Multiplying huge integers using Fourier transforms

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Outline

The problem

How does FFT Multiplication work?

The implementation

Results

Questions

Outline

- The problem
- How does FFT Multiplication work?
- The implementation
- Results
- Questions

Outline

The problem

How does FFT Multiplication work?

The implementation

Results

Questions

The problem

$$\begin{array}{r} 820348901038490238478324 \\ \underline{\quad 17394232497289349328 94} \times \\ ??? \end{array}$$

Multiplying huge integers occurs in many fields of Computational Science:

- Cryptography
- Number theory
- ...

The problem

Traditional approaches to multiplication require $\boldsymbol{O}(n^2)$ multiplications.

A simple example

$$\begin{aligned} & 123 \\ & \underline{456} \times \\ & 6 \cdot 3 + 6 \cdot 20 + 6 \cdot 100 + \\ & 50 \cdot 3 + 50 \cdot 20 + 50 \cdot 100 + \\ & \underline{400 \cdot 3 + 400 \cdot 20 + 400 \cdot 100} = \\ & 56088 \end{aligned}$$

Two integers of length 3, $3 \times 3 = 9$ operations, this is costly for very large integers.

Can this be done faster?

One way to do this faster is by using Fourier transforms, reducing the arithmetic complexity to $\boldsymbol{O}(n \log n)$.

How does FFT Multiplication work?

Multiplying polynomials

Coefficient form

A polynomial represented in coefficient form is described by a coefficient vector $\mathbf{a} = [a_0, a_1, \ldots, a_{n-1}]$ as follows:

$$p(x) = \sum_{i=0}^{n-1} a_i x^i.$$

Degree

The degree of such a polynomial is the largest index of a nonzero coefficient a_i.

Outline
The problem
How does FFT Multiplication work?
Multiplying polynomials
Vector convolution
The Interpolation Theorem for Polynomials
Calculating the polynomial product in a different form
How to find $2n$ distinct points that can be quickly evaluated?
The Discrete Fourier Transform
The Cooley-Tukey algorithm for FFT
Why is this faster?
The implementation
Results
Questions

How does FFT Multiplication work?

Vector convolution

Multiplying polynomials in coefficient form

Like multiplying integers, takes $\Theta(n^2)$ time:

$$p(x)q(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots + a_{n-1}b_{n-1}x^{2n-2}.$$

Degree of polynomial product

The highest power of x of polynomials p and q having n coefficients is x^{n-1}, so their product has a highest power $x^{(n-1)+(n-1)} = x^{2n-2}$, but for symmetry reasons we define their product of size $2n$, with coefficient vector $c = [c_0, c_1, \ldots, c_{2n-1}]$, defining $c_{2n-1} = 0$.

Multiplying huge integers using Fourier transforms

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Outline

The problem

How does FFT Multiplication work?

Multiplying polynomials

Vector convolution

The Interpolation Theorem for Polynomials

Calculating the polynomial product in a different form

How to find $2n$ distinct points that can be quickly evaluated?

The Discrete Fourier Transform

The Cooley-Tukey algorithm for FFT

Why is this faster?

The implementation

Results

Questions

How does FFT Multiplication work?

The Interpolation Theorem for Polynomials

We have to find a different way to represent a polynomial other than the coefficient form, a representation that allows for faster multiplication than $\Theta(n^2)$.

The Interpolation Theorem for Polynomials

Given a set of n points in the plane, $S = (x_0, y_0), (x_1, y_1), (x_2, y_2), ..., (x_n - 1, y_n - 1)$, such that the x_i's are all distinct, there is a unique degree-$(n-1)$ polynomials $p(x)$ with $p(x_i) = y_i$, for $i = 0, 1, \ldots, n-1$.

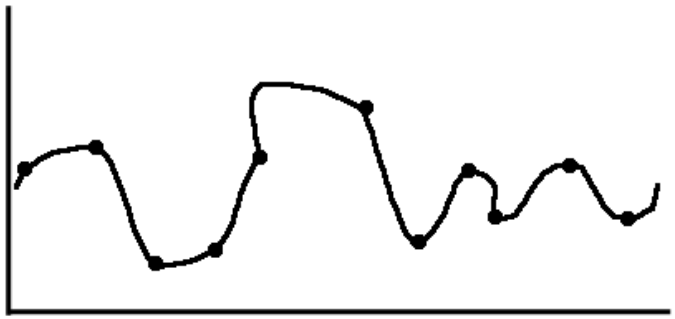

Figure: A set of n distinct points in the plane uniquely defines an $(n-1)$-degree polynomial.

Multiplying huge integers using Fourier transforms

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Outline

The problem

How does FFT Multiplication work?

Multiplying polynomials

Vector convolution

The Interpolation Theorem for Polynomials

Calculating the polynomial product in a different form

How to find $2n$ distinct points that can be quickly evaluated?

The Discrete Fourier Transform

The Cooley-Tukey algorithm for FFT

Why is this faster?

The implementation

Results

Questions

How does FFT Multiplication work?

Calculating the polynomial product in a different form

Idea

We know that the product of polynomials p and q of degree $n-1$ is a polynomial of degree $2n-1$, so:

- We evaluate p in $2n$ points;
- We evaluate q in the same $2n$ points;
- So if we compute the products of p and q evaluated at these $2n$ points, we have uniquely defined the polynomial of their product (albeit not in coefficient form).

Not there yet

Evaluating $2n$ different inputs will take $O(n^2)$ time. The challenge is to find a set of inputs that has specific properties so that some of the outputs can be used to quickly evaluate other parts of the input.

How does FFT Multiplication work?

How to find $2n$ distinct points that can be quickly evaluated?

Primitive Roots of Unity

A number ω is a primitive nth root of unity, for $n \geq 2$, if it satisfies the following properties:

- $\omega^n = 1$, that is, ω is an nth root of 1.
- The numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are distinct.

Special properties

- (Reflective property) If ω is a primitive nth root of unity and $n \geq 2$ is even, then $\omega^{k+n/2} = -\omega^k$.
- (Reduction property) If ω is a primitive $(2n)$th root of unity, then ω^2 is a primitive nth root of unity. So if $1, \omega, \omega^2, \ldots, \omega^{2n-1}$ are distinct, then $1, \omega^2, (\omega^2)^2, \ldots, (\omega^2)^{n-1}$ are also distinct (since the latter set is a subset of the former set).

Outline
The problem
How does FFT Multiplication work?
Multiplying polynomials
Vector convolution
The Interpolation Theorem for Polynomials
Calculating the polynomial product in a different form
How to find $2n$ distinct points that can be quickly evaluated?
The Discrete Fourier Transform
The Cooley-Tukey algorithm for FFT
Why is this faster?
The implementation
Results
Questions

How does FFT Multiplication work?

The Discrete Fourier Transform

DFT

The Discrete Fourier Transform of an $(n-1)$-degree polynomial $p(x)$, is its evaluation at the nth roots of unity, $\omega^0, \omega^1, \omega^2, ..., \omega^{n-1}$.

Note

Done naively, this can still take $O(n^2)$ time. That is why we have the Fast Fourier Transform.

Different methods

There are many ways in which an FFT can be implemented, in our implementation we used a recursive divide-and-conquer approach. This method is relatively simple but its use is widespread. In the following, we will explain this method in more detail.

How does FFT Multiplication work?

The Cooley-Tukey algorithm for FFT

A divide and conquer approach

If n is even, we can divide a degree-$(n-1)$ polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

into two degree-$(n/2-1)$ polynomials

$$p^{\text{even}}(x) = a_0 + a_2 x + a_4 x^2 + \cdots + a_{n-2} x^{n/2-1}$$
$$p^{\text{odd}}(x) = a_1 + a_3 x + a_5 x^2 + \cdots + a_{n-1} x^{n/2-1}$$

which we can combine into p using the equation

$$p(x) = p^{\text{even}}(x^2) + x p^{\text{odd}}(x^2).$$

Multiplying huge integers using Fourier transforms

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Outline

The problem

How does FFT Multiplication work?

Multiplying polynomials

Vector convolution

The Interpolation Theorem for Polynomials

Calculating the polynomial product in a different form

How to find $2n$ distinct points that can be quickly evaluated?

The Discrete Fourier Transform

The Cooley-Tukey algorithm for FFT

Why is this faster?

The implementation

Results

Questions

How does FFT Multiplication work?

Why is this faster?

Reuse of computations

By the reduction property, the values $(\omega^2)^0, \omega^2, (\omega^2)^2, (\omega^2)^3, \ldots, (\omega^2)^{n-1}$ are $(n/2)$th roots of unity. Thus, we can evaluate each of $p^{\text{even}}(x)$ and $p^{\text{odd}}(x)$ at these values, and we can reuse those same computations in evaluating $p(x)$.

Further reducing computation

If we use an even n, then by the reflective property, we know that $\omega^{k+n/2} = -\omega^k$. So we can reuse the computations of half the number of points to derive the values of the other half.

Outline
The problem
How does FFT Multiplication work?
Multiplying polynomials
Vector convolution
The Interpolation Theorem for Polynomials
Calculating the polynomial product in a different form
How to find $2n$ distinct points that can be quickly evaluated?
The Discrete Fourier Transform
The Cooley-Tukey algorithm for FFT
Why is this faster?
The implementation
Results
Questions

The implementation

Pseudo-code FFT algorithm

```
Algorithm FFT(a, omega)
  Input: An n-length coefficient vector a = [a_0,a_1,...,a_(n-1)]
         and a primitive nth root of unity omega (n = a power of 2)
  Output: A vector y of values of the polynomial for a
          at the nth roots of unity.
  if n=1 then
    return y = a.
  end
  // divide step
  a_even = [a_0,a_2,a_4,...,a_(n-2)]
  a_odd  = [a_1,a_3,a_5,...,a_(n-1)]
  // recursive calls with omega^2 as n/2th root of unity
  y_even = FFT(a_even, omega^2)
  y_odd  = FFT(a_odd,  omega^2)
  x = 1 // storing powers of omega
  // combine step, using x = omega^i
  for (i=0;i<n/2,i++)
    y[i]     = y_even[i]+x*y_odd[i]
    y[i+n/2] = y_even[i]-x*y_odd[i] // because of reflective prop.
    x = x*omega
  end
  return y
end
```


Outline
The problem
How does FFT Multiplication work?
The implementation
Pseudo-code FFT algorithm
How to multiply two polynomials?
The Inverse Discrete Fourier Transform
Multiplying big integers
Correctness
Complexity of multiplication with FFT
Results
Questions

The implementation

How to multiply two polynomials?

Convolution

Now that we know how the FFT works, we can use it to multiply two polynomials given as coefficient vectors **a** and **b**.

- Pad **a** and **b** with n zeros so that they will be of size $2n$: $\mathbf{a}' = [a_0, a_1, \ldots, a_{n-1}, 0, 0, \ldots, 0]$.
- Compute the FFT's **y**=FFT(**a**) and **z**=FFT(**b**).
- Multiply the vectors **y** and **z** component-wise $\mathbf{m} = \mathbf{y} \times \mathbf{z} = [y_0 z_0, y_1 z_1, \ldots, y_{2n-1} z_{2n-1}]$.
- Compute the inverse FFT of **m**.

Note

In practice, instead of $2n$, the next power of 2 is used instead. This is required by the divide and conquer algorithm.

The implementation

The Inverse Discrete Fourier Transform

Definition

We use the Inverse Discrete Fourier Transform to recover the coefficients from a vector **y** of the values of a degree-$(n-1)$ polynomial p at the nth roots of unity, $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

$$a_i = \sum_{j=0}^{n-1} y_j \omega^{-ij} / n.$$

Note

We will not derive this formula here, but it follows from the formulation of the DFT as a matrix multiplication of the column vector **a** and a matrix of powers of ω. The above formula can then be derived from multiplying **a** by the inverse of this matrix.

Outline
The problem
How does FFT Multiplication work?
The implementation
Pseudo-code FFT algorithm
How to multiply two polynomials?
The Inverse Discrete Fourier Transform
Multiplying big integers
Correctness
Complexity of multiplication with FFT
Results
Questions

The implementation

Multiplying big integers

The Base

An integer can be expressed as a polynomial of a certain base B.

Example

The most simple implementation would be of base $B = 10$.

$$\begin{matrix} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$

But different bases can also be used, for example $B = 100$.

$$\begin{matrix} b_2 & b_1 & b_0 \\ 12 & 34 & 56 \end{matrix}$$

Any positive integer can be used as base, but for the sake of simplicity, in our implementation we only used bases that are powers of 10.

Outline
The problem
How does FFT Multiplication work?
The implementation
Pseudo-code FFT algorithm
How to multiply two polynomials?
The Inverse Discrete Fourier Transform
Multiplying big integers
Correctness
Complexity of multiplication with FFT
Results
Questions

The implementation

Correctness

Mathematica as verification

Mathematica supports multiplication of Big Integers.

Note

Note that when we multiply big integers, the coefficients a_i that we get after the Inverse Fourier Transform are supposed to be integers. Because we divide by n, we incur a certain rounding error. In our results we will analyse this error.

Outline
The problem
How does FFT Multiplication work?
The implementation
Pseudo-code FFT algorithm
How to multiply two polynomials?
The Inverse Discrete Fourier Transform
Multiplying big integers
Correctness
Complexity of multiplication with FFT
Results
Questions

The implementation

Complexity of multiplication with FFT

Arithmetic complexity

Because the algorithm `FFT()` calls itself two times with half the input, and then performs the merge in a loop of $O(n \log(n))$, so according to the Master Theorem, the complexity is $O(n \log(n))$.

Actual complexity

The actual complexity of the algorithm in terms of the number of operations is higher, because then we would have to include ordinary multiplications as a factor in the complexity. This also shows in practice, because the overhead of these multiplications is significant, especially when working with a large base number.

Outline
The problem
How does FFT Multiplication work?
The implementation
Pseudo-code FFT algorithm
How to multiply two polynomials?
The Inverse Discrete Fourier Transform
Multiplying big integers
Correctness
Complexity of multiplication with FFT
Results
Questions

Results

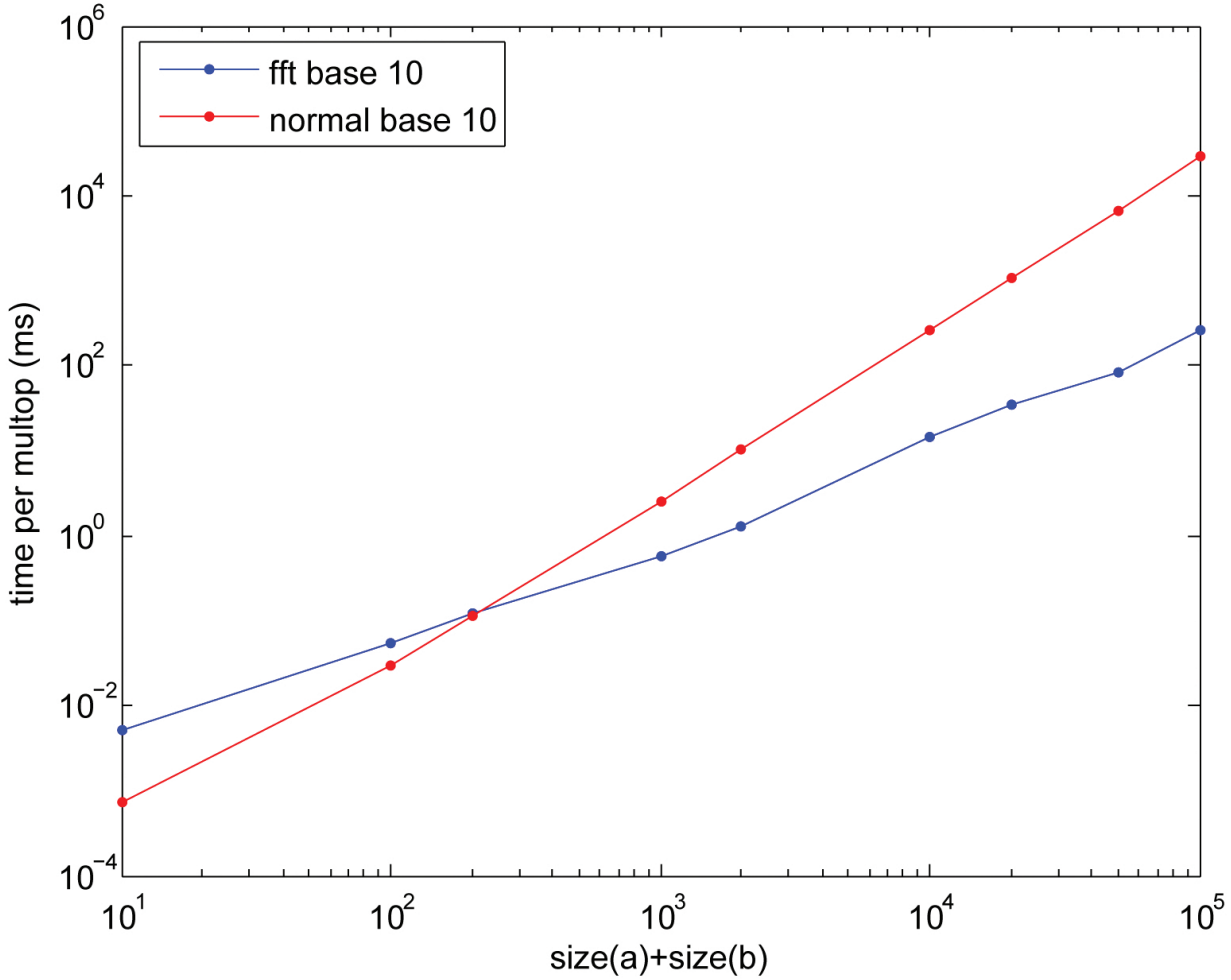
Experimental settings

- We have written an implementation that compares the speed of a naive multiplication implementation versus an FFT one.
- Both the FFT and the naive approach first split the integers up in coefficients to a certain base B, as shown before.
- Where we let the program perform the same multiplication for a duration of 500ms and then count how many times the multiplication has been performed to get a good average of the time a single multiplication of the polynomials took.
- We performed tests on nine multiplications of different lengths (in decimal form): 5×5, 50×50, 100×100, 500×500, 1000×1000, 5000×5000, 10000×10000, 25000×250000, 50000×50000.

Outline
The problem
How does FFT Multiplication work?
The implementation
Results
Experimental settings
Base 10
Base 100
Base 1000
Base 10000
Comparing FFT's
Comparing Normal Multiplications
Comparing both multiplications
Maximum Square Error
Conclusions
Questions

Results

Base 10 FFT Multiplication vs. Base 10 Normal Multiplication



time(ms)	10	100	200	1000	2000	10000	20000	50000	100000
FFT	0.005	0.055	0.122	0.580	1.281	14.156	35.125	85.75	258.0
Normal	0.0007	0.031	0.115	2.563	10.25	265.5	1070.5	6836.0	28211.0

Multiplying huge integers using Fourier transforms

Ando Emerencia (s1283936)

Outline

The problem

How does FFT Multiplication work?

The implementation

Results

Experimental settings

Base 10

Base 100

Base 1000

Base 10000

Comparing FFT's

Comparing Normal Multiplications

Comparing both multiplications

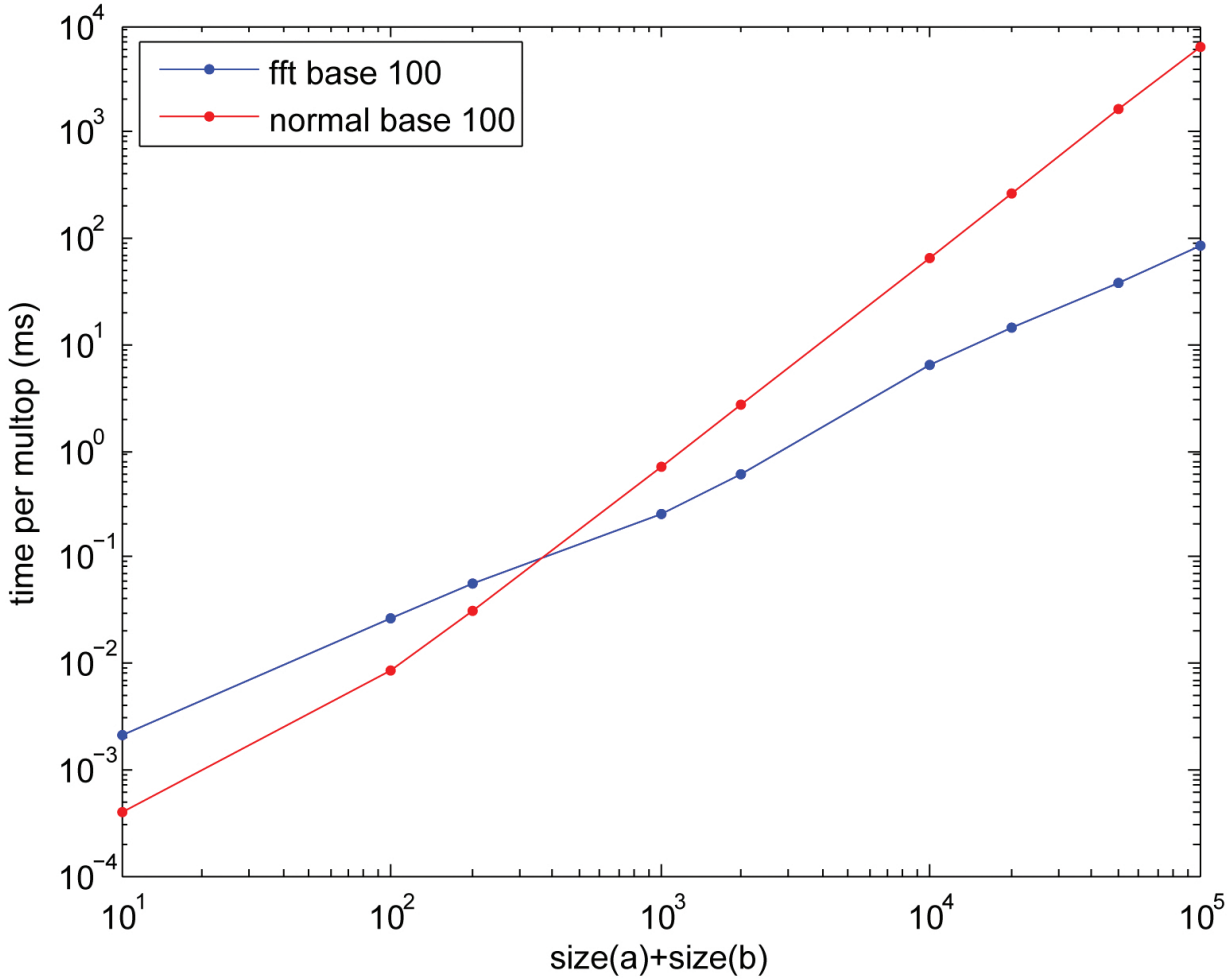
Maximum Square Error

Conclusions

Questions

Results

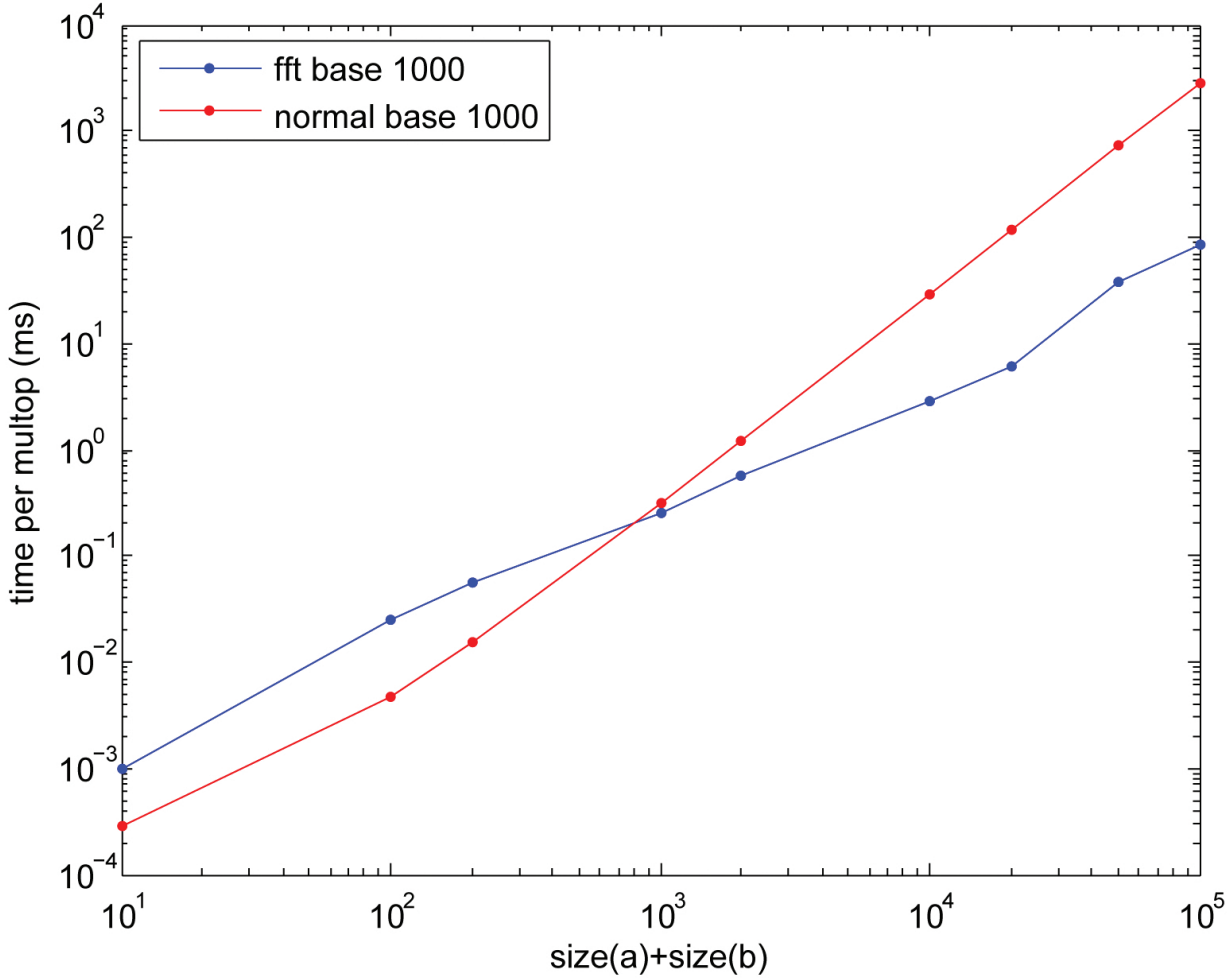
Base 100 FFT Multiplication vs. Base 100 Normal Multiplication



time(ms)	10	100	200	1000	2000	10000	20000	50000	100000
FFT	0.0021	0.0258	0.057	0.259	0.611	6.344	14.156	37.0	86.0
Normal	0.0004	0.0086	0.031	0.701	2.688	66.5	265.5	1680.0	6398.5

Results

Base 1000 FFT Multiplication vs. Base 1000 Normal Multiplication



time(ms)	10	100	200	1000	2000	10000	20000	50000	100000
FFT	0.0010	0.0248	0.055	0.259	0.580	2.930	6.094	37.0	86.0
Normal	0.0003	0.0048	0.015	0.320	1.219	28.313	117.25	742.5	2820.0

Multiplying huge integers using Fourier transforms

Ando Emerencia (s1283936)

Outline

The problem

How does FFT Multiplication work?

The implementation

Results

Experimental settings

Base 10

Base 100

Base 1000

Base 10000

Comparing FFT's

Comparing Normal Multiplications

Comparing both multiplications

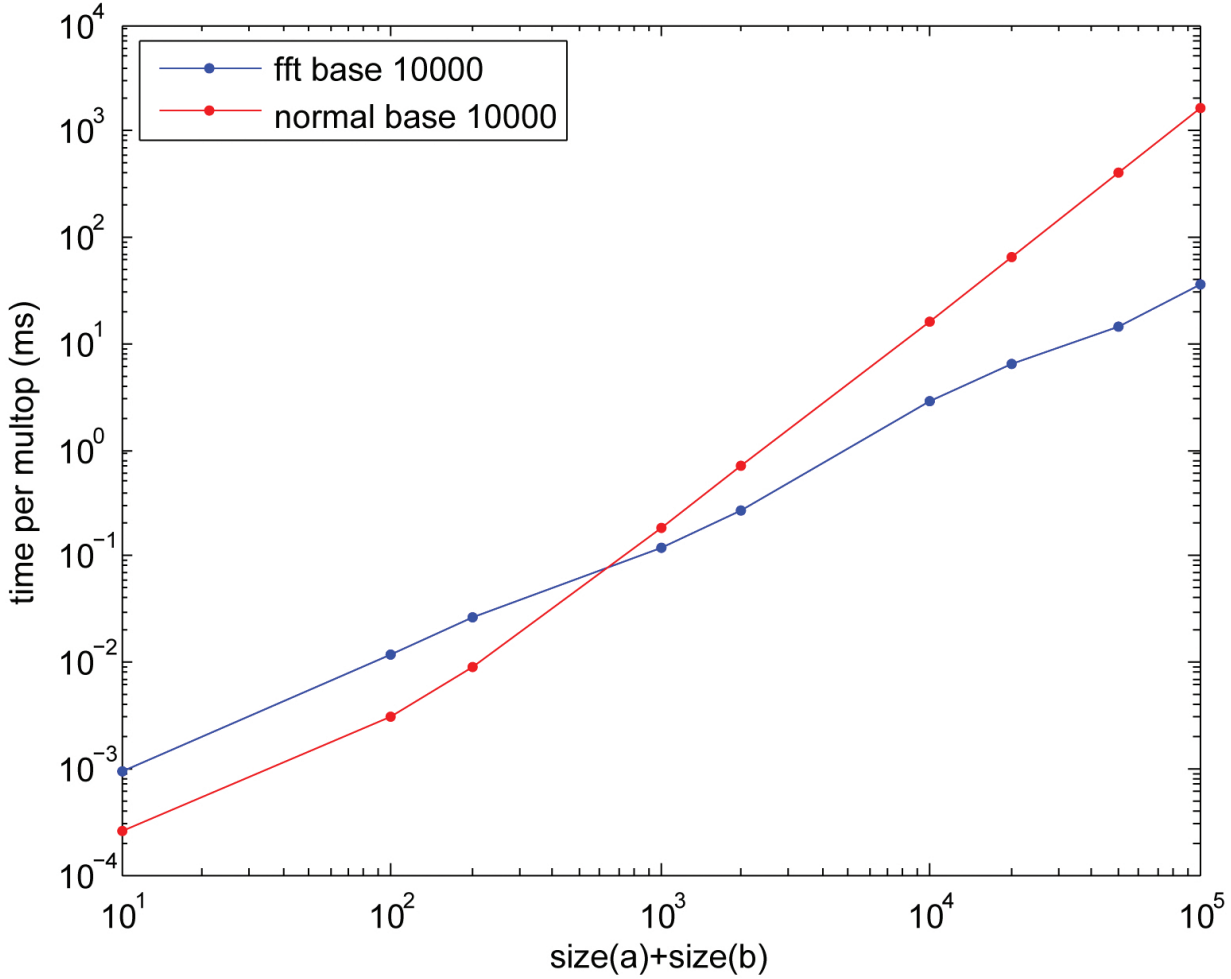
Maximum Square Error

Conclusions

Questions

Results

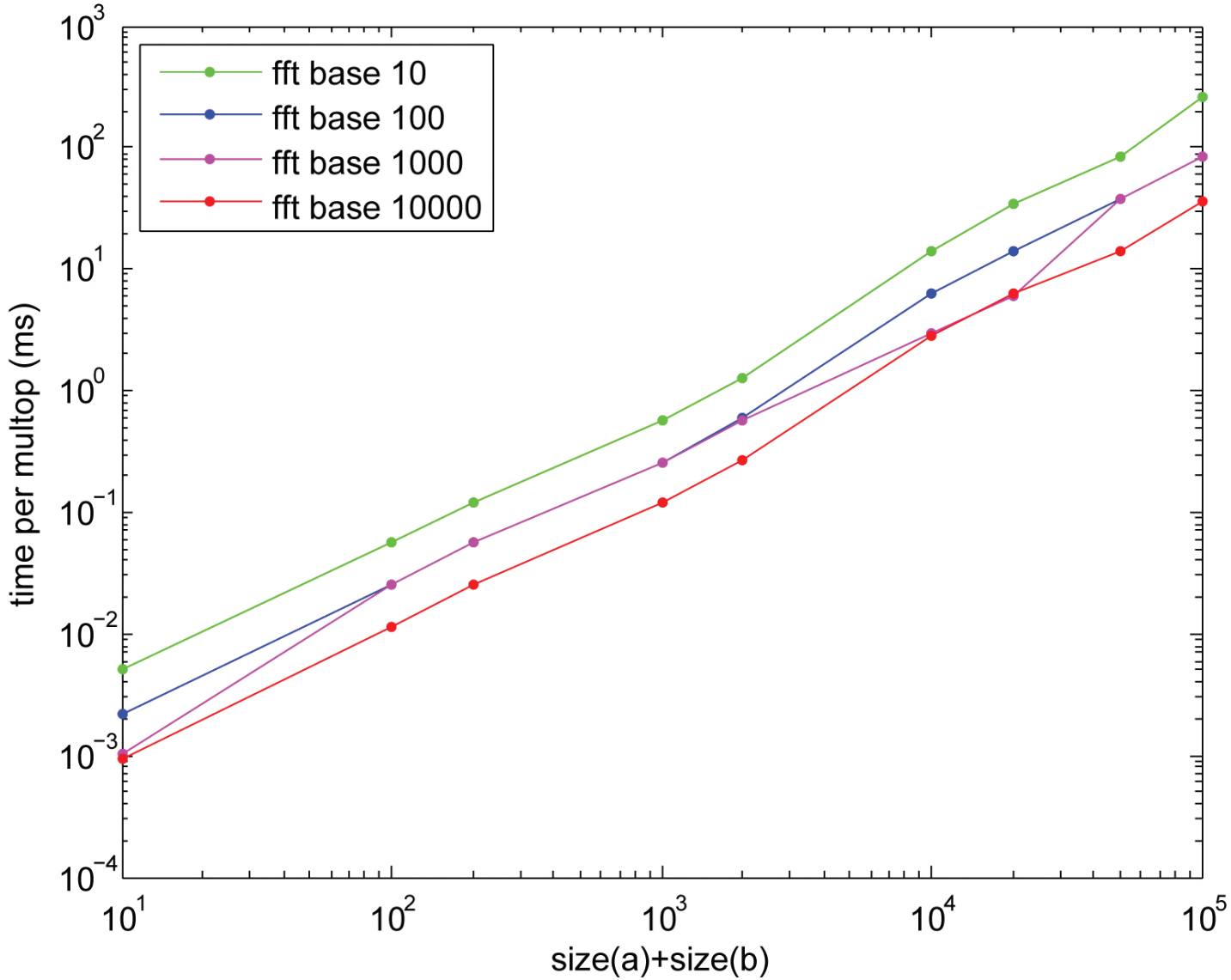
Base 10000 FFT Multiplication vs. Base 10000 Normal Multiplication



time(ms)	10	100	200	1000	2000	10000	20000	50000	100000
FFT	0.0010	0.0115	0.026	0.122	0.269	2.831	6.25	14.05	35.9
Normal	0.0003	0.0031	0.009	0.183	0.708	15.65	65.6	415.6	1671.8

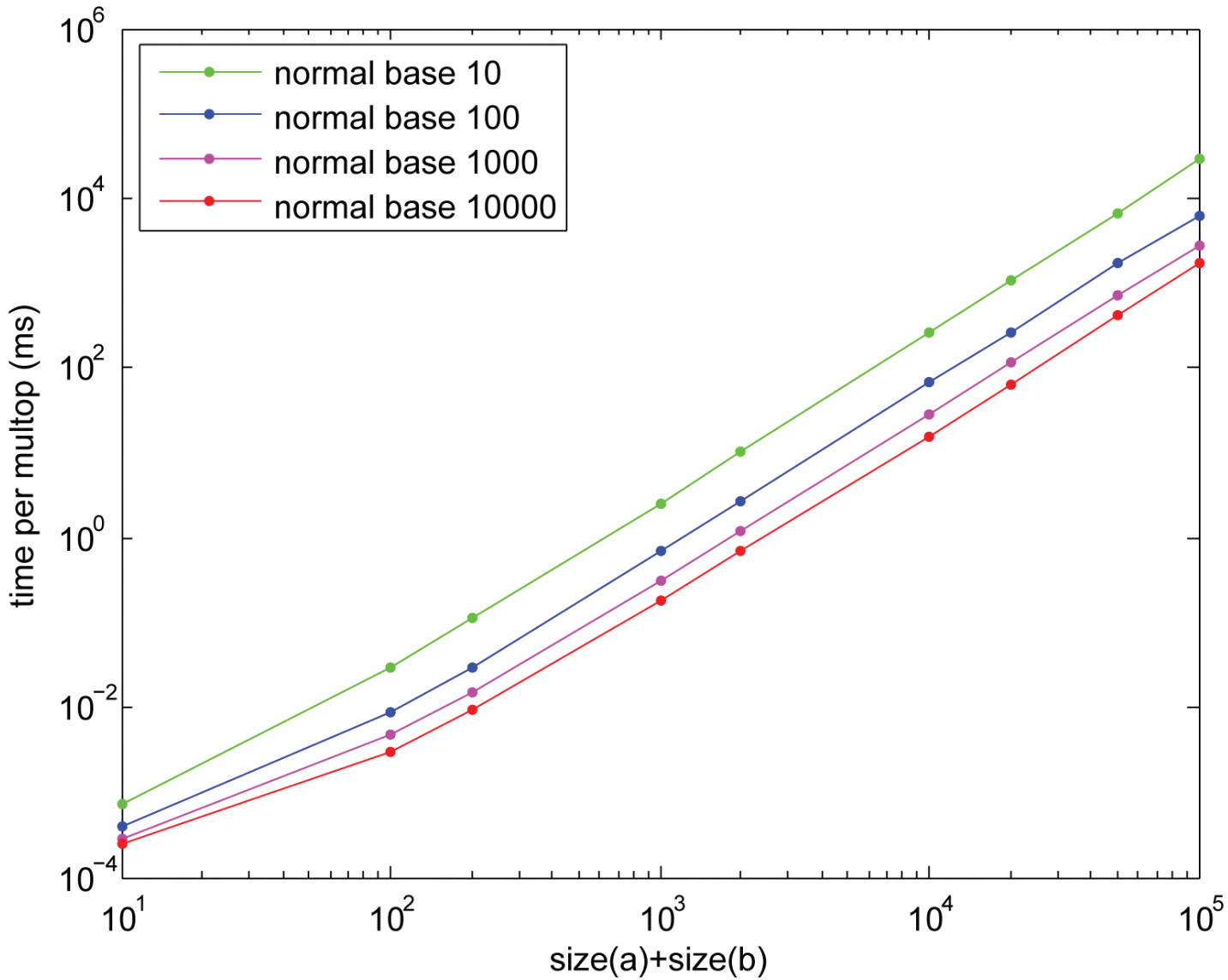
Results

Comparing FFT's of various bases



Results

Comparing Normal Multiplications at various bases



Outline

The problem

How does FFT Multiplication work?

The implementation

Results

Experimental settings

Base 10

Base 100

Base 1000

Base 10000

Comparing FFT's

Comparing Normal Multiplications

Comparing both multiplications

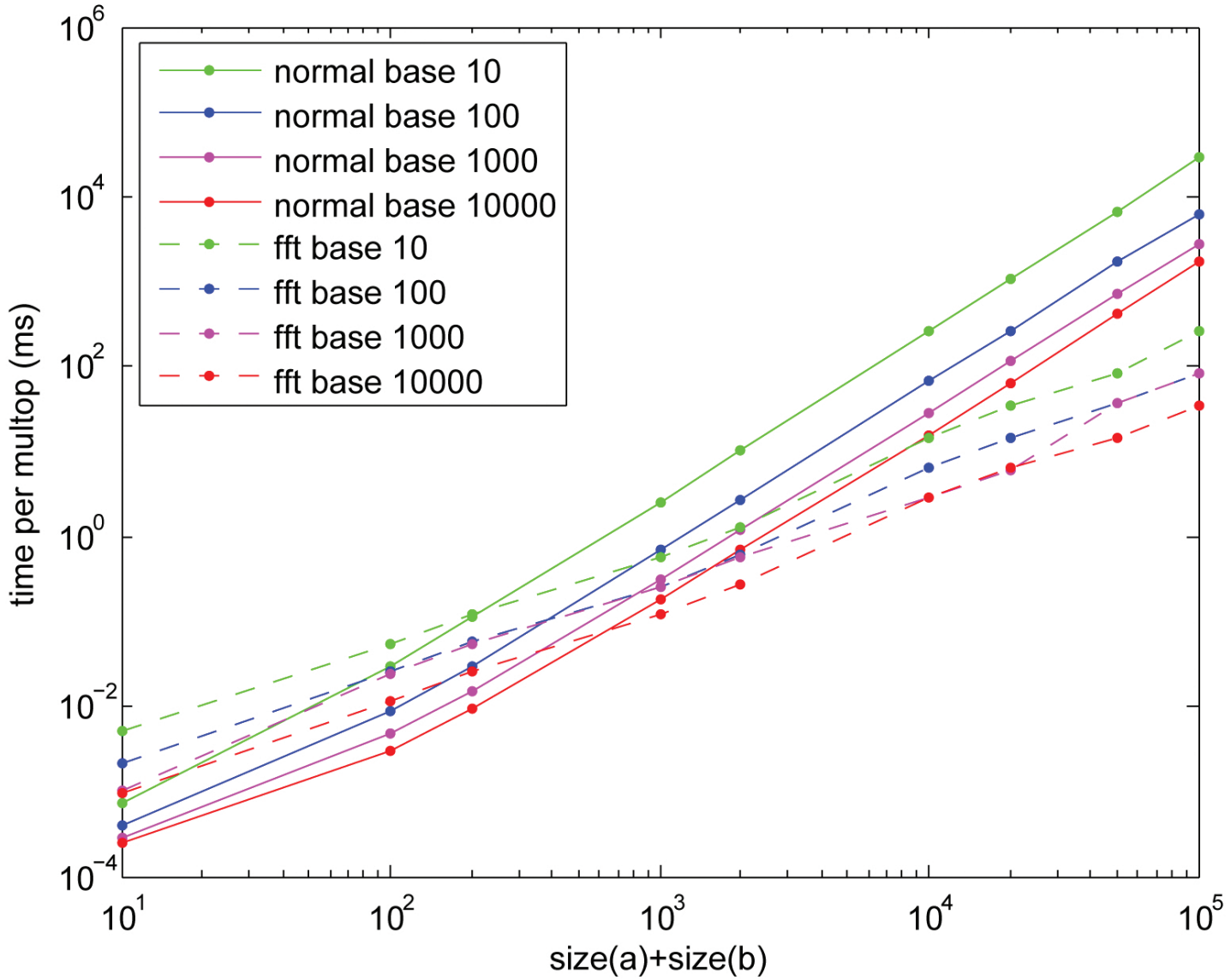
Maximum Square Error

Conclusions

Questions

Results

Comparing Normal Multiplications with FFT Multiplications at various bases



Outline

The problem

How does FFT Multiplication work?

The implementation

Results

Experimental settings

Base 10

Base 100

Base 1000

Base 10000

Comparing FFT's

Comparing Normal Multiplications

Comparing both multiplications

Maximum Square Error

Conclusions

Questions

Results

Maximum Square Error

Definition

In the inverse FFT procedure, when rounding down from doubles to integers, we keep track of the square of the highest deviation, that is:

$$\text{SquareError} = (\lfloor x + 0.5 \rfloor - x)^2$$

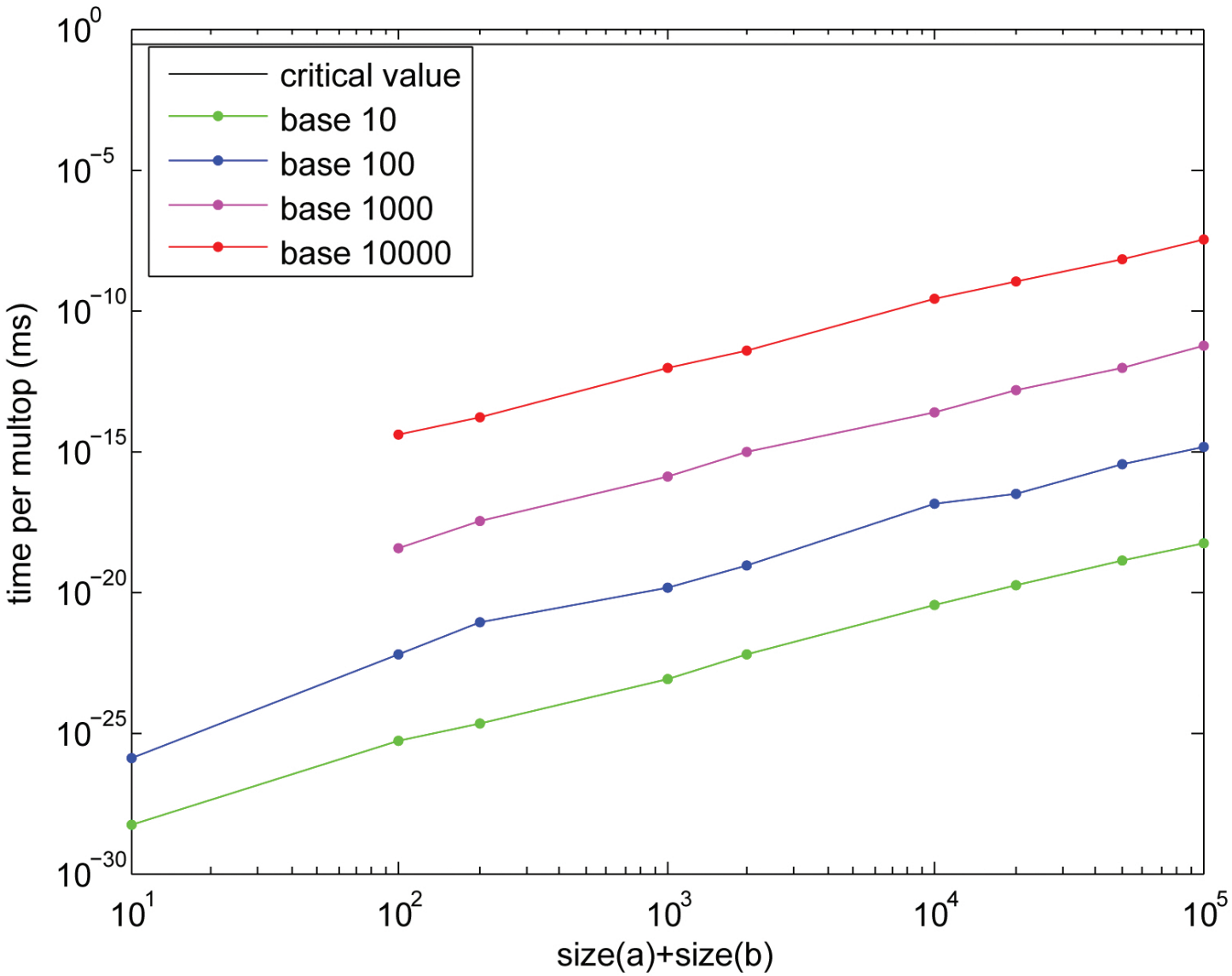
Critical value

Note that if the Maximum Square Error comes close to 0.25, then the real error comes close to 0.5 and then there is a chance that we make rounding errors and end up with an incorrect answer.

Outline
The problem
How does FFT Multiplication work?
The implementation
Results
Experimental settings
Base 10
Base 100
Base 1000
Base 10000
Comparing FFT's
Comparing Normal Multiplications
Comparing both multiplications
Maximum Square Error
Conclusions
Questions

Results

Maximum Square Error of the FFT multiplication at various bases



Outline

The problem

How does FFT Multiplication work?

The implementation

Results

Experimental settings

Base 10

Base 100

Base 1000

Base 10000

Comparing FFT's

Comparing Normal Multiplications

Comparing both multiplications

Maximum Square Error

Conclusions

Questions

Results

Conclusions

- Choosing a larger base will speed up the calculation of both the normal and the FFT multiplications.
- Choosing a larger base will increase the Maximum Square Error of the inverse FFT.
- Increasing the size of the input will increase the Maximum Square Error for the FFT.
- For number sizes of around 10000 and higher (depending on the base used), FFT multiplication can easily be more than 100 times as fast as a normal multiplication implementation.
- When working with numbers of 10^7 or higher, we may have to choose a smaller base in order to avoid rounding errors from causing an incorrect answer.

Outline
The problem
How does FFT Multiplication work?
The implementation
Results
Experimental settings
Base 10
Base 100
Base 1000
Base 10000
Comparing FFT's
Comparing Normal Multiplications
Comparing both multiplications
Maximum Square Error
Conclusions
Questions

Questions?

References

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