space defined by $\mathbf{a} \cdot \mathbf{b} \geq \|\mathbf{a}\|^2$. We now define the meet of a set of elements as the element closest to the origin in/on the closed convex hull of the set. If the convex hull includes the origin, this is the origin itself (and in this case there is indeed no other lower bound of the entire set). If the origin is outside the convex hull, the meet is still well-defined (minimization of a strictly convex function over a convex set) and must lie on the boundary of the convex hull. It is possible to see that the original points must thus be upper bounds of the meet. Also, since the meet is in the closed convex hull of the original points, and the set of upper bounds of any element is closed and convex, any element which was a lower bound of all of the original points must still be a lower bound of the meet. Based on Theorem 3, we can now conclude that – based on the meet described above – we have a conditionally complete sponge with the origin as its least element.

The "geodesic ordering" on the Poincaré upper-half plane developed by Angulo and Velasco-Forero [3] can be shown to form a conditionally complete sponge along much the same lines as the inner product sponge.

4.2 Angle sponge

Another problem area for the lattice formalism is that of periodic values, like angles. Several solutions [2, 4, 21, 25, 29] have been proposed to deal with angles, but none of them really deal with the inherent periodicity of angles. This is not by accident: it is impossible to have a periodic lattice.

Interestingly, we *can* create a periodic sponge: consider an angle a to be less than an angle b (both considered to be in the interval $[0, 2\pi)$) if and only if $b - \pi < a \leq b$ or $b - \pi < a - 2\pi \leq b$. In other words: a is less than b if and only if a is less than $180°$ clockwise from b.

It is clear that the above gives an orientation (the relation is reflexive and antisymmetric). Furthermore, if a set of angles has a common upper bound, all angles must lie on some arc of less than 180 degrees, and there must be a unique supremum (similarly for the infimum). We can thus conclude that we have defined a (periodic) conditionally complete sponge on angles.

4.3   A non-sponge: The Loewner order

The Loewner order [6] considers a (symmetric) matrix $\mathbf{A}$ less than or equal to another (symmetric) matrix $\mathbf{B}$ if the difference $\mathbf{B} - \mathbf{A}$ is positive semidefinite. This is a partial order compatible with the vector space structure of (symmetric) matrices, but it does not give rise to a lattice, or even a sponge. Any join/meet based on the Loewner order cannot satisfy both the absorption property and part preservation at the same time (we gave an example where part preservation breaks down in previous work [19]). As a partial fix, Burgeth et al. originally [9] computed the meet as the matrix inverse of the join of the inverses, so at least positive semidefiniteness would be preserved. However, matrix inversion does not reverse the order, and this still does not solve the problem that no upper bound of two matrices can be a lower bound of all common upper bounds.

In later work Burgeth et al. compute both the join and meet in a way that is compatible with the Loewner order [7, 8], but as a result they have to be careful not to get values outside the original range. The resulting structure is likely to be a so-called χ-lattice [23], but it is not yet clear how important this is from a morphological perspective. Based on the arguments presented by Pasternak et al. [24] and some preliminary experimentation, we expect it could be interesting to simply use the inner product sponge directly on the vector space of symmetric tensors (which would still preserve positive semidefiniteness).

5   Operators

One of the main advantages of the lattice-theoretical framework is that it allows us to classify operators into various categories based on certain lattice-related properties, and that these classes often have useful and intuitive interpretations. Although it remains to be seen to what extent existing classes carry over to the sponge case, here we show that at least one crucial property is preserved when we directly translate so-called "structural openings" (and in the process, that we can reason about such things for sponges in general).

We can try to translate structural dilations and erosions on images defined on a (translation-invariant) domain E to the sponge case. We then get a dilation-like operator defined by $\delta_A(f)(x) = J(\{f(y) \mid x \in A_y\})$ and an erosion-like operator $\varepsilon_A(f)(x) = M(\{f(y) \mid y \in A_x\})$ (where $x \in E$ and A_y is taken to be the structuring element translated by y). These operators need not commute with taking the join or meet, respectively, nor do they need to satisfy $\delta_A(f) = M(\{g \mid f \leq \varepsilon_A(g)\})$ like in a complete lattice [22, Prop. 3.14]. It is an open question whether there exist different definitions that recover a bit more of the traditional

properties (while remaining compatible with the lattice case). Nevertheless, we can use these operators to define an operator that behaves a bit like a lattice-based structural opening (and is a structural opening if the sponge is a lattice):

$$\gamma(f)(x) = J(\{M(\{f(z) \mid z \in A_y\}) \mid x \in A_y\}).$$

It is immediately obvious that the resulting operator is still guaranteed to be anti-extensive (due to each of the $M(\{f(z) \mid z \in A_y\})$ being a lower bound of $f(x)$), something which is not guaranteed by Loewner-based operators [19]. The operator may no longer be idempotent though. Increasingness is also potentially violated, but this is mostly due to the meet and join not necessarily being increasing in a sponge, so it may make sense to look for a different property. In any case, the above shows that for any image, the set of lower bounds that can be written as a dilation of some other image is non-empty, so it should be possible to implement some sort of projection onto this set, giving an operator that would clearly be anti-extensive *and* idempotent.

6   Conclusion

There is a need for mathematical morphology beyond what can be done with lattices. To this end, we have proposed a novel algebraic structure (closely related to the notion of a weakly associative lattice). This structure generalizes lattices not by forgoing (unique) meets and joins, but rather by letting go of having a (transitive) order. It preserves the absorption property though, as well as a property called "part preservation". These properties are important in the intuitive interpretation of joins and meets: absorption guarantees that the meet of a set is a lower bound of the set, while part preservation guarantees that a meet is truely the "greatest" lower bound, in the sense that it is an upper bound of all other lower bounds.

To demonstrate the potential relevance of this new framework, we give several examples of sponges, some of which were actually defined in earlier work (but not recognized as fitting into some general framework). One of our example sponges operates on a periodic space (something which is utterly impossible with a lattice). One advantage of recognizing these as sponges rather than ad-hoc constructions is that sponges guarantee relatively intuitive interpretations of joins and meets, and allow us to recover at least some properties of familiar operators like structural openings. Also, sponges allow us to reason about the structure itself. For example: we can recognize conditionally complete semisponges and see that any conditionally complete semisponge is a full (conditionally complete) sponge. It may also be possible to give characterizations of certain classes of sponges, as has been done for weakly associative lattices [26].

In terms of future work, the field is wide open: are sponges the "right" generalization of lattices? What kinds of sponges give what opportunities? What classes of operators can be defined? The list of open questions is endless. Also, it would be interesting to see if more interesting sponges can be constructed or found in the literature. For example, one may wonder whether some of the more algebraic methods given by Angulo [1], Burgeth et al. [10] give rise to sponges.

7   Acknowledgements

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