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APPROXIMATE SOLUTION METHODS FOR LINEAR STOCHASTIC DIFFERENCE EQUATIONS

II. INHOMOGENEOUS EQUATIONS, MULTI-TIME AVERAGES, BIOLOGICAL APPLICATION

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The cumulant expansion for linear stochastic difference equations introduced in part I is applied to the general case, where the equation contains multiplicative, additive and initial value terms which are all random and statistically interdependent. Also the two-time correlation functions of the solution are discussed. Finally the expansion for the probability density functions is studied. In the case that the coefficient matrix constitutes a Markov process, an exact equation for the joint probability density function of the solution of the difference equation and the random coefficient matrix is derived. From this equation the moments of the solution can be obtained in a simple way. As an application we consider the growth of a biological population with two age classes in a random environment, which itself is modelled by a two-state Markov chain. The exact results are compared with those of the cumulant expansion and with previous findings of Tuljapurkar.

1. Introduction

This is the second part of our treatment of vector stochastic difference equations of the form

$$u(t) = A(t,\omega)u(t-1) + f(t,\omega) \quad (t \in I_{t_0+1}), \tag{1.1a}$$

$$u(t_0) = u_0(\omega), \tag{1.1b}$$

where $u(t)$ is a vector, $A(t,\omega)$ and $f(t,\omega)$ a random matrix and vector, respectively, and $u_0(\omega)$ a random initial condition. For any integer a, I_a denotes the set of integers $\{a, a+1, a+2, \ldots\}$. The stochastic nature of the above quantities is indicated by the parameter $\omega \in \Omega$, where (Ω, Σ, P) is a probability space. Averages with respect to the probability measure P are indicated by angular brackets $\langle \ldots \rangle$.

In part I[1]) (hereafter referred to as I) we considered (1.1) in the homogeneous case ($f \equiv 0$) and u_0 non-random. We assumed $A(t,\omega)$ to be of the form

$$A(t,\omega) = A_0(t) + \alpha A_1(t,\omega), \tag{1.2}$$

where $A_0(t)$ is a deterministic matrix, $A_1(t,\omega)$ a random matrix with short correlation time τ_c, and α a small parameter. Then a systematic perturbation expansion was developed, assuming that $\alpha\tau_c \ll 1$. The result was that the average of $u(t)$ obeys itself a first order difference equation (assuming that A_0 is nonsingular),

$$\langle u(t)\rangle = \{A_0 + K(t/t_0)\}\langle u(t-1)\rangle\,, \tag{1.3}$$

where the matrix $K(t/t_0)$ is an infinite sum of terms which contain the so-called time ordered cumulants of $A_1(t)$. If $t - t_0 \gg \tau_c$ this matrix is independent of t_0. In the case of a singular A_0 the derivation of (1.3) has to be modified, but it was found that, taking into account terms up to order α^2 only and after a transient time, $\langle u(t)\rangle$ obeys again an equation of the form (1.3).

In this paper we consider again (1.1) in the general case where A, f and u_0 are all random and mutually correlated (section 2). Our treatment closely follows our previous study of stochastic differential equations[2,3]) (henceforth abbreviated as s.d.e.), so we will only emphasize those points where essential differences compared to the latter case occur.

In section 3 we consider the correlation function $\langle u(t)\otimes u(t')\rangle$ of $u(t)$, where $\otimes$ denotes a Kronecker product. Homogeneous as well as inhomogeneous equations are discussed.

We also derive an expansion for the probability density functions of $u(t)$ (subsection 4.1). An application to a one-dimensional case which results in the well-known lognormal distribution is given in subsection 4.2. If the matrix A in (1.1) is a Markov process and f is identically zero, an exact equation for the probability density function of the joint process $\{u(t), A(t)\}$ is derived, from which also exact expressions for the moments of $u(t)$ can be inferred (subsection 4.3).

In the final section 5, an example from population biology is examined. Here one has a model for populations with distinct age (size, . . .) classes in a random environment. We take a population with two age classes and model the environment by a two state Markov chain. The cumulant expansion to second order in α is applied to this model to obtain the average number of individuals in each age class (subsection 5.1). We compare the results with those of Tuljapurkar[4,9]) (subsection 5.2) and exact results (subsection 5.3) obtained by application of the method of subsection 4.3.

2. The expansion in the general case

In this section we extend the cumulant expansion of part I to the inhomogeneous equation (1.1) with random initial condition. There are three correlation times of importance now, the autocorrelation time of $A_1(t)$, and the cross correlation times of $A_1(t)$ with $f(t)$ and with u_0. We assume all these to be finite and denote the largest of them by τ_c. The result will be obtained by transforming (1.1) with sure initial condition first to a homogeneous equation, for which the expansion developed in I is applicable and subsequently extracting from the result a difference equation for $\langle u(t)\rangle$ itself (subsection 2.1). In subsection 2.2 the case where in addition the initial value $u_0(\omega)$ is random, is handled by reducing it to an inhomogeneous equation with sure initial condition.

2.1. *The inhomogeneous case*

We first consider (1.1) with nonrandom u_0. Define an interaction representation (denoted by superscripts (1)) via

$$u(t)=\mathscr{X}_{A_0}(t/t_0)u^{(1)}(t)\quad (t\in I_{t_0+1}), \tag{2.1a}$$

$$A_1^{(1)}(t)=\mathscr{X}_{A_0}^{-1}(t/t_0)A_1(t)\mathscr{X}_{A_0}(t-1/t_0),\quad g(t)=\mathscr{X}_{A_0}^{-1}(t/t_0)f(t), \tag{2.1b}$$

where $A_0(t)$ is assumed to be non-singular on I_{t_0+1}. Here we define for an arbitrary matrix $A(t)$ the matrizants

$$\mathscr{X}_A(t/t_0)=\overleftarrow{T}\prod_{s=t_0+1}^{t}A(s),\quad X_A^{-1}(t/t_0)=\overrightarrow{T}\prod_{s=t_0+1}^{t}A^{-1}(s)\quad (t\in I_{t_0+1}), \tag{2.2a}$$

$$\mathscr{X}_A(t_0/t_0)=\mathscr{X}_A^{-1}(t_0/t_0)=1\,, \tag{2.2b}$$

where $\mathscr{X}_A^{-1}$ only exists if A is non-singular. Here $\overleftarrow{T}$ and $\overrightarrow{T}$ are time- and anti-time ordering operators (latest times to the left and right, respectively). Then (1.1) leads to

$$\Delta u^{(1)}(t-1)=V(t)u^{(1)}(t-1)+g(t)\,, \tag{2.3a}$$

$$u^{(1)}(t_0)=u_0\,, \tag{2.3b}$$

where Δ denotes the difference operator ($\Delta f(t)=f(t+1)-f(t)$ for arbitrary $f(t)$), and

$$V(t)=\alpha A_1^{(1)}(t)\,. \tag{2.4}$$

Now define the enlarged state vector

$$w(t)=\begin{pmatrix}u^{(1)}(t)\\ z(t)\end{pmatrix},$$

where $z(t)$ is a scalar function with constant value 1. From (2.3) one obtains for $w(t)$

$$\Delta w(t-1)=B(t)w(t-1),\quad w(t_0)=\begin{pmatrix}u_0\\ 1\end{pmatrix}, \tag{2.5}$$

where $B(t)$ is the matrix

$$B(t)=\left(\begin{array}{c|c}V(t) & g(t)\\ \hline \emptyset & 0\end{array}\right). \tag{2.6}$$

The symbol $\emptyset$ denotes a matrix or vector with all elements zero. Applying the result of section 2 of I we have

$$\Delta\langle w(t-1)\rangle = K_B(t/t_0)\langle w(t-1)\rangle\,, \tag{2.7}$$

where

$$K_B(t/t_0) = \left\langle B(t)\colon \overleftarrow{T} \prod_{s=t_0+1}^{t-1} \{1+\hat{B}(s)\}\colon \right\rangle_{\mathrm{p}} \tag{2.8a}$$

and

$$\hat{B}(t) = B(t)\{1-B(t)\}^{-1}\,, \tag{2.8b}$$

which exists if α is small enough. The colons in (2.8a) indicate that one should first expand the expression between them in powers of B and subsequently take the partially time ordered cumulant* $\langle\ldots\rangle_{\mathrm{p}}$ of each term. Any p-cumulant which contains m matrices B can be expressed in the moments of B of order $\leqslant m$, according to the rules given in the appendix of I. The reason for writing $1+\hat{B}(s)$ in (2.8a) instead of $(1-B(s))^{-1}$ is that in this form the analogy with the case of s.d.e.[2,3]) is most clearly displayed, which is useful in extending the result of I, as will become clear in this and the next section.

For convenience we define for an arbitrary matrix $A(t)$,

$$Q_A(t/t_0) = \mathscr{X}_{1+A}(t/t_0)\,, \tag{2.9}$$

where $\mathscr{X}_{1+A}$ is given by (2.2a) (with A replaced by $1+A$). Making use of the identity

$$Q_A(t/t_0) = 1 + \sum_{s=t_0+1}^{t} Q_A(t/s)A(s)\,, \tag{2.10}$$

where the sum in the r.h.s. of (2.10) is by definition zero if $t=t_0$, we can express K_B in the form

$$K_B(t/t_0) = \langle B(t)\rangle + \sum_{s=t_0+1}^{t-1} \langle B(t)\colon Q_{\hat{B}}(t-1/s)\{1-B(s)\}^{-1}\colon B(s)\rangle_{\mathrm{p}}\,. \tag{2.11}$$

After expanding the expression within the brackets $\langle\ldots\rangle_{\mathrm{p}}$ in (2.11) in powers of B, one gets products of the form $(p_1, p_2\ldots = 0,1,2\ldots)$

$$\begin{aligned} &B(t)B^{p_1}(t_1)\ldots B(t_m)^{p_m}B(s)\\ &\quad = \left(\begin{array}{c|c} V(t)V(t_1)^{p_1}\ldots V(t_m)^{p_m}V(s) & V(t)V(t_1)^{p_1}\ldots V(t_m)^{p_m}g(s) \\ \hline \emptyset & 0 \end{array}\right), \end{aligned} \tag{2.12}$$

using the form (2.6) of the matrix B. Combining eqs. (2.11) and (2.12) with (2.7)

* Often abbreviated as p-cumulant.

we find the following equation for the average of $u^{(1)}(t)$:

$$\Delta\langle u^{(1)}(t-1)\rangle = K_V(t/t_0)\langle u^{(1)}(t-1)\rangle + G_V(t/t_0)\,, \tag{2.13}$$

where

$$K_V(t/t_0) = \langle V(t)\rangle + \sum_{s=t_0+1}^{t-1} \langle V(t){:}\, Q_V(t-1/s)(1-V(s))^{-1}{:}\, V(s)\rangle_{\mathrm{p}} \tag{2.14a}$$

and

$$G_V(t/t_0) = \langle g(t)\rangle + \sum_{s=t_0+1}^{t-1} \langle V(t){:}\, Q_V(t-1/s)(1-V(s))^{-1}{:}\, g(s)\rangle_{\mathrm{p}}\,. \tag{2.14b}$$

Thus, the only difference between (2.14a) and (2.14b) is that in G_V all terminal factors are $g(s)$ instead of $V(s)$ as in K_V. The formal expressions (2.14) should again be expanded in powers of V and subsequently the p-cumulants have to be computed. For the validity of these expansions we again require that $\alpha\tau_c \ll 1$, that is both the autocorrelation time of $V(t)$ and the crosscorrelation time of $V(t)$ with $g(t)$ should be short. If this is the case then again both K_V and G_V in (2.14) are independent of t_0 if $t - t_0 \gg \tau_c$. If $V(t)$ and $g(t)$ are statistically independent, and moreover* $\langle g(t)\rangle = 0$, then $G_V(t/t_0) = 0$. This can be checked via the decomposition of the p-cumulant in t-cumulants (appendix of I).

Let us give the result (2.13) to second order in the original representation, regarding A_1 and f to be of the same order of magnitude and assuming $t - t_0 \gg \tau_c$†:

$$\langle u(t)\rangle = \left[A_0 + \alpha\langle A_1(t)\rangle + \alpha^2 \sum_{\tau=1}^{\infty} \langle\langle A_1(t)\{A_0^{\tau-1}A_1(t-\tau)\}\rangle\rangle A_0^{-\tau}\right]\langle u(t-1)\rangle$$
$$+ \langle f(t)\rangle + \sum_{\tau=1}^{\infty} \langle\langle A_1(t)\{A_0^{\tau-1} f(t-\tau)\}\rangle\rangle\,. \tag{2.15}$$

This equation is valid for a time-independent, nonsingular unperturbed matrix A_0 in (1.2). If A_0 is singular, we can proceed as follows. Define an enlarged state vector $w(t)$ as before. Then (1.1) with sure initial condition leads to

$$w(t) = [B_0 + B_1(t)]w(t-1)\,, \tag{2.16a}$$

where

$$B_0 = \begin{pmatrix} A_0 & \emptyset \\ \emptyset & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \alpha A_1(t) & f(t) \\ \emptyset & 0 \end{pmatrix}. \tag{2.16b}$$

Clearly B_0 is singular. Therefore we can apply eq. (5.10) of I to (2.16) and extract

* This point was overlooked in ref. 2.

† Double brackets denote ordinary cumulants.

from it an equation for $\langle u(t)\rangle$. One can easily convince oneself that again (2.15) applies, but with the matrix $A_0^{-\tau}$ in the first line replaced by $(A_{0,N}^{-1})^{\tau}$, where the prescription to calculate the latter matrix can be found in section 5 of I.

Remark. The matrix B_0 in (2.16b) is singular even if A_0 is nonsingular. However, the case where A_0 is nonsingular was handled by first transforming to the interaction representation before introducing the enlarged state vector. Obviously this is impossible in the singular case.

2.2. *Random initial conditions*

To deal with the case of a random initial condition $u_0(\omega)$ we transform (2.3) to an inhomogeneous equation with zero initial condition. Define

$$v(t) = u^{(1)}(t) - u_0(\omega)\,. \tag{2.17}$$

Then (2.3) becomes

$$\Delta v(t-1) = V(t)v(t-1) + V(t)u_0 + g(t), \quad v(t_0) = 0 \tag{2.18}$$

and the result of the previous subsection is immediately applicable. So

$$\Delta\langle v(t-1)\rangle = K_V(t/t_0)\langle v(t-1)\rangle + G_V(t/t_0) + I'_V(t/t_0)\,,$$

where K_V and G_V are the same as in (2.14) and I'_V is obtained from G_V by replacing $g(s)$ by $V(s)u_0$. For $\langle u^{(1)}(t)\rangle$ itself then follows

$$\Delta\langle u^{(1)}(t-1)\rangle = K_V(t/t_0)\langle u^{(1)}(t-1)\rangle + G_V(t/t_0) + I_V(t/t_0)\,, \tag{2.19}$$

where

$$I_V(t/t_0) = \langle V(t)(u_0 - \langle u_0\rangle)\rangle + \sum_{s=t_0+1}^{t-1} \langle V(t){:}\, Q_V(t-1/s)(1-V(s))^{-1}{:}\, [V(s)(u_0 - \langle u_0\rangle)]\rangle_{\mathrm{p}}\,. \tag{2.20}$$

As in the case of a s.d.e.[2]) one shows that $I_V(t/t_0) = 0$ if u_0 is nonrandom or statistically independent of V, and in general

$$I(t/t_0) = 0 \quad \text{if } t - t_0 \gg \tau_c\,.$$

Transforming back to the original representation, and neglecting terms of third or higher order, one finds an equation like (2.15) with two additional inhomogeneous terms, which can be obtained by replacing $f(\cdot)$ in the second line of (2.15) by $A_1(\cdot)(u_0 - \langle u_0\rangle)$. If A_0 is singular one again has to modify the result in the same way as before.

3. The correlation function

Until now only one-time averages of the solution $u(t)$ of (1.1) were considered. In this section we develop an expansion for multi-time averages. We restrict ourselves here to a derivation of the two-time correlation function $C_u(t, t') = \langle u(t) \otimes u(t') \rangle$. The method is the same as used for s.d.e.[3]). Higher order correlation functions can be handled in an analogous way (compare appendix A of ref. 3). First the homogeneous case is discussed, and subsequently the inhomogeneous case, both with sure initial condition. The case of a random initial condition is not discussed. It can be reduced to the inhomogeneous case as in subsection 2.2.

3.1. *The homogeneous case*

We start again with the homogeneous equation in the interaction representation,

$$\Delta u^{(1)}(t-1) = V(t) u^{(1)}(t-1), \quad u^{(1)}(t_0) = u_0 , \tag{3.1}$$

where $V(t)$ is defined in (2.4). The formal solution can be written as

$$u^{(1)}(t) = Q_V(t/t_0) u_0 \tag{3.2}$$

using definition (2.9). The average $\langle u^{(1)}(t) \rangle$ obeys (2.7) with B replaced by V, and from it one obtains

$$\langle u^{(1)}(t) \rangle = \left[\bar{T} \prod_{s=t_0+1}^{t} \{1 + K_V(s/t_0)\} \right] u_0 , \tag{3.3}$$

where K_V is defined (for arbitrary V) by

$$K_V(t/t_0) = \langle V(t) \colon Q_{\hat{V}}(t-1/t_0) \colon \rangle_{\mathrm{p}} . \tag{3.4}$$

The definition of $\hat{V}$ is analogous to (2.8b). From (3.2) and (3.3) one obtains the identity

$$\left\langle \bar{T} \prod_{s=t_0+1}^{t} \{1 + V(s)\} \right\rangle = \bar{T} \prod_{s=t_0+1}^{t} \{1 + K_V(s/t_0)\} , \tag{3.5}$$

where the time ordering operator in the r.h.s. acts on the first time variable of K_V. Now replace $V(t)$ by a matrix function $C(t)$ with finite support on I_{t_0}. Then (3.5) yields

$$\left\langle \bar{T} \prod_{s=t_0+1}^{\infty} \{1 + C(s)\} \right\rangle = \bar{T} \prod_{s=t_0+1}^{\infty} \{1 + K_c(s/t_0)\} . \tag{3.6}$$

Next we write the correlation function $C_{u^{(1)}}(t, t+\tau)$ as

$$\begin{aligned}\langle u^{(1)} \otimes u^{(1)}(t+\tau)\rangle &= \langle Q_V(t/t_0) \otimes Q_V(t+\tau/t_0)\rangle u_0 \otimes u_0 \\ &= \langle Q_{V'}(t/t_0) Q_{V''}(t+\tau/t_0)\rangle u_0 \otimes u_0 \quad (\tau = 1, 2, \dots)\,, \end{aligned} \tag{3.7}$$

where we define for arbitrary matrix or vector C

$$C' = C \otimes \mathbf{1}\,, \quad C'' = \mathbf{1} \otimes C\,. \tag{3.8}$$

Here $\mathbf{1}$ denotes the unit matrix of the same dimension as C. Making use of the fact that $C'(t)$ and $C''(t')$ commute for all t and t' we have

$$\begin{aligned} Q_{V'}(t/t_0) Q_{V''}(t+\tau/t_0) &= \tilde{T}\left[\prod_{s=t_0+1}^{t} \{1 + V'(s)\} \prod_{s=t_0+1}^{t+\tau} \{1 + V''(s)\}\right] \\ &= \tilde{T} \prod_{s=t_0+1}^{\infty} \{1 + C(s)\} \equiv Q_c(\infty/t_0)\,, \end{aligned} \tag{3.9}$$

with

$$C(s) = \{\theta(t-s) V'(s)(1 + V''(s)) + \theta(t+\tau-s) V''(s)\}\theta(t - t_0 - 1)\,. \tag{3.10}$$

The discrete stepfunction θ is defined by

$$\theta(t) = \begin{cases} 1, & t = 0, 1, 2, \dots, \\ 0, & t = -1, -2, \dots . \end{cases} \tag{3.11}$$

Now we can apply (3.6) to (3.9). From (3.10) it follows that

$$C(s) = \begin{cases} V'(s)\{1 + V''(s)\} + V''(s) \equiv \tilde{V}(s), & t_0 < s \leqslant t\,, \\ V''(s), & t < s \leqslant t + \tau\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{3.12}$$

Our next aim is to express $K_c(s/t_0)$ in terms of the ordered cumulants of $\tilde{V}$ and/or V''. Since this point was not carefully treated in ref. 3 we consider it here in more detail. For example, calculate the third order p-cumulant $\langle C(s)C(s_1)C(s_2)\rangle_p$, where $t + \tau > s > s_1 > t \geqslant s_2$. Using the decomposition in t-cumulants[1]) one has

$$\langle C(s)C(s_1)C(s_2)\rangle_p = \langle C(s)C(s_1)C(s_2)\rangle_t - \langle C(s)C(s_2)\rangle_t \langle C(s_1)\rangle\,. \tag{3.13}$$

The definition of the t-cumulant $\langle \dots \rangle_t$ is given in the appendix of I. The important point here is that in all terms contributing to a given t-cumulant the order of the time variables is the same (in contrast to a p-cumulant, see (3.13)). Using (3.12) we can therefore write (3.13) as

$$\langle C(s)C(s_1)C(s_2)\rangle_p = \langle V''(s)V''(s_1)\tilde{V}(s_2)\rangle_t - \langle V''(s)\tilde{V}(s_2)\rangle_t \langle V''(s_1)\rangle\,. \tag{3.14}$$

Now we cannot write (3.14) as $\langle V''(s)V''(s_1)\tilde{V}(s_2)\rangle_p$, since according to the rules in I only time-variables are permuted if one compares the different terms contributing to the p-cumulant, but not the matrices themselves. For this reason we introduce here a new partially ordered cumulant, denoted by $\langle \ldots \rangle_{\bar{p}}$, which can be obtained by the same rules as the p-cumulant, except that the matrices themselves are permuted, and not only their time variables (i.e. each matrix keeps its original time variable). So (3.14) becomes

$$\langle C(s)C(s_1)C(s_2)\rangle_p = \langle V''(s)V''(s_1)\tilde{V}(s_2)\rangle_{\bar{p}}\,. \tag{3.15}$$

If all matrices involved are the same functions of time, the two types of partially ordered cumulants are identical. Moreover, the distinction only becomes apparent in third or higher order cumulants.

Now the reasoning leading to (3.15) can be repeated for all higher order cumulants contributing to K_c in (3.6). Therefore we conclude that

$$K_c(s/t_0) = \begin{cases} K_{\hat{\tilde{V}}}(s/t_0), & t_0 < s \leqslant t\,, \\ M_V(s;t/t_0), & t < s \leqslant t+\tau\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{3.16}$$

where (for arbitrary V) we define M_V by

$$M_V(s;t/t_0) = \langle V''(s)\colon Q_{\hat{V}''}(s-1/t)Q_{\hat{\tilde{V}}}(t/t_0)\colon\rangle_{\bar{p}}\,. \tag{3.17}$$

Again for arbitrary C

$$\hat{C}(t) = C(t)\{1 - C(t)\}^{-1} \tag{3.18}$$

and $K_{\hat{\tilde{V}}}$ is defined in analogy with (3.4). Note that in the case of $K_{\hat{\tilde{V}}}$ we can use p- or p̄-cumulants since only one type of matrix, viz. $\tilde{V}$, is involved. The colons in (3.17) mean again that one should first expand in powers of V'' and $\hat{\tilde{V}}$ before computing the p̄-cumulants.

Combining (3.6) and (3.16) we get

$$\begin{aligned}\langle Q_c(\infty/t_0)\rangle &= \left[\overleftarrow{T}\prod_{s=t+1}^{t+\tau}\{1 + M_V(s;t/t_0)\}\right]\left[\overleftarrow{T}\prod_{s=t_0+1}^{t}\{1 + K_{\hat{\tilde{V}}}(s;t_0)\}\right] \\ &\equiv Q_{M_v}(t+\tau/t)Q_{K_{\hat{\tilde{v}}}}(t/t_0)\,. \end{aligned}\tag{3.19}$$

From (3.7), (3.9) and (3.19) one arrives at the following expression for the correlation function $C_{u^{(1)}}$

$$C_{u^{(1)}}(t, t+\tau) = Q_{M_v}(t+\tau/t)Q_{K_{\hat{\tilde{v}}}}(t/t_0)(u_0 \otimes u_0)\,. \tag{3.20}$$

Adhering to our general strategy we now rewrite the result (3.20) in the form of a first order difference equation.

3.1.1. *The difference equation for C_u*

First the partial difference operators Δ_τ and Δ_t are introduced, defined by their action on an arbitrary function $f(t, t+\tau)$:

$$\Delta_\tau f(t, t+\tau) = f(t, t+\tau+1) - f(t, t+\tau)\,,$$
$$\Delta_t f(t, t+\tau) = f(t+1, t+\tau+1) - f(t, t+\tau)\,. \tag{3.21}$$

Operating with Δ_τ on (3.20) the following equation can be inferred, making use of (3.19)

$$\Delta_\tau C_{u^{(1)}}(t, t+\tau) = M_V(t+\tau+1; t/t_0) C_{u^{(1)}}(t, t+\tau)\,, \tag{3.22}$$

where the matrix M_V is given by (3.17). This difference equation in τ has to be solved subject to the initial condition $C_{u^{(1)}}(t, t) = \langle u^{(1)}(t) \otimes u^{(1)}(t)\rangle$, which in its turn can be obtained from the cumulant expansion for one-time averages (part I) as the solution of

$$\Delta_t C_{u^{(1)}}(t, t) = K_V(t+1/t_0) C_{u^{(1)}}(t, t)\,. \tag{3.23}$$

The initial condition corresponding to (3.23) is $C_{u^{(1)}}(t_0, t_0) = u_0 \otimes u_0$.

In contrast to the case of a s.d.e.[3]) we have not been able to find a closed equation for $C_{u^{(1)}}$ when acting with the difference operator Δ_t on (3.20).

3.1.2. *The result to second order*

Here we give the result of the cumulant expansion to second order in α, after transforming back to the original representation (1.1) via (2.1) and (2.4) (thus in particular assuming that $\det A_0 \neq 0$). One finds from (3.22)

$$\begin{aligned} C_u(t, t+\tau+1) = \Big[& A_0'' + \alpha\langle A_1''(t+\tau+1)\rangle \\ & + \alpha^2 \sum_{s=t_0+1}^{t+\tau} \langle\langle A_1''(t+\tau+1)\{(A_0'')^{t+\tau-s} A_1''(s)\}\rangle\rangle (A_0''^{-1})^{t+\tau+1-s} \\ & + \alpha^2 \sum_{s=t_0+1}^{t} \langle\langle A_1''(t+\tau+1)\{(A_0')^{t-s} A_1'(s)\}\rangle\rangle (A_0'^{-1})^{t+1-s} \Big] \\ & \times C_u(t, t+\tau) \end{aligned} \tag{3.24}$$

and from (3.23)

$$\begin{aligned} C_u(t, t) = \Big[& A_0' A_0'' + \alpha\langle A_1''(t)\rangle A_0'' + \alpha A_0'\langle A_1''(t)\rangle + \alpha^2\langle A_1'(t) A_1''(t)\rangle \\ & + \alpha^2 \sum_{s=t_0+1}^{t-1} \{\langle\langle (A_1'(t) A_0'' + A_1''(t) A_0')(A_0'^{t-1-s} A_1'(s))\rangle\rangle (A_0'^{-1})^{t-s} \\ & + \langle\langle (A_1'(t) A_0'' + A_1''(t) A_0')(A_0''^{t-1-s} A_1''(s))\rangle\rangle (A_0''^{-1})^{t-s}\} \Big] \\ & \times C_u(t-1, t-1)\,. \end{aligned} \tag{3.25}$$

The case that A_0 is singular can be handled by a combination of the methods in I, section 5 and in ref. 3, section 8. Again one can show that in this case the matrix A_0^{-1} in (3.24) and (3.25) has to be replaced by $A_{0,N}^{-1}$ as defined in section 5 of I. The resulting equations are only valid after a transient time large compared to τ_c, which implies that in (3.24) and (3.25) one should put $t_0 = -\infty$.

3.2. *The inhomogeneous case*

To study the correlation function in the inhomogeneous case, we start again with (2.3) and consider eq. (2.5) for the enlarged state vector $w(t)$. Applying the result of the previous subsection, one finds from (3.22)

$$\Delta_\tau C_w(t, t+\tau) = M_B(t+\tau+1; t/t_0) C_w(t, t+\tau) \tag{3.26}$$

with

$$M_B(t+\tau+1; t/t_0) = \langle B''(t+\tau+1)\colon Q_{\tilde{B}'}(t+\tau/t) Q_{\tilde{B}}(t/t_0)\colon\rangle_{\mathrm{p}}, \tag{3.27}$$

where B is defined in (2.6), and $\tilde{B}$ as in (3.12). The quantity M_B in (3.27) can be expressed as an infinite sum of terms which involve (summations over) products of joint moments of B' and B''. Each such product can be written as a Kronecker product. For example

$$\langle B''(t_1)B'(t_2)\rangle\langle B'(t_3)B''(t_4)\rangle = \{B(t_2)B(t_3)\}\otimes\{B(t_1)B(t_4)\}, \tag{3.28}$$

where we used horizontal brackets instead of angular brackets to indicate which matrices belong to the same moment.

To obtain an equation for the correlation function of $u^{(1)}$ from (3.26) we use the same method as in ref. 3, section 4. For the components of $\langle w(t) \otimes w(t+\tau)\rangle$ one has (suppose that the dimension of $V(t)$ in (2.3) is n):

$$\Delta_\tau \langle w_i(t) w_k(t+\tau)\rangle = \sum_{j,l=1}^{n+1} (M_B)_{ij,kl} \langle w_j(t) w_l(t+\tau)\rangle, \tag{3.29}$$

where the time indices of M_B are suppressed. A matrix element $(.)_{ij,kl}$ is calculated by first expressing M_B in the joint moments of B' and B'', writing each term as a Kronecker product, say $D\otimes E$, and taking the matrix element according to

$$(D\otimes E)_{ij,kl} = D_{ij} E_{kl}. \tag{3.30}$$

Each matrix D or E is either the unit matrix of dimension $n+1$, or a (summation over a) product of one or more matrices $B(.)$ on which a certain averaging operation is performed, as in (3.28). We saw already that a product of B-matrices has the form (2.12). Therefore the matrices D and E in (3.30) have the following structure, if they contain at least one matrix B:

$$D = \left(\begin{array}{c|c} D^{(1)} & D^{(2)} \\ \hline \emptyset & 0 \end{array}\right), \quad E = \left(\begin{array}{c|c} E^{(1)} & E^{(2)} \\ \hline \emptyset & 0 \end{array}\right),$$

where $D^{(1)}$ and $E^{(1)}$ are the same as D and E which each matrix $B(.)$ replaced by $V(.)$, while $D^{(2)}$ (or $E^{(2)}$) is obtained by replacing the last matrix $V(t_i)$ in $D^{(1)}$ (or $E^{(1)}$) by $g(t_i)$.

Now restrict the range of i and k in (3.29) to values between 1 and n. The summation over j and l can be split up in four regions: $\{1 \leqslant j \leqslant n;\ 1 \leqslant l \leqslant n\}$; $\{1 \leqslant l \leqslant n;\ j = n+1\}$; $\{1 \leqslant j \leqslant n;\ l = n+1\}$; $\{j = l = n+1\}$. If we take only a typical term (3.30) into account, (3.29) yields

$$\Delta_\tau \langle u_i^{(1)}(t) u_k^{(1)}(t+\tau)\rangle = \sum_{j,l=1}^{n} D_{ij}^{(1)} E_{kl}^{(1)} \langle u_j^{(1)}(t) u_l^{(1)}(t+\tau)\rangle + \sum_{l=1}^{n} D_i^{(2)} E_{kl}^{(1)} \langle u_l^{(1)}(t+\tau)\rangle + \sum_{j=1}^{n} D_{ij}^{(1)} E_k^{(2)} \langle u_j^{(1)}(t)\rangle + D_i^{(2)} E_k^{(2)} \tag{3.31}$$

or in vector notation

$$\Delta_\tau C_{u^{(1)}}(t, t+\tau) = G^{(1,1)} C_{u^{(1)}}(t, t+\tau) + G^{(2,1)} \langle u^{(1)}(t+\tau)\rangle + G^{(1,2)} \langle u^{(1)}(t)\rangle + G^{(2,2)}, \tag{3.32}$$

where

$$G^{(i,j)} = D^{(i)} \otimes E^{(j)} \quad (i, j = 1, 2).$$

If the matrix D or E in (3.30) is the $(n+1)$-dimensional unit matrix, we have that $D^{(1)}$ or $E^{(1)}$ is equal to the n-dimensional unit matrix, while $D^{(2)}$ or $E^{(2)}$ is a null vector. So for each term in $G^{(1,1)}$ there is a corresponding contribution to $G^{(2,1)}$, $G^{(1,2)}$ and $G^{(2,2)}$ (unless $D^{(2)}$ and/or $E^{(2)}$ is zero) which can be obtained by replacing the last matrix $V(t_i)$ in $D^{(1)}$, $E^{(1)}$ or both by a vector $g(t_i)$, respectively. For example if

$$G^{(1,1)} = \langle V(t_3) \otimes \{V(t_1)V(t_2)\}\rangle = \langle V''(t_1)V''(t_2)V'(t_3)\rangle,$$

then

$$G^{(2,1)} = \langle g(t_3) \otimes \{V(t_1)V(t_2)\}\rangle = \langle V''(t_1)V''(t_2)g'(t_3)\rangle,$$
$$G^{(1,2)} = \langle V(t_3) \otimes \{V(t_1)g(t_2)\}\rangle = \langle V''(t_1)g''(t_2)V(t_3)\rangle,$$
$$G^{(2,2)} = \langle g(t_3) \otimes \{V(t_1)g(t_2)\}\rangle = \langle V''(t_1)g''(t_2)g(t_3)\rangle.$$

From these formulas one sees that it is not necessary to write the expression $G^{(1,1)}$ first as a Kronecker product. To obtain $G^{(2,1)}$, $G^{(1,2)}$ and $G^{(2,2)}$ one may just as well start from $G^{(1,1)}$ in terms of products of moments (or t-cumulants, but not p̄-cumulants) of V' and V'', and replace in each term the last matrices $V'(.)$, $V''(.)$ or both by $g'(.)$, $g''(.)$ or both respectively*. However, there is the additional

* If a term of $G^{(1,1)}$ contains no matrix V' (or V''), there is no corresponding contribution to $G^{(2,1)}$ and $G^{(2,2)}$ (or $G^{(1,2)}$ and $G^{(2,2)}$).

prescription that any matrix C' or C'' (where C can be a matrix or vector) succeeding a matrix g' or g'' has to be replaced by C.

We can repeat the above procedure for each term of M_B. Thus if all terms are included, $C_{u(1)}$ still obeys an equation of the form (3.32). If $g \equiv 0$ in (2.3), (3.32) should reduce to (3.22), so one can identify

$$G^{(1,1)} = M_V(t+\tau+1; t/t_0), \tag{3.33}$$

where M_V is given by (3.17). To obtain $G^{(1,2)}$, $G^{(2,1)}$ and $G^{(2,2)}$ one should expand $G^{(1,1)}$ in the joint moments of V' and V'' and subsequently apply the rules given in the preceding paragraph. The matrix $G^{(1,1)}$ contains the moments of $V(.)$ alone, $G^{(2,1)}$ and $G^{(1,2)}$ those of $V(.)$ with one $g(.)$, and $G^{(2,2)}$ those of V with two g's.

If we regard V and g to be of the same order of magnitude, we find that the matrices $G^{(i,j)}$ in (3.32) to second order are given by

$$G^{(1,1)} = \langle V''(t+\tau+1)\rangle + \sum_{s=t_0+1}^{t+\tau} \langle\langle V''(t+\tau+1)V''(s)\rangle\rangle$$
$$+ \sum_{s=t_0+1}^{t} \langle\langle V''(t+\tau+1)V'(s)\rangle\rangle, \tag{3.34a}$$

$$G^{(2,1)} = \sum_{s=t_0+1}^{t} \langle\langle V''(t+\tau+1)g'(s)\rangle\rangle, \tag{3.34b}$$

$$G^{(1,2)} = \langle g''(t+\tau+1)\rangle + \sum_{s=t_0+1}^{t+\tau} \langle\langle V''(t+\tau+1)g''(s)\rangle\rangle$$
$$+ \sum_{s=t_0+1}^{t} \langle\langle g''(t+\tau+1)V(s)\rangle\rangle, \tag{3.34c}$$

$$G^{(2,2)} = \sum_{s=t_0+1}^{t} \langle\langle g''(t+\tau+1)g(s)\rangle\rangle. \tag{3.34d}$$

By means of (2.1) and (2.4) one can transform (3.32) and (3.34) back to the original representation in the case of a nonsingular A_0. If A_0 is singular only $G^{(1,1)}$ has to be modified as indicated in the previous subsection.

4. Probability density functions

In this section we investigate the applicability of the cumulant expansion to find the probability density function (p.d.f.) of the solution $u(t, \omega)$ of (1.1). To this end we first derive an analog of the "stochastic Liouville equation" in the case of an s.d.e.[2]). The derivation is only formal in the sense that possible difficulties concerning existence, differentiability etc. of the p.d.f. are ignored. The method can be extended to obtain multi-time probability density functions (compare appendix A of ref. 3), and also to nonlinear difference equations.

In subsection 4.1 the expansion for the p.d.f. is derived. Subsection 4.2 shows a one-dimensional example which results in the well known lognormal distribution. Finally we consider in subsection 4.3 the case that the matrix $A(t)$ in (1.1) constitutes a Markov process. This case is exactly solvable for arbitrary correlation time of $A(t)$. In particular an exact equation for the first moment of the solution $u(t)$ is derived.

4.1. *The cumulant expansion of the p.d.f.*

We start by defining a density $\rho(\boldsymbol{u}, t)$* in phase space by

$$\rho(\boldsymbol{u}, t) = \delta(\boldsymbol{u}(t) - \boldsymbol{u}) \equiv \prod_{j=1}^{n} \delta(u_j(t) - u_j) . \tag{4.1}$$

Now apply the difference operator to $\rho(\boldsymbol{u}, t)$ and use the integral representation of the delta function to obtain

$$\begin{aligned} \Delta\rho(\boldsymbol{u}, t) &= \Delta \frac{1}{(2\pi)^n} \int \mathrm{d}\boldsymbol{k}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{u}(t)-\boldsymbol{u})} \\ &= \frac{1}{(2\pi)^n} \int \mathrm{d}\boldsymbol{k}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{u}(t)-\boldsymbol{u})}(\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\Delta\boldsymbol{u}(t)} - 1) \\ &= \left[\left(\frac{1}{2\pi}\right)^n \int \mathrm{d}\boldsymbol{k}\ \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{u}(t)-\boldsymbol{u})} \prod_{j=1}^{n} \sum_{m_j=0}^{\infty} \frac{(\mathrm{i}k_j)^{m_j}}{m_j!} (\Delta u_j(t))^{m_j}\right] - \rho(\boldsymbol{u}, t) \\ &= \left[\prod_{j=1}^{n} \left\{\sum_{m_j=0}^{\infty} \frac{(-)^{m_j}}{m_j!} \frac{\partial^{m_j}}{\partial u_j^{m_j}} \left(\int \frac{\mathrm{d}k_j}{2\pi} \mathrm{e}^{\mathrm{i}k_j(u_j(t)-u_j)}(\Delta u_j(t))^{m_j}\right)\right\}\right] - \rho(\boldsymbol{u}, t) . \end{aligned}$$

Finally use the definition (4.1) and eq. (1.1) for $\Delta u_j(t)$† to get

$$\begin{aligned} \Delta\rho(\boldsymbol{u}, t) &= \left[\prod_{j=1}^{n} \left\{\sum_{m_j=0}^{\infty} \frac{(-)^{m_j}}{m_j!} \frac{\partial^{m_j}}{\partial^{m_j}_{u_j}} \delta(u_j(t) - u_j)(\Delta u_j(t))^{m_j}\right\}\right] - \rho(\boldsymbol{u}, t) \\ &= \sum_{\boldsymbol{m}}{}^{*} \frac{(-)^m}{\boldsymbol{m}!} \frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{u}^{\boldsymbol{m}}} [\{(A(t+1) - 1)\boldsymbol{u} + \boldsymbol{f}(t+1)\}^{\boldsymbol{m}} \rho(\boldsymbol{u}, t)] \qquad (4.2) \\ &\equiv L(\boldsymbol{u}, t, \omega)\rho(\boldsymbol{u}, t) , \end{aligned}$$

where we have introduced the abbreviated notation

$$m = m_1 + m_2 + \cdots m_n, \quad \boldsymbol{m}! = m_1! \ldots m_n! ,$$

$$(\boldsymbol{x})^{\boldsymbol{m}} = x_1^{m_1} \ldots x_n^{m_n} \quad \text{for any vector } \boldsymbol{x} ,$$

$$\frac{\partial^{\boldsymbol{m}}}{\partial \boldsymbol{u}^{\boldsymbol{m}}} = \frac{\partial^{m_1}}{\partial^{m_1}_{u_1}} \cdots \frac{\partial^{m_n}}{\partial u_n^{m_n}}$$

* In this second vectors will be printed in bold-face type.

† In the nonlinear case one should substitute at this point a nonlinear function $f_j(\boldsymbol{u}, t) = \Delta u_j(t)$.

and the asterisk on the summation symbol indicates that one should sum over all nonnegative integer values of $m_1, m_2, \ldots, m_n$ not all equal to zero. The stochastic differential operator L is defined by the second line of (4.2).

Now the cumulant expansion of I (section 2) can be applied to (4.2). Using the fact that

$$\langle \rho(\boldsymbol{u}, t) \rangle = P(\boldsymbol{u}, t) \tag{4.3}$$

gives the probability density to find $u_j(t)$ between the values u_j and $u_j + \mathrm{d}u_j$ at time $t (j = 1, \ldots, n)$, we find

$$\Delta P(\boldsymbol{u}, t-1) = K_L(t/t_0) P(\boldsymbol{u}, t-1), \tag{4.4}$$

where K_L is the deterministic differential operator

$$K_L(t/t_0) = \left\langle L(\boldsymbol{u}, t) \colon \left[\overleftarrow{T} \prod_{s=t_0+1}^{t-1} \{1 + \hat{L}(\boldsymbol{u}, s)\} \right] \colon \right\rangle_{\mathrm{p}} \tag{4.5}$$

and again $\hat{L}$ is defined as in (2.8b).

If $A(t)$ in (1.1) contains an unperturbed matrix A_0, one can apply the method given above to eq. (2.3) in the interaction representation. This yields the probability density $P^{(1)}(\boldsymbol{v}, t)$ of $u^{(1)}(t)$. From this one obtains $P(\boldsymbol{u}, t)$ by*

$$P(\boldsymbol{u}, t) = P^{(1)}(A_0^{-t}\boldsymbol{u}, t) |A_0^{-t}|$$

assuming A_0 is nonsingular. If A_0 is singular one cannot expect the present method to work, since in this case the unperturbed evolution does not conserve the number of phase points and a continuity equation like (4.2) cannot be valid.

4.2. *Example*

As an illustration consider the scalar equation

$$u(t) = \{1 + \alpha\xi(t)\} u(t-1),$$
$$u(0) = u_0, \tag{4.6}$$

where $\{\xi(t)\}$ is a stationary process with autocorrelation time τ_c, and such that $1 + \alpha\xi(t)$ takes only positive values. Moreover assume $u_0 > 0$. Then it is easier to take logarithms:

$$y(t) = y(t-1) + \eta(t), \tag{4.7}$$

where

$$y(t) = \ln u(t), \quad \eta(t) = \ln(1 + \alpha\xi(t)).$$

* Compare section 21 of ref. 7 for the case of an s.d.e.

So $\eta(t)$ is of order α. Again we introduce

$$\rho(y, t) = \delta(y(t) - y),$$

and consider the Fourier transform* of ρ:

$$\hat{\rho}(k, t) = \int \mathrm{d}y\, \mathrm{e}^{iky} \rho(y, t) = \int \mathrm{d}y\, \mathrm{e}^{iky} \delta(y(t-1) + \eta(t) - y)$$
$$= e^{ik\eta(t)} \hat{\rho}(k, t-1).$$

So

$$\Delta\hat{\rho}(k, t-1) = (\mathrm{e}^{ik\eta(t)} - 1)\hat{\rho}(k, t-1). \tag{4.8}$$

This equation is of the form considered in part I. Applying the results of section 2 of I, we obtain

$$\Delta\langle\hat{\rho}(k, t-1)\rangle = K_\eta \langle\hat{\rho}(k, t-1)\rangle, \tag{4.9}$$

where

$$K_\eta = \left[\langle \mathrm{e}^{ik\eta(t)} - 1\rangle + \sum_{t_1=1}^{t-1} \langle\langle(\mathrm{e}^{ik\eta(t)} - 1)(\mathrm{e}^{ik\eta(t_1)} - 1)\rangle\rangle + \mathcal{O}(\alpha^3)\right]$$
$$\overset{t \gg \tau_c}{=} \left[ik\langle\eta(t)\rangle + \frac{(ik)^2}{2!}\langle\eta^2\rangle + \sum_{\tau=1}^{\infty} \langle\langle\eta(\tau)\eta(0)\rangle\rangle(ik)^2 + \mathcal{O}(\alpha^3)\right]$$
$$= \exp\left[ik\langle\eta\rangle + \frac{(ik)^2}{2!}\left\{\langle\langle\eta^2\rangle\rangle + 2\sum_{\tau=1}^{\infty}\langle\langle\eta(\tau)\eta(0)\rangle\rangle\right\} + \mathcal{O}(\alpha^3)\right] - 1. \tag{4.10}$$

Hence (we neglect the mismatch of u_0 (ref. 7))

$$\langle\hat{\rho}(k, t)\rangle = (1 + K_\eta)^t\, \mathrm{e}^{iky_0}, \quad y_0 = \ln u_0.$$

By inverse Fourier transformation one obtains the p.d.f. of $y(t)$,

$$\tilde{P}(y, t) = \langle\rho(y, t)\rangle = (2\pi\sigma^2 t)^{-1/2} \exp\left[\frac{-(y - y_0 - at)^2}{2\sigma^2 t}\right], \tag{4.11}$$

where

$$a = \langle\eta\rangle = \alpha\langle\xi\rangle - \tfrac{1}{2}\alpha^2\langle\xi^2\rangle + \mathcal{O}(\alpha^3),$$

$$\sigma^2 = \langle\langle\eta^2\rangle\rangle + 2\sum_{\tau=1}^{\infty}\langle\langle\eta(\tau)\eta(0)\rangle\rangle \tag{4.12a}$$

$$= \alpha^2\left\{\langle\langle\xi^2\rangle\rangle + 2\sum_{\tau=1}^{\infty}\langle\langle\xi(\tau)\xi(0)\rangle\rangle\right\} + \mathcal{O}(\alpha^3). \tag{4.12b}$$

* One could also start from (4.4) neglecting terms of $\mathcal{O}(\alpha^3)$ and take the Fourier transform of the resulting equation to obtain the solution.

Finally we find the p.d.f. of $u(t)$ by

$$P(u, t) = \tilde{P}(\ln u, t)\frac{\mathrm{d}\ln u}{\mathrm{d}u}$$
$$= (2\pi\sigma^2 t)^{-1/2} u^{-1} \exp\left[-\frac{(\ln(u/u_0) - at)^2}{2\sigma^2 t}\right], \tag{4.13}$$

which is the lognormal distribution. The result (4.13) holds if $t \gg \tau_c$ and is correct up to $\mathcal{O}(\alpha^2)$. The form of $P(u, t)$ could have been anticipated, since by (4.7) $y(t)$ is a sum of weakly dependent stochastic variables. So by the central limit theorem the distribution of y is Gaussian for large t.

The mean and variance of $u(t)$ are given by

$$\langle u(t)\rangle = \mathrm{e}^{(\sigma^2/2+a)t}u_0, \tag{4.14a}$$
$$\langle\langle u^2(t)\rangle\rangle = \mathrm{e}^{2(\sigma^2/2+a)t}(\mathrm{e}^{\sigma^2 t} - 1)u_0^2. \tag{4.14b}$$

Thus $\langle u(t)\rangle \to \infty$ as $t\to\infty$, but also $\langle\langle u^2(t)\rangle\rangle \to \infty$. And in fact the modus u_m of $P(u, t)$ is

$$u_m = \mathrm{e}^{(a-\sigma^2)t}u_0 \xrightarrow{t\to\infty} 0 \quad \text{if } a < \sigma^2. \tag{4.14c}$$

That is, although the average grows without bound, the probability to take on a non-zero value may go to zero for certain values of the parameters. To be more precise, for the probability that $u(t)$ takes a value smaller than $\epsilon > 0$, one finds[4])

$$P[u(t) < \epsilon] = \mathrm{Erf}[\ln(\epsilon/u_0 - at)/\sigma t^{1/2}],$$

where Erf denotes the error function. So as $t\to\infty$

$$P[u(t) < \epsilon] = \begin{cases} 0, & \text{if } a > 0, \\ \frac{1}{2}, & \text{if } a = 0, \\ 1, & \text{if } a < 0, \end{cases}$$

independent of σ^2.

Finally we remark that the same lognormal distribution (4.13) is obtained from the cumulant expansion* to $\mathcal{O}(\alpha^2)$ for the s.d.e.

$$\dot{u} = \alpha\xi(t)u. \tag{4.15}$$

In this case

$$a = \alpha\langle\xi\rangle, \quad \sigma^2 = 2\alpha^2\int_0^\infty \mathrm{d}\tau\langle\langle\xi(t)\xi(t-\tau)\rangle\rangle.$$

* For the method, see ref. 2, section 7.

The lognormal distribution (4.13) is exact for all t when $\xi(t)$ is Gaussian white noise[5]) (and if (4.15) is interpreted in the Stratonovich sense).

4.3. *The case where the multiplicative noise is a Markov chain*

In this subsection we consider the stochastic difference equation

$$\boldsymbol{u}(t) = A(\xi(t))\boldsymbol{u}(t-1)\,, \tag{4.16}$$

where the matrix A is a function of a Markov chain $\{\xi(t)\}_{t=-\infty}^{\infty}$. The probability density $\pi(\xi, t)$ of $\xi(t)$ obeys the master equation

$$\pi(\xi, t+1) = \sum_{\xi'} T(\xi/\xi')\pi(\xi', t)\,. \tag{4.17}$$

The summation runs over all states of ξ, and $T(\xi/\xi')$ is the transition matrix (if the state space of ξ is continuous*, the summation in (4.17) becomes an integral). The joint process $\{\boldsymbol{u}(t), \xi(t)\}$ is again a Markov process with joint p.d.f. $P(\boldsymbol{u}, \xi, t)$ defined by

$$P(\boldsymbol{u}, \xi, t) = \langle \delta(\boldsymbol{u}(t) - \boldsymbol{u})\delta(\xi(t) - \xi)\rangle\,.$$

For $\rho(\boldsymbol{u}, t) = \delta(\boldsymbol{u}(t) - \boldsymbol{u})$ one has the equation

$$\Delta\rho(\boldsymbol{u}, t-1) = L(\boldsymbol{u}, \xi(t))\rho(\boldsymbol{u}, t-1)\,, \tag{4.18}$$

where from (4.2)

$$L(\boldsymbol{u}, \xi)\cdots = \sum_m{}^{*} \frac{(-)^m}{m!}\frac{\partial^m}{\partial \boldsymbol{u}^m}[\{(A(\xi) - 1)\boldsymbol{u}\}^m \ldots]\,. \tag{4.19}$$

Let $\sigma(\xi, t) = \delta(\xi(t) - \xi)$. Then we can combine the continuity equation (4.18) for $\rho(\boldsymbol{u}, t)$ with the master equation (4.17) as follows†:

$$\begin{aligned} \Delta P(\boldsymbol{u}, \xi, t-1) &= \langle \rho(\boldsymbol{u}, t)\sigma(\xi, t)\rangle - \langle \rho(\boldsymbol{u}, t-1)\sigma(\xi, t-1)\rangle \\ &= \langle \{\Delta\rho(\boldsymbol{u}, t-1)\}\sigma(\xi, t)\rangle + \langle \rho(\boldsymbol{u}, t-1)\{\Delta\sigma(\xi, t-1\}\rangle \\ &= \langle L(\boldsymbol{u}, \xi(t))\rho(\boldsymbol{u}, t-1)\sigma(\xi, t)\rangle \\ &\quad + \sum_{\xi'} W(\xi/\xi')\langle \rho(\boldsymbol{u}, t-1)\sigma(\xi', t-1)\rangle\,, \end{aligned} \tag{4.20}$$

where

$$W(\xi/\xi') = T(\xi/\xi') - \delta_{\xi,\xi'}\,. \tag{4.21}$$

Here we used the fact that the process $\xi(t)$ evolves independent of $\boldsymbol{u}(t)$, so if the

* For an example, see ref. 6.

† Compare the corresponding case of an s.d.e. in ref. 7.

value of $\boldsymbol{u}$ at time $t-1$ is fixed, the second term in the r.h.s. of (4.20) evolves in the same way as $\pi(\xi, t)$ itself. However the evolution of $\boldsymbol{u}(t)$ does depend on $\xi(t)$, as is expressed by the dependence of L on $\xi(t)$. Using the above argument in the first term too, in addition to the fact that we can put $\xi(t)$ in $L(\xi(t))$ equal to ξ (due to $\sigma(\xi, t)$), it follows that

$$\begin{aligned} \Delta P(\boldsymbol{u}, \xi, t-1) &= \langle L(\boldsymbol{u}, \xi)\rho(\boldsymbol{u}, t-1)\sigma(\xi, t)\rangle + WP(\boldsymbol{u}, \xi, t-1) \\ &= TL(\boldsymbol{u}, \xi)\langle \rho(\boldsymbol{u}, t-1)\sigma(\xi, t-1)\rangle + WP(\boldsymbol{u}, \xi, t-1) \\ &= \{TL(\boldsymbol{u}, \xi) + W\}P(\boldsymbol{u}, \xi, t-1)\,, \end{aligned}$$

where $WP(\boldsymbol{u}, \xi, t) \equiv \Sigma_{\xi'} W(\xi/\xi')P(\boldsymbol{u}, \xi', t)$, etc. Using (4.21) we finally arrive at

$$P(\boldsymbol{u}, \xi, t) = T\{1 + L(\boldsymbol{u}, \xi)\}P(\boldsymbol{u}, \xi, t-1)\,. \tag{4.22}$$

Now it is easy to obtain equations from which the moments of $\boldsymbol{u}(t)$ can be obtained. Define the marginal averages

$$\langle \boldsymbol{u}(t)\rangle_\xi = \int \mathrm{d}\boldsymbol{u}\, \boldsymbol{u}P(\boldsymbol{u}, \xi, t)\,. \tag{4.23}$$

From (4.22) and (4.19) (of which only the term with $m = 1$ contributes) one has

$$\begin{aligned} \langle \boldsymbol{u}(t)\rangle_\xi &= \sum_{\xi'} T(\xi/\xi')\{\langle \boldsymbol{u}(t-1)\rangle_\xi + (A(\xi') - 1)\langle \boldsymbol{u}(t-1)\rangle_{\xi'}\} \\ &= \sum_{\xi'} T(\xi/\xi')A(\xi')\langle \boldsymbol{u}(t-1)\rangle_{\xi'}\,. \end{aligned} \tag{4.24}$$

After solving this linear set of equations for $\langle u_i(t)\rangle_\xi$, $i = 1, \ldots, n$, the average of $\boldsymbol{u}$ can be obtained as

$$\langle \boldsymbol{u}(t)\rangle = \sum_\xi \langle \boldsymbol{u}(t)\rangle_\xi\,.$$

Higher moments of $\boldsymbol{u}(t)$ can be obtained in a similar way. An application of eq. (4.24) will be discussed in the next section.

5. Application: a biological population with two age classes in a random environment

Recently[4,8,9]) equations of the form (1.1) have been used to describe the growth of a biological population in a random environment. The discrete time formulation is appropriate for a population with non-overlapping generations. Moreover, the members of the population are assumed to be in one of n age (or size/stage) classes of equal length, hence the population is modeled by a vector difference equation. Linear equations are appropriate as long as density dependent effects are unimportant. So if the original model is nonlinear, (1.1) can be viewed as its linearization around equilibrium.

In the biological literature models of type (1.1) have been widely used with a special matrix form of A, a so called Leslie matrix. This is a matrix which has only non-zero elements on the first row (these are the fecundities of each age class) and the subdiagonal (these are the transition rates from one age class to the next). The vector of age classes $(u_1, u_2, \ldots, u_n)$ is chosen such that u_1 is the youngest age class, u_2 the next youngest, etc.

Here we consider the simplest case of a population with two age classes: $u = (u_1, u_2)$. The corresponding difference equation is

$$u(t) = A(t)u(t-1), \quad A(t) = \begin{pmatrix} m_0 & m_1 + \alpha\xi(t) \\ l & 0 \end{pmatrix}. \tag{5.1}$$

The matrix A is of Leslie type, and only one element is random, due to the presence of a random perturbation $\alpha\xi(t)$. All constants m_0, m_1 and l are positive. The model describes a situation where the birth rate of the older age class fluctuates, e.g. because it is more prone to diseases.

We take $\{\xi(t)\}$ to be a two state Markov chain (values ± 1) already introduced in I, subsection 3.2. Its transition matrix is

$$T = \begin{pmatrix} 1-v & v \\ v & 1-v \end{pmatrix}, \quad 0 < v < 1\,. \tag{5.2}$$

If one assumes the initial distribution $\pi(\xi, 0)$ to be the equilibrium distribution, the first two moments of $\xi(t)$ are

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(t') \rangle = \rho^{|t-t'|}, \quad \rho = 1 - 2v\,. \tag{5.3}$$

This example has been studied before by Tuljapurkar[9]) by means of a different method. We compare our results (subsection 5.1) with his ones in subsection 5.2, and with the exact expressions (subsection 5.3) which can be obtained in this case by the method of subsection 4.3.

5.1. *Results of the cumulant expansion*

To calculate the average of $u(t)$ we use the approximate equation (2.15), which is valid if $\alpha\tau_c \ll 1$, where $\tau_c = -(\ln|\rho|)^{-1}$. So

$$\langle u(t) \rangle = K\langle u(t-1) \rangle, \quad t \gg \tau_c\,, \tag{5.4}$$

where

$$K = A_0 + \alpha^2 \sum_{\tau=1}^{\infty} \rho^\tau A_1 A_0^{\tau-1} A_1 A_0^{-\tau} + \mathcal{O}(\alpha^3) \tag{5.5a}$$

and

$$A_0 = \begin{pmatrix} m_0 & m_1 \\ l & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{5.5b}$$

The eigenvalues of A_0 are

$$\lambda_\pm = \tfrac{1}{2}\{m_0 \pm \sqrt{m_0^2 + 4m_1 l}\} \tag{5.6}$$

and

$$A_0^t = \frac{1}{l(\lambda_+ - \lambda_-)} \begin{pmatrix} l(\lambda_+^{t+1} - \lambda_-^{t+1}) & \lambda_+\lambda_-(-\lambda_+^t + \lambda_-^t) \\ l^2(\lambda_+^t - \lambda_-^t) & l\lambda_+\lambda_-(-\lambda_+^{t-1} + \lambda_-^{t-1}) \end{pmatrix}. \tag{5.7}$$

From (5.6) it is seen that $\lambda_+ > 0$, $\lambda_- < 0$ and $|\lambda_+| > |\lambda_-|$. Calculating the matrix products in (5.5a) and carrying out the summation we find

$$K = \begin{pmatrix} m_0 & m_1 \\ l & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} m_0' & m_1' \\ 0 & 0 \end{pmatrix}, \tag{5.8}$$

where

$$m_0' = \frac{l^2}{(\lambda_+ - \lambda_-)^2} \frac{1}{\lambda_+\lambda_-} \left\{ (\lambda_+ + \lambda_-) \frac{\rho}{1-\rho} - \lambda_- \frac{\rho\lambda_+/\lambda_-}{1 - \rho\lambda_+/\lambda_-} - \lambda_+ \frac{\rho\lambda_-/\lambda_+}{1 - \rho\lambda_-/\lambda_+} \right\}, \tag{5.9a}$$

$$m_1' = \frac{l}{(\lambda_+ - \lambda_-)^2} \frac{1}{\lambda_+\lambda_-} \left\{ -(\lambda_+^2 + \lambda_-^2) \frac{\rho}{1-\rho} + \lambda_+\lambda_- \left(\frac{\rho\lambda_+/\lambda_-}{1 - \rho\lambda_+/\lambda_-} - \frac{\rho\lambda_-/\lambda_+}{1 - \rho\lambda_-/\lambda_+} \right) \right\}. \tag{5.9b}$$

Since $|\lambda_+| > |\lambda_-|$ we have to require $|\rho| < |\lambda_-/\lambda_+|$ for the sums in (5.5a) to converge*. K has the same form as A_0, so the eigenvalues $\lambda_\pm'$ of K can be immediately inferred from (5.6)

$$\begin{aligned} \lambda_\pm' &= \tfrac{1}{2}\{m_0 + \alpha^2 m_0' \pm \sqrt{(m_0 + \alpha^2 m_0')^2 + 4(m_1 + \alpha^2 m_1')l}\} \\ &= \lambda_\pm \pm \alpha^2 \left(\frac{\lambda_\pm m_0' + l m_1'}{\lambda_+ - \lambda_-} \right) + \mathcal{O}(\alpha^3). \end{aligned} \tag{5.10}$$

Inserting the expressions (5.9) for m_0' and m_1' in (5.10), one finds after some algebra

$$\lambda_\pm' = \lambda_\pm + \alpha^2 \frac{l^2\rho^2}{1-\rho} \{\lambda_\pm(\lambda_\pm - \lambda_\mp)(\lambda_\pm - \rho\lambda_\mp)\}^{-1}. \tag{5.11}$$

For all $\rho \in (-1, 1)$ it is the case that $\lambda_+' > \lambda_+$, $\lambda_-' < \lambda_-$. The solution of (5.4) is

$$\langle u(t) \rangle = K^t \langle u(0) \rangle,$$

* In the second and third term of (5.9) the effective autocorrelation time is increased ($\tau_c' = -(\ln|\rho\lambda_+/\lambda_-|)^{-1}$), resp. decreased. See also ref. 10.

where K' is the same as (5.7) with $\lambda_\pm$ replaced by $\lambda'_\pm$. As $t \to \infty$ the terms with $(\lambda'_+)^t$ dominate. Qualitatively the asymptotic growth rate λ'_+ is larger than λ_+, increases with increasing variance α^2 of ξ, and rapidly increases as ρ runs from -1 to $+1$, where positive ρ has a much larger effect than negative ρ (the condition $\alpha\tau_c \ll 1$ implies that for fixed α only values of $|\rho|$ much smaller than α^{-1} are allowed). Second moments can be obtained in the same way from (3.25).

5.2. *Comparison with previous results of Tuljapurkar*

Now our results are compared with those of Tuljapurkar[9]). His method applies to equations of the form (1.1) where ξ is an ergodic Markov chain with a finite number of states, so (5.1) falls within this class. Results are obtained for the asymptotic growth rate μ of the total number of individuals $u_1(t) + u_2(t)$, which should correspond to our λ'_+. We now apply his general results to our example (5.1).

The lowest order contribution to μ is $\mu^{(1)} = \lambda^*$, where λ^* is the dominant eigenvalue of A_0, i.e. λ_+ in our notation. The first order correction $\mu^{(1)}$ vanishes (since $\langle \xi(t) \rangle = 0$), and formula (A.18) of ref. 9 gives the second order correction $\mu^{(2)}$. In the case of a 2×2-matrix A it yields

$$\mu^{(2)} = \frac{1}{\lambda^*} \frac{v_1}{1 - v_1} \frac{\bar{v}' \otimes \bar{v}' (D_1 \otimes E_1) \bar{u} \otimes \bar{u}}{(\bar{v}'\bar{u})^2} + \frac{v_1}{\lambda^* - v_1 \kappa_1} \frac{\bar{v}' E_1 \bar{u}_1 \bar{v}'_1 D_1 \bar{u}}{(\bar{v}'\bar{u})(\bar{v}'_1 \bar{u}_1)} . \tag{5.12}$$

Here v_1 is the serial autocorrelation time, i.e. ρ; κ_1 is the second eigenvalue of A_0, i.e. $\kappa_1 = \lambda_-$; $\bar{u}$ and $\bar{u}_1$ are the right eigenvectors of A_0 corresponding to λ_+ and λ_-, respectively; and $\bar{v}$ and $\bar{v}_1$ are the corresponding left eigenvectors. Unprimed vectors are column vectors, primed ones are row vectors. The matrices E_1 and D_1 are defined by (A.17) as[9])

$$E_1 = \pi_1(0) H_0 + \pi_1(1) H_1 ; \quad D_1 = \pi(0)\psi_1(0) H_0 + \pi(1)\psi_1(1) H_1 . \tag{5.13}$$

The matrices H_m are defined by $H_m = A_m - A_0$, where A_m is the value of the matrix A in (5.1) corresponding to the mth value of the Markov chain. In our notation, $H_0 = \alpha A_1$, $H_1 = -\alpha A_1$, corresponding to the values ± 1 of ξ. The vectors $\bar{\pi}$ and $\bar{\pi}_1$ are the right eigenvectors of the transition matrix T (eq. (5.2)) with components $\pi(m)$, $\pi_1(m)$, $m = 0, 1$; $\bar{\psi}_1$ is the left eigenvector of T; $\bar{\pi}$ corresponds to the eigenvalue 1, $\bar{\pi}_1$ and $\bar{\psi}_1$ to the second eigenvalue ρ of T. The components of $\bar{\pi}$ sum to 1, and $\bar{\psi}'_1 \bar{\pi}_1 = 1$. From (5.2) one has

$$\bar{\pi} = \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{\pi}_1 = \tfrac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \bar{\psi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.14}$$

where the normalization conditions given above are satisfied. Now E_1 and D_1 can be calculated from (5.13) and (5.14),

$$E_1 = \tfrac{1}{2}(H_0 - H_1) = \alpha A_1, \quad D_1 = \tfrac{1}{2}(H_0 - H_1) = \alpha A_1 .$$

The (non-normalized) eigenvectors of A_0 are

$$\bar{u} = \begin{pmatrix} \lambda_+ \\ l \end{pmatrix}, \quad \bar{v} = \begin{pmatrix} \lambda_+ \\ m_1 \end{pmatrix}, \quad \bar{u}_1 = \begin{pmatrix} \lambda_- \\ l \end{pmatrix}, \quad \bar{v}_1 = \begin{pmatrix} \lambda_- \\ m_1 \end{pmatrix}.$$

Inserting all this in (5.12) we get in our notation

$$\mu^{(2)} = \alpha^2 \left\{ \frac{1}{\lambda_+} \frac{\rho}{1-\rho} \frac{(\bar{v}' \otimes \bar{v}')(A_1 \otimes A_1)(\bar{u} \otimes \bar{u})}{(\bar{v}'\bar{u})^2} + \frac{\rho}{\lambda_+ - \rho\lambda_-} \frac{\bar{v}' A_1 \bar{u}_1 \bar{v}_1' A_1 \bar{u}}{(\bar{v}'\bar{u})(\bar{v}_1'\bar{u}_1)} \right\}. \tag{5.15}$$

Since

$$\bar{v} A_1 \bar{u} = \lambda_+ l = \bar{v}' A_1 \bar{u}_1, \quad \bar{v}_1' A_1 \bar{u} = \lambda_- l$$

and

$$\bar{v}'\bar{u} = \lambda_+^2 + m_1 l = \lambda_+(\lambda_+ - \lambda_-), \quad \bar{v}_1'\bar{u}_1 = \lambda_-^2 + m_1 l = \lambda_-(\lambda_- - \lambda_+),$$

it follows from (5.15) that

$$\mu^{(2)} = \frac{\alpha^2 l^2}{\lambda_+(\lambda_+ - \lambda_-)^2} \left\{ \frac{\rho}{1-\rho} - \frac{\rho}{1 - \rho\lambda_-/\lambda_+} \right\}$$

$$= \frac{\alpha^2 l^2 \rho^2}{1-\rho} \{\lambda_+(\lambda_+ - \lambda_-)(\lambda_+ - \rho\lambda_-)\}^{-1} \tag{5.16}$$

in complete agreement with the second order correction to λ_+' as given by (5.11).

5.3. *Comparison with the exact result*

Finally we illustrate here the method of subsection 4.3 and calculate the exact moments of the solution of (5.1). It is shown that the exact results up to $\mathcal{O}(\alpha^2)$ agree with (5.11) and (5.16), which were obtained by perturbation theory.

The marginal moments $\langle u_i(t) \rangle_\xi$, $i = 1, 2$, $\xi = \pm 1$, as defined by (4.23), satisfy (4.24), where A is given in (5.1) and T in (5.2). This results in the following system of linear difference equations

$$\begin{aligned}
\langle u_1(t) \rangle_+ &= (1-\nu)\{m_0 \langle u_1(t-1) \rangle_+ + (m_1 + \alpha)\langle u_2(t-1) \rangle_+\} \\
&\quad + \nu\{m_0 \langle u_1(t-1) \rangle_- + (m_1 - \alpha)\langle u_2(t-1) \rangle_-\}, \\
\langle u_1(t) \rangle_- &= \nu\{m_0 \langle u_1(t-1) \rangle_+ + (m_1 + \alpha)\langle u_2(t-1) \rangle_-\} \\
&\quad + (1-\nu)\{m_0 \langle u_1(t-1) \rangle_- + (m_1 - \alpha)\langle u_2(t-1) \rangle_-\}, \\
\langle u_2(t) \rangle_+ &= (1-\nu) l \langle u_1(t-1) \rangle_+ + \nu l \langle u_1(t-1) \rangle_-, \\
\langle u_2(t) \rangle_- &= \nu l \langle u_1(t-1) \rangle_+ + (1-\nu) l \langle u_1(t-1) \rangle_- .
\end{aligned}$$

By the introduction of the column vector $U(t) = \mathrm{Col}\{\langle u_1(t) \rangle_+, \langle u_1(t) \rangle_-, \langle u_2(t) \rangle_+,$

$\langle u_2(t)\rangle_-\}$ we can write this set in matrix form,

$$U(t) = MU(t-1)\,, \tag{5.17}$$

where the matrix M is given by

$$M = \begin{pmatrix} m_0(1-v) & m_0 v & (m_1+\alpha)(1-v) & (m_1-\alpha)v \\ m_0 v & m_0(1-v) & (m_1+\alpha)v & (m_1-\alpha)(1-v) \\ l(1-v) & lv & 0 & 0 \\ lv & l(1-v) & 0 & 0 \end{pmatrix}. \tag{5.18}$$

To obtain the eigenvalues of M, we have to solve the secular equation

$$\mu^4 - m_0(1+\rho)\mu^3 + \mu^2\{m_0^2\rho - lm_1(1+\rho^2)\} + \mu\{m_0 lm_1\rho(1+\rho)\} + l^2\rho^2(m_1^2-\alpha^2) = 0\,, \tag{5.19}$$

where ρ is given in (5.3). If the eigenvalues and eigenvectors of M are determined, one can of course construct the complete solution of (5.17) under the initial condition $\langle u_1(0)\rangle_+ = \langle u_1(0)\rangle_- = \frac{1}{2}u_1(0)$, $\langle u_2(0)\rangle_+ = \langle u_2(0)\rangle_- = \frac{1}{2}u_2(0)$, and from that the moments $\langle u_1(t)\rangle$, $\langle u_2(t)\rangle$ via the prescription given in subsection 4.3. However, we restrict ourselves here to a calculation of the eigenvalues of M to $\mathcal{O}(\alpha^2)$ and compare the results with (5.11) and (5.16).

The four eigenvalues of M are easily obtained if $\alpha = 0$. In that case the secular equation (5.19) factorizes as

$$(\mu^2 - \mu m_0 - lm_1)(\mu^2 - \mu m_0\rho - lm_1\rho^2) = 0\,,$$

so the unperturbed roots $\mu_1^{(0)}, \ldots, \mu_4^{(0)}$ are given by

$$\mu_{1,2}^{(0)} = \lambda_\pm\,, \quad \mu_{3,4}^{(0)} = \rho\lambda_\pm\,,$$

where $\lambda_\pm$ is given by (5.6). The roots of (5.19) for α unequal to zero are now written to second order as

$$\mu_{1,2} = \lambda_\pm + x_\pm\alpha^2 + \mathcal{O}(\alpha^4), \quad \mu_{3,4} = \rho\lambda_\pm + y_\pm\alpha^2 + \mathcal{O}(\alpha^4)\,. \tag{5.20}$$

Before we proceed to calculate $x_\pm$ and $y_\pm$ we note that the solution of (5.17) involves four eigenvalues, whereas the solution of (5.4) involves only two eigenvalues. The reason is the following: the perturbation method of subsection 5.1 (and 5.2) presupposes a small correlation time, i.e. small ρ (for higher orders in (5.5a) are neglected and the initial time t_0 has disappeared). And if ρ is small, the roots $\mu_{3,4}$ in (5.20) correspond to rapidly decaying transients which are neglected anyway in (5.4) (there is one exception, viz. $\alpha = 0$: in this case (5.4) is exact; and indeed it is readily checked that the roots $\mu_{3,4}^{(0)}$ don't contribute to the solution of (5.17) if $\alpha = 0$, due to the special initial distribution, i.e. $u(0)$ fixed and $\xi(0)$ in equilibrium).

So for comparison with the previous results we only calculate the corrections to $\lambda_\pm$ in (5.20). Inserting the expression for $\mu_{1,2}$ in (5.19) and collecting all terms of order α^2 one gets

$$x_\pm = l^2\rho^2[4\lambda_\pm^3 - m_0(1+\rho)3\lambda_\pm^2 + 2\lambda_\pm\{m_0^2\rho - lm_1(1+\rho^2)\} + m_0 l m_1 \rho(1+\rho)]^{-1} . \quad (5.21)$$

First all quantities in $x_\pm$ can be expressed in terms of $\lambda_\pm$ and ρ via

$$m_0 = \lambda_+ + \lambda_-, \quad lm_1 = -\lambda_+\lambda_-$$

as is easily verified from (5.6). After some algebraic manipulations we find

$$x_\pm = l^2\rho^2[\lambda_\pm(1-\rho)(\lambda_\pm - \lambda_\mp)(\lambda_\pm - \rho\lambda_\mp)]^{-1}$$

in agreement with (5.11) and (5.16).

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