The Sum and Product puzzle

There are two natural numbers m, n greater than 1: their sum does not exceed 100. Now Sarah knows the sum $s = m + n$, and Paul knows the product $p = m \cdot n$. They have the following dialogue:

> Paul: I don't know what m and n are.
> Sarah: No, I knew that you didn't know.
> Paul: Oh, but now I do know!
> Sarah: And so do I!

What are m and n?

The solution

After Paul's first statement, we know that there is not a unique decomposition of p in $m, n > 1$ with $m + n \leq 100$. From this we conclude

> p is not the product of two primes;
> p is not the third power of a prime;
> $p \neq 2^{11}$;
> p does not contain a prime factor greater that 50.

From Sarah's first statement, we learn that *all* splittings of s in m and n satisfy these four conditions. This implies that s is odd (for Goldbach's conjecture that all even numbers except 2 and 4 are the sum of two primes, holds below 100), s is not of the form $q + 2$ with q prime, and $s < 55$ (for 53 is the smallest prime greater that 50). In other words:

$$s \in S = \{11, 17, 23, 27, 29, 35, 37, 41, 47, 51, 53\}$$

By Paul's second statement, we know that p has exactly one decomposition m, n with $m + n \in S$. We call such products p *good* products. Before we go on, we make the following observation:

$$\text{if } p = 2^k \cdot q \text{ with } q \text{ prime, } k > 1 \text{ and } 2^k + q \in S\text{, then } p \text{ is good.} \tag{1}$$

This follows from the fact that all other decompositions of p have an even sum.

Sarah's second statement implies that s admits exactly one good p. Here s admits p means: there are $m, n < 1$ with $m + n = s$ and $m \cdot n = p$. We shall show that 17 admits only 52, which is good (by (1)); all other elements of S admit at least two good p's.

First we observe that the other products admitted by 17 are not good. This follows from

$$\begin{array}{ll}
2 \cdot 15 = 30 = 5 \cdot 6 & 5 + 6 = 11 \in S \\
3 \cdot 14 = 42 = 2 \cdot 21 & 2 + 21 = 23 \in S \\
5 \cdot 12 = 60 = 3 \cdot 20 & 3 + 20 = 23 \in S \\
6 \cdot 11 = 66 = 2 \cdot 33 & 2 + 33 = 35 \in S \\
7 \cdot 10 = 70 = 2 \cdot 35 & 2 + 35 = 37 \in S \\
8 \cdot 9 = 72 = 3 \cdot 24 & 3 + 24 = 27 \in S
\end{array}$$

Finally we show that all elements of S except 17 admit at least two good products. We observe

$$\begin{aligned}
11 &= 4+7 = 8+3\\
23 &= 4+19 = 16+7\\
27 &= 4+23 = 8+19\\
29 &= 16+13\\
35 &= 4+31 = 16+19\\
37 &= 8+29 = 32+5\\
41 &= 4+37\\
47 &= 4+43 = 16+31\\
51 &= 4+47 = 8+43\\
53 &= 16+37
\end{aligned}$$

so by applying (1) we see that all elements of S admit at least one good product; moreover, 11, 23, 27, 35, 37, 47 and 51 admit at least two good products. To see that 29, 41 and 53 each also admit a second good product, we observe

$$\begin{array}{llll}
29 = 2+27 & 2\cdot 27 = 54 = 3\cdot 18 = 6\cdot 9 & 3+18 = 21 \notin S & 6+9 = 15 \notin S\\
41 = 7+34 & 7\cdot 34 = 238 = 14\cdot 17 & 14+17 = 31 \notin S & \\
53 = 10+43 & 10\cdot 43 = 430 = 5\cdot 86 & 5+86 = 91 \notin S &
\end{array}$$

so 29 admits 54, 41 admits 238, 53 admits 430, and these three products are good.

So the solution of the puzzle is 4,13, with $s = 17$ and $p = 52$.