Similarity Measures for Convex Polyhedra Based on Minkowski Addition *

Alexander V. Tuzikov

Institute of Engineering Cybernetics
Academy of Sciences of Republic Belarus
Surganova 6, 220012 Minsk, Belarus

Jos B.T.M. Roerdink

Department of Mathematics and Computing Science
University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands

Henk J.A.M. Heijmans

Centre for Mathematics and Computer Science
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Abstract

In this paper we introduce and investigate similarity measures for convex polyhedra based on Minkowski addition and inequalities for the mixed volume, volume and surface area related to the Brunn-Minkowski theory. All measures considered are invariant under translations; furthermore, they may also be invariant under subgroups of the affine transformation group. For the case of rotation and scale invariance, we prove that to obtain the measures based on (mixed) volume, it is sufficient to compute certain functionals only for a finite number of critical rotations. Extensive use is made of the slope diagram representation of convex polyhedra.

AMS Subject Classification (1991): 52A38, 52B15, 68T10, 68U05

Keywords & Phrases: similarity measure, convex set, convex polyhedron, Minkowski addition, slope diagram representation, affine transformation, rotation, reflection, multiplication, similitude, volume, mixed volume, Brunn-Minkowski inequality, critical rotation, critical angle.

1 Introduction

In this report we consider the problem of comparing convex polyhedra using Minkowski addition. The Brunn-Minkowski theory [25] allows one to introduce several similarity measures

*Report CS-R9708, Department of Computing Science, University of Groningen, September 1997. Postscript version obtainable at `ftp.cs.rug.nl/pub/cs-reports/CS-R9708.ps.gz`

for convex shapes based on inequalities for the volume, mixed volume and surface area. We consider similarity measures for convex shapes which are invariant under subgroups of the group of affine transformations on $\mathbb{R}^3$ and follow the outline of the report [12] devoted to 2D convex polygons. All these similarity measures are translation-invariant. If one considers the measures which are invariant under the group of orthogonal transformations, the direct computation of similarity measures in the 3D case becomes very time consuming. Every orthogonal transformation with positive determinant can be considered as a rotation about some axis by a fixed angle. Therefore the optimization should be performed for all possible positions of rotation axes and rotation angles.

Data representation is a very important part of every computation. A spherical representation of convex polyhedra is most suitable while dealing with Minkowski addition. One of the simplest of such spherical representations is the Extended Gaussian Image (EGI). According to this representation every polyhedral facet is given by a point on the unit sphere having the same unit normal vector as the corresponding facet. A weight is assigned to such a point which equals the area of the corresponding facet. If follows from the Minkowski existence theorem [25] that the discrete distribution of these weights uniquely defines a convex polyhedron. The representation is translation-invariant and if the polyhedron rotates its EGI rotates in the same way. Due to these properties the EGI representation is often used in computer vision for solving problems of recognition and pose determination of 3D shapes [3,13,16,18].

Although the EGI defines a unique convex polyhedron, the reconstruction of a polyhedron itself from its EGI is a difficult problem. Several algorithms have been developed for this reconstruction. Little [17] suggested an iterative algorithm which finds the distances of the polyhedral facets from the origin. Recently Moni [22] proposed an algorithm which first establishes an adjacency relation of facets and then finds directions and lengths of polyhedral edges. The time complexity of the polytope reconstruction problem from its EGI was investigated in [8]. Since the EGI is limited to convex shapes, several extensions of it have been proposed in the literature to deal with non-convex shapes as well [11,14,23].

This paper deals only with convex polyhedra and uses the slope diagram representation [7]. The facets, edges and vertices of a polyhedron are represented on the unit sphere in $\mathbb{R}^3$ by spherical points, spherical arcs and spherical polygons, respectively. Additionally, we keep information about areas of facets and lengths of polyhedral edges. This representation is unique for convex polyhedra, allows easy polyhedron reconstruction and computation of Minkowski addition of polyhedra. This representation is redundant in comparison to EGI which contains only spherical points and areas of corresponding polyhedral facets. As will be shown later, spherical arcs play also an important role in computing similarity measures for convex polyhedra. Although in fact they can be derived from spherical points using time consuming reconstruction algorithms, we prefer to have them explicitly in the polyhedron representation.

If one restricts oneself to comparing convex polyhedra then it is possible to prove that the volume, mixed volume and surface area (which will be referred to as 'objective functionals') of a Minkowski sum of polyhedra are piecewise concave functions of the rotation angle of one polyhedron with a fixed axis of rotation. This implies that, for every fixed rotation axis, there is only a finite number of rotation angles at which it is necessary to compute the objective functionals in order to obtain the similarity measure. We also show that the set of rotation axes to be checked can be found using only information about the orientation of facets of polyhedra and the position of their edges. This set depends also on the similarity measure under consideration. Moreover we show that for the case of (mixed) volume the set of rotation axes to be checked is finite.

The paper is organized in the following way. In Section 2 we briefly discuss the approaches for Minkowski addition of convex polyhedra, and introduce the slope diagram representation of convex polyhedra, as well as some facts about the affine transformation group and its subgroups. Properties of mixed volumes and main inequalities related to the Brunn-Minkowski theory needed in the paper are given in Section 3. To compare convex polyhedra we introduce in Section 4 the notion of similarity measures and define a number of such measures based on inequalities for the volume, mixed volume and surface area. In Section 5 similarity measures based on (mixed) volume are investigated which are invariant under rotations and scaling. Given any axis of rotation, it is proved that it is sufficient to compute the objective functionals needed to obtain these measures only for a finite number of critical rotations, thus generalizing a similar result for the 2D case [12]. Moreover it is proved for the case of (mixed) volume that only a finite number of rotation axes has to be checked. Section 6 contains conclusions and a discussion of future work.

2 Preliminaries

This section presents some basic notation and other prerequisites needed in the sequel of the paper. Also, the representation of convex polyhedra using slope diagrams is introduced, as well as some facts about the affine transformation group and its subgroups.

By $\mathcal{K}(\mathbb{R}^3)$, or briefly $\mathcal{K}$, we denote the family of all nonempty compact subsets of $\mathbb{R}^3$. Provided with the Hausdorff distance [25] this is a metric space. The compact convex subsets of $\mathbb{R}^3$ are denoted by $\mathcal{C} = \mathcal{C}(\mathbb{R}^3)$, and the convex polyhedra by $\mathcal{P}(\mathbb{R}^3)$. In this paper, we are not interested in the location of a shape $A \subseteq \mathbb{R}^3$; in other words, two shapes A and B are said to be *equivalent* if they differ only by translation. We denote this as $A \equiv B$.

2.1 Minkowski addition of convex polyhedra

Minkowski addition of two sets $A, B \subseteq \mathbb{R}^n$ is defined by

$$A \oplus B = \{a + b \mid a \in A,\ b \in B\}.$$

It is well-known [25] that every element A of $\mathcal{C}$ is uniquely determined by its *support function* given by:

$$h(A, u) = \sup\{\langle a, u\rangle \mid a \in A\}, \qquad u \in S^2.$$

Here $\langle a, u\rangle$ is the inner product of vectors a and u, and S^2 denotes the unit sphere in $\mathbb{R}^3$. It is also known that [25]:

$$h(A \oplus B, u) = h(A, u) + h(B, u), \qquad u \in S^2, \tag{2.1}$$

for $A, B \in \mathcal{C}$. The *support set* $F(A, u)$ of A at $u \in S^2$ consists of all points $a \in A$ for which $\langle a, u\rangle = h(A, u)$. Support sets can be of dimension $0, 1, 2$. The support set of dimension k, $(k = 0, 1, 2)$ is called a *k-face* and denoted by F^k. If A is a convex polyhedron, then 0-faces, 1-faces and 2-faces are called *vertices*, *edges* and *facets* of A, respectively. Henceforth, a facet will be denoted by F_i, and its area by $S(F_i)$.

It is known from Minkowski's existence theorem [20] (see also [25, p. 390] for a discussion of the n-dimensional case as well as a general concept of surface measures for convex sets) that a convex polyhedron is uniquely determined by areas and normal vector directions of its facets.

Theorem 2.1 (Minkowski's existence theorem) *Let $u_1, \ldots, u_k \in S^2$ be distinct vectors linearly spanning $\mathbb{R}^3$, and let $m_1, \ldots, m_k$ be positive real numbers such that*

$$\sum_{i=1}^{k} m_i u_i = 0.$$

Then there exists a convex polyhedron P in $\mathbb{R}^3$ having k facets with normal vectors u_i and area m_i, i.e.,

$$S(F(P, u_i)) = m_i$$

for $i = 1, \ldots, k$.

This theorem is true for n-dimensional polytopes as well.

Several equivalent ways are known to define Minkowski addition [9] for convex polyhedra using representations based on vertices or facets. These are especially helpful for the actual computation of Minkowski sums. Let $p_i,\ i = 1, \ldots, n_P$ be the vertices of P and $q_i,\ i = 1, \ldots, n_Q$ be those of Q. Then

$$P \oplus Q = \mathrm{conv}\ \{p_i + q_j \mid i = 1, \ldots, n_P,\ q_i,\ j = 1, \ldots, n_Q\}.$$

Here conv$\{\ \cdot\ \}$ denotes the convex hull.

Theorem 2.2 *Let P and Q be two convex polyhedra in $\mathbb{R}^3$. Then for every $u \in S^2$,*

$$F(P \oplus Q, u) = F(P, u) \oplus F(Q, u). \tag{2.2}$$

This theorem is valid for the n-dimensional case as well [25, Thm.1.7.5].

Equation (2.2) is the basis for computing Minkowski addition of convex polyhedra. We follow here the outline of [7] and refer to it for a more detailed discussion.

Since a convex polyhedron is defined by its oriented facets, it is sufficient for computation of $P \oplus Q$ to find only the facets of polyhedron $P \oplus Q$. For every facet $F(P \oplus Q, u)$ the normal unit vector u is either orthogonal to a facet of P or/and Q, or there exist non-parallel edges of P and Q for which u is a normal vector. Therefore the facets of $P \oplus Q$ can be obtained by [7,19]:

1. *Minkowski addition of two facets:* addition of a facet of P and a facet of Q;

2. *Minkowski addition of a facet and an edge:* addition of a facet of one of the two summands and an edge of the other;

3. *Minkowski addition of a facet and a vertex:* addition of a facet of one of the two summands and a vertex of the other;

4. *Minkowski addition of two non-parallel edges:* addition of non-parallel edges of P and Q.

Here the added facets, edges, and vertices lie in supporting planes with parallel outward normals.

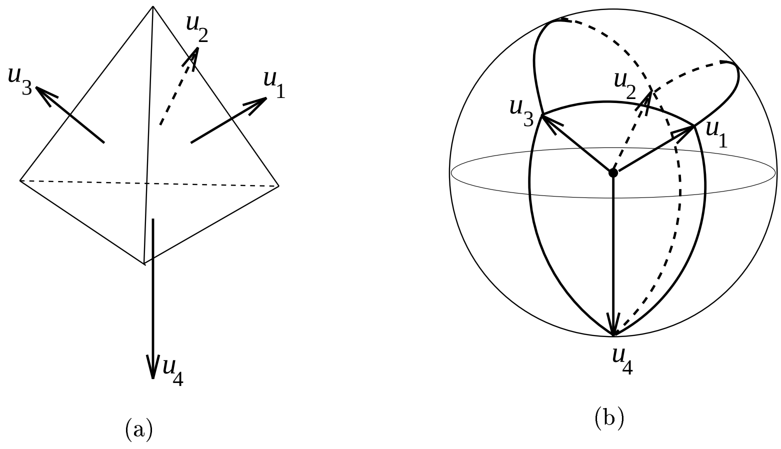


Figure 1: *Polyhedron (a) and its slope diagram representation (b).*

2.2 Polyhedra representation

The sequel of the paper makes use of the slope diagram representation (SDR) of convex polyhedra [7]. According to this representation, facets, edges and vertices of a polyhedron are given by points, spherical arcs and convex spherical polygons of the unit sphere S^2, see Fig. 1.

- *Facet representation.* A facet F_i of a polyhedron which is orthogonal to the unit vector u_i is represented on the sphere S^2 by the end point of this vector;
- *Edge representation.* Each edge is represented by the arc of the great circle (spherical arc) joining the two points corresponding to the two adjacent facets of the edge;
- *Vertex representation.* The region (called the spherical polygon) of the sphere bounded by the spherical arcs corresponding to the edges which are adjacent to a polyhedral vertex, represents this vertex on the sphere S^2. The spherical arcs are included in the region.

Sometimes we speak about spherical points and arcs of a polyhedron, meaning spherical points and arcs of its slope diagram representation. Also, weights of spherical points and spherical arcs are used. The weight of a spherical point or arc equals the area of the corresponding polyhedral facet, or the length of the corresponding polyhedral edge, respectively.

Therefore the SDR of a polyhedron P is a triple $SDR(P) = (\mathcal{V}, \mathcal{A}, \mathcal{W})$. Here $\mathcal{V} = \{u_1, u_2, \ldots, u_{n_P}\}$ is the set of spherical points, for which the same notation is used as for the corresponding unit vectors $\{u_i\}$ of P. $\mathcal{A} \subset \mathcal{V} \times \mathcal{V}$ is the set of spherical arcs. An arc from $\mathcal{A}$ connecting points u_i and u_j is denoted by (u_i, u_j). $\mathcal{W}$ denotes the weights of points and arcs, i.e., $a_P(u_i)$ (or $a(u_i)$) equals the area of the corresponding facet F_i and $l_P(u_i, u_j)$ (or simply $l(u_i, u_j)$) equals the length of the edge between facets F_i and F_j of the polyhedron P.

In the two dimensional case, i.e., in the case of convex polygons, the slope diagram can be considered also as a function $M(P,u)$ defined on the unit circle S^1. Given a polygon $P \subseteq \mathbb{R}^2$, denote by l_i the length of edge i and by u_i the vector orthogonal to this edge. Then

$$M(P,u) = \begin{cases} l_i, & \text{if } u = u_i, \\ 0, & \text{otherwise.} \end{cases}$$

This representation is also called in [12] a perimetric measure representation.

As follows from (2.2), Minkowski addition of two convex polygons can be computed by merging their respective slope diagrams. Mathematically, this amounts to the following relation [6, 9]:

$$M(P \oplus Q, u) = M(P,u) + M(Q,u), \quad \text{for } P, Q \in \mathcal{P}(\mathbb{R}^2) \text{ and } u \in S^1. \tag{2.3}$$

Let us denote by $\alpha_i = \angle u_i$ the angle between the positive x-axis and u_i. Then, given a slope diagram representation $M(P,u)$ of a convex polygon P, its area $S(P)$ can be computed as follows [12]:

$$S(P) = \sum_{i=1}^{n} l_i \sin\alpha_i \sum_{j=1}^{i} l_j \cos\alpha_j - \frac{1}{2}\sum_{i=1}^{n} l_i^2 \sin\alpha_i \cos\alpha_i. \tag{2.4}$$

Here n is the number of vertices of polygon P.

Now we have all the necessary tools to find Minkowski addition of two convex polyhedra P and Q by merging their slope diagram representations. The following three cases need special attention:

1. A spherical arc of one polyhedron intersects a spherical arc of the other;
2. A spherical point of one polyhedron lies on a spherical arc of the other;
3. Two spherical points coincide.

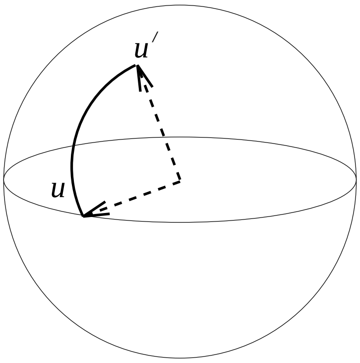
Let us consider these cases in more detail.

Case 1 Let two spherical arcs (u,u') and (v,v') intersect at the point $w \in S^2$ (see Fig. 2(c)). Point w represents a facet of $P \oplus Q$. This point is adjacent to u, u', v, v' and the weights of the corresponding spherical arcs are computed as follows (see Fig. 2(d-f) for illustration):

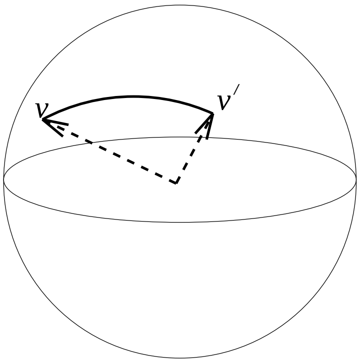
$$l_{P\oplus Q}(w,u) = l_{P\oplus Q}(w,u') = l_P(u,u') \;\text{ and }\; l_{P\oplus Q}(w,v) = l_{P\oplus Q}(w,v') = l_Q(v,v').$$

For, the edges corresponding to arcs (u,u') and (v,v') will be the edges of a facet of $P \oplus Q$ corresponding to w. The normal vectors $u'' = \frac{u\times u'}{|u\times u'|}$ and $v'' = \frac{v\times v'}{|v\times v'|}$ are parallel to the corresponding edges of polyhedra P and Q represented by the arcs (u,u') and (v,v'), respectively. Directions and lengths of all edges of the facet corresponding to the point w being known, one can find the area of this facet by (2.4).

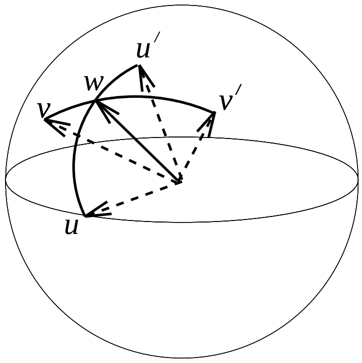
Case 3 Let us consider now an example of case 3. Denote the coinciding spherical points of P and Q by s (see Fig. 3(a), (b), (c)). Suppose also that point s is adjacent to spherical points u_1, u_2, u_3 of P and spherical points v_1, v_2, v_3, v_4 of Q. Point s represents a facet



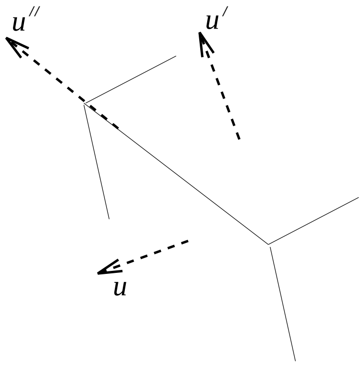
(a) Spherical arc (u, u') of P.



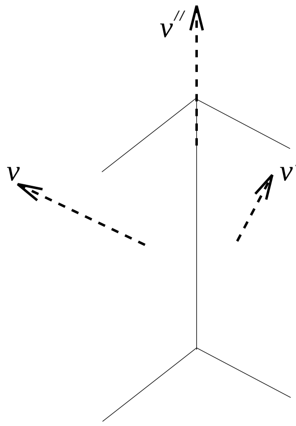
(b) Spherical arc (v, v') of Q.



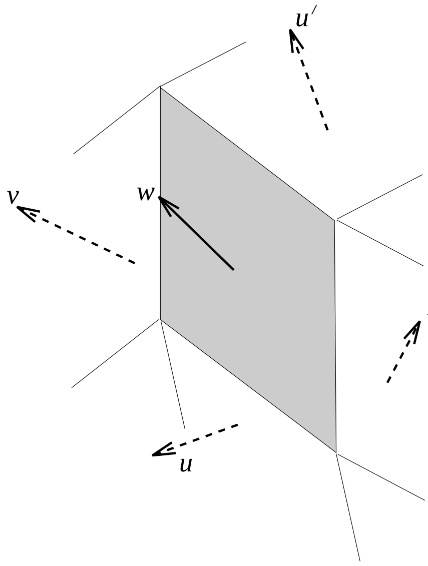
(c) SDR of $P \oplus Q$ with new spherical point w.



(d) Edge of P corresponding to (a).

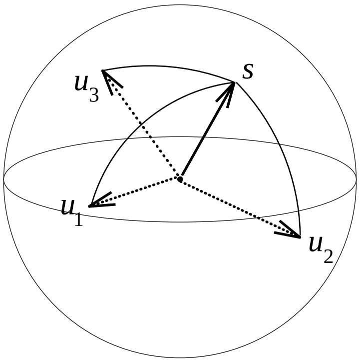


(e) Edge of Q corresponding to (b).

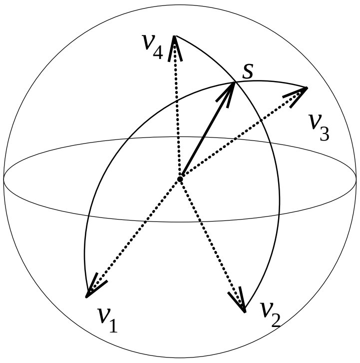


(f) Facet of $P \oplus Q$ corresponding to (c).

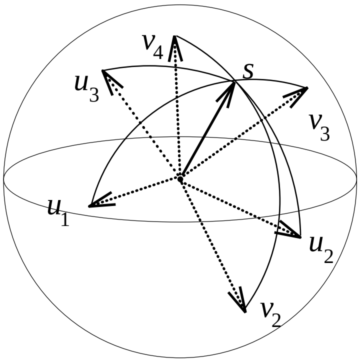
Figure 2: *Minkowski addition of two convex polyhedra P and Q with intersecting spherical arcs.*



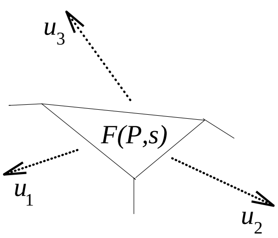
(a) Spherical point s of P connected to u_1, u_2, u_3.



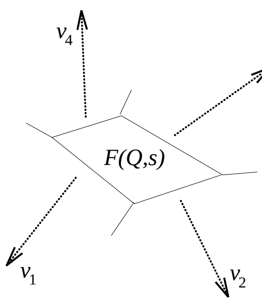
(b) Spherical point s of Q connected to v_1, v_2, v_3, v_4.



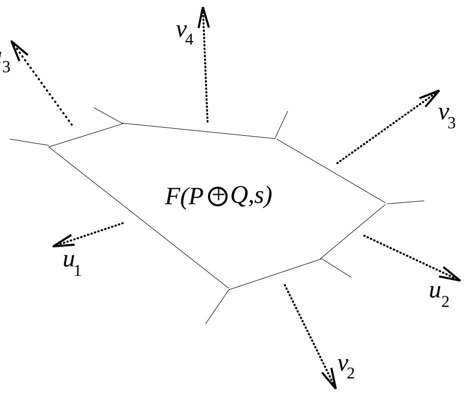
(c) SDR of $P \oplus Q$.



(d) Facet of P corresponding to (a).



(e) Facet of Q corresponding to (b).



(f) Facet of $P \oplus Q$ corresponding to (c).

Figure 3: *Minkowski addition of two convex polyhedra P and Q with coinciding spherical points.*

of polyhedron $P \oplus Q$. The arcs (s, u_2), (s, u_3), (s, v_2) and (s, v_4) are assumed to belong to different great circles. Therefore there will be arcs (s, u_2), (s, u_3), (s, v_2) and (s, v_4) in the SDR of $P \oplus Q$ with lengths determined by the SDR of P and Q, respectively. For, the edges corresponding to these spherical arcs will be the edges of the polyhedral facet corresponding to s in $P \oplus Q$. The arcs (s, u_1), (s, v_1) and (s, v_3) are assumed to belong to the same great circle, such that the arcs (s, u_1) and (s, v_1) have the same direction and the arc (s, u_1) is shorter than (s, v_1). Therefore the spherical point s in $P \oplus Q$ will be adjacent to u_1 and v_3 and $l_{P\oplus Q}(s, u_1) = l_P(s, u_1) + l_Q(s, v_1)$ and $l_{P\oplus Q}(s, v_3) = l_Q(s, v_3)$. That is, the edges e_1, e_2 corresponding to the arcs (s, u_1) and (s, v_1) on the same great circle are parallel, with the length of the corresponding edge of polyhedron $P \oplus Q$ being equal to the sum of the lengths of the edges e_1, e_2. This rule of changing weights is illustrated in Fig. 3(f). Similarly to case 1 we can compute the area of the facet of $P \oplus Q$ corresponding to the spherical point s by (2.4).

Case 2 This is similar to case 3. Suppose that a spherical point u lies on a spherical arc (v, v_1). Let us introduce a new spherical point v' on the arc (v, v_1) at the same position as u having weight zero, i.e. corresponding to a rectangular facet of zero area. This brings us back to case 3.

2.3 Transformation groups

Consider subgroups of the group G' of *affine transformations* on $\mathbb{R}^3$. If $g \in G'$ and $A \in \mathcal{K}$, then $g(A) = \{g(a) \mid a \in A\}$. We write $g \equiv g'$ if $g(A) \equiv g'(A)$ for every $A \in \mathcal{K}$. This is equivalent to saying that $g^{-1}g'$ is a translation. We denote by G the subgroup of G' containing all linear transformations, i.e., transformations g with $g(0) = 0$.

The following result is obvious.

Lemma 2.3 *For any two sets $A, B \subseteq \mathbb{R}^3$ and for every $g \in G$,*

$$g(A \oplus B) = g(A) \oplus g(B). \tag{2.5}$$

We introduce the following notations for subsets of G:

M: multiplications with respect to the origin by a positive factor;
R: rotations about an axis (passing through the origin);
E: (plane) reflections (planes passing through the origin);
I: isometries (distance preserving transformations);
S: similitudes (rotations, reflections, multiplications).

Observe that I, R, M and S are subgroups of G [26]. For every transformation $g \in G$ one can compute its *determinant* '$\det g$' which is, in fact, the determinant of the matrix corresponding to g. If g is an isometry then $|\det g| = 1$; the converse is not true, however. If H is a subgroup of G, then H_+ denotes the subgroup of H containing all transformations with positive determinant. For example, $I_+ = R$ and S_+ comprises all multiplications and rotations. If H is a subgroup of G, then the set $\{mh \mid h \in H,\ m \in M\}$ is also a subgroup, which will be denoted by MH. Rotations in $\mathbb{R}^3$ are denoted as follows. When ℓ is an axis (i.e., directed line) passing through the coordinate origin, $r_{\ell,\alpha}$ means a rotation about ℓ through angle α in a counter-clockwise direction.

At several instances in this paper, the following concept will be needed.

Definition 2.4 Let $H \subseteq G$ and $\mathcal{J} \subseteq \mathcal{K}$. We say that H is *$\mathcal{J}$-compact* if, for every $A \in \mathcal{J}$ and every sequence $\{h_n\}$ in H, the sequence $\{h_n(A)\}$ has a limit point of the form $h(A)$, where $h \in H$.

It is easy to verify that R is $\mathcal{K}$-compact. However, the subcollection $\{r^m \mid m \in \mathbb{Z}\}$, where $r = r_{\ell,\alpha} \in R$ is a rotation with α/π irrational, is not $\mathcal{K}$-compact for fixed axis ℓ. The following result is easy to prove.

Lemma 2.5 *Assume that H is $\mathcal{J}$-compact and let $f : \mathcal{J} \to \mathbb{R}$ be a continuous function. If $A \in \mathcal{J}$ and $f_0 := \sup_{h \in H} f(h(A))$ is finite, then there exists an element $h_0 \in H$ such that $f(h_0(A)) = f_0$.*

3 Mixed volumes

This section briefly describes properties of volumes and mixed volumes of compact sets in $\mathbb{R}^3$. For a comprehensive treatment the reader may consult [25].

The following theorem is due to Minkowski for $n = 3$ [5, p. 353]. Here $\mathcal{C}(\mathbb{R}^n)$ is the space of compact convex subsets of $\mathbb{R}^n$.

Theorem 3.1 (Minkowski theorem on mixed volumes) *The volume of the Minkowski sum $A = \lambda_1 A_1 \oplus \cdots \oplus \lambda_m A_m$ of convex sets from $\mathcal{C}(\mathbb{R}^n)$, where m is a positive integer and $\lambda_i \geq 0$, is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_m$. That is*

$$V(A) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(A_{i_1}, \ldots, A_{i_n}),$$

where the coefficients $V(A_{i_1}, \ldots, A_{i_n})$ are invariant under permutations of their arguments. The coefficient $V(A_{i_1}, \ldots, A_{i_n})$ is called the mixed volume *of the convex sets $A_{i_1}, \ldots, A_{i_n}$.*

For our purposes the case $n = 3$, $m = 2$ is the most interesting one. Thus, for convex sets A, B in $\mathbb{R}^3$ and $\lambda, \mu \geq 0$ one has

$$V(\lambda A \oplus \mu B) = \lambda^3 V(A) + 3\lambda^2 \mu V(A, A, B) + 3\lambda\mu^2 V(A, B, B) + \mu^3 V(B). \qquad (3.1)$$

Let us present some useful properties of mixed volumes [24, 25]:

$$V(A, A, A) = V(A) \qquad (3.2)$$

$$V(A, B, C) \geq 0; \text{ if } V(A), V(B), V(C) > 0, \text{ then } V(A, B, C) > 0 \qquad (3.3)$$

$$V(\lambda A, B, C) = \lambda V(A, B, C) \text{ for every } \lambda > 0 \qquad (3.4)$$

$$\text{If } x \in \mathbb{R}^3, \text{ then } V(A + x, B, C) = V(A, B, C) \qquad (3.5)$$

$$\text{If } A_1 \subset A_2, \text{ then } V(A_1, B, C) \leq V(A_2, B, C) \qquad (3.6)$$

$$V(g(A), g(B), g(C)) = |\det g| \cdot V(A, B, C), \text{ for every affine transformation } g \qquad (3.7)$$

$$V(A, B, C) \text{ is continuous in } A, B, C \text{ with respect to the Hausdorff metric.} \qquad (3.8)$$

Note that the fundamental relation

$$V(g(A)) = |\det g| \cdot V(A), \qquad (3.9)$$

holding for every affine transformation g, is in agreement with (3.2) and (3.7).

If P is a convex polyhedron with facets F_i and corresponding outward unit normal vectors u_i, $i = 1, \ldots, k$, then [25]

$$V(A, P, P) = \frac{1}{3} \sum_{i=1}^{k} h(A, u_i) S(F_i), \tag{3.10}$$

where $S(F_i)$ is the area of the facet F_i of P and $h(A, u_i)$ is the value of the support function of A for the normal vector u_i.

In this paper the following inequalities play a central role, see Hadwiger [10] or Schneider [25] for a comprehensive discussion.

Brunn-Minkowski inequality. For two arbitrary compact sets $A, B \subset \mathbb{R}^3$ the following inequality holds:

$$V(A \oplus B)^{\frac{1}{3}} \geq V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}}, \tag{3.11}$$

with equality if and only if A and B are convex and homothetic modulo translation, i.e., $B \equiv \lambda A$ for some $\lambda > 0$.

Minkowski inequality. For convex sets $A, B \in \mathcal{C}(\mathbb{R}^3)$

$$V(A, A, B)^3 \geq V(A)^2 V(B), \tag{3.12}$$

and as before equality holds if and only if $B \equiv \lambda A$ for some $\lambda > 0$.

It follows from the generalized Brunn-Minkowski inequality [15, p. 304] for convex sets $A, B \in \mathcal{C}(\mathbb{R}^3)$ that

$$S(A \oplus B)^{\frac{1}{2}} \geq S(A)^{\frac{1}{2}} + S(B)^{\frac{1}{2}}, \tag{3.13}$$

where $S(A) = 3V(A, A, \boldsymbol{\mathcal{B}})$ is the surface area of the set A, and $\boldsymbol{\mathcal{B}}$ is the unit ball in $\mathbb{R}^3$. If $V(A)$ and $V(B)$ are positive, then equality is achieved if and only if $B \equiv \lambda A$ for some $\lambda > 0$.

Using the fact that for two arbitrary real numbers x, y one has $(x+y)^2 \geq 4xy$ with equality *iff* $x = y$, one derives from the Brunn-Minkowski inequality that

$$V(A \oplus B) \geq 8V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}, \tag{3.14}$$

with equality if and only if $A \equiv B$ and both sets are convex. Similarly, one gets from (3.13) for convex sets A and B

$$S(A \oplus B) \geq 4S(A)^{\frac{1}{2}} S(B)^{\frac{1}{2}}, \tag{3.15}$$

with equality if and only if $A \equiv B$.

Alexandrov-Fenchel inequality. For convex sets $A, B, C \subset \mathbb{R}^3$,

$$V(A, B, C)^2 \geq V(A, A, C) V(B, B, C) \tag{3.16}$$

with equality if A and B are homothetic. Taking for C the unit ball $\boldsymbol{\mathcal{B}}$, we get that

$$V(A, B, \boldsymbol{\mathcal{B}})^2 \geq V(A, A, \boldsymbol{\mathcal{B}}) V(B, B, \boldsymbol{\mathcal{B}}) = \frac{1}{9} S(A) S(B) \tag{3.17}$$

with equality if and only if A and B are homothetic. Note that

$$\begin{aligned} S(A \oplus B) &= 3V(A \oplus B, A \oplus B, \boldsymbol{\mathcal{B}}) \\ &= 3\Big(V(A, A, \boldsymbol{\mathcal{B}}) + 2V(A, B, \boldsymbol{\mathcal{B}}) + V(B, B, \boldsymbol{\mathcal{B}})\Big) \\ &= S(A) + 6V(A, B, \boldsymbol{\mathcal{B}}) + S(B). \end{aligned}$$

Therefore,

$$V(A, B, \boldsymbol{\mathcal{B}}) = \frac{1}{6}\Big(S(A \oplus B) - S(A) - S(B)\Big). \tag{3.18}$$

There exist several formulas, based on the support function and areas of facets, that can be used to calculate the volume of convex polyhedra (see, for example, [9, p. 324]). Let $u_1, u_2, \ldots, u_k$ be unit normal vectors of the facets $F_1, F_2, \ldots, F_k$ of a 3-dimensional convex polyhedron P, and let $h(P, u)$ be the value of the support function of P for the unit vector u. Then, from (3.10), the volume of P can be calculated as follows

$$V(P) = \frac{1}{3}\sum_{i=1}^{k} h(P, u_i) S(F_i). \tag{3.19}$$

Here $S(F_i)$ is the area of the facet F_i. Other formulas for convex polyhedra can be found in [2] and for non-convex ones in [1].

4 Similarity measures

This section adopts the approach developed in [12] to compare different shapes in such a way that this comparison is invariant under a given group H of transformations. For example, if one takes for H all rotations, then the comparison should return the same outcome for A and B as for A and $r(B)$, where r is some rotation. In this section, we consider subgroups of the group G of linear transformations on $\mathbb{R}^3$, as introduced in Section 2.3.

To compare different shapes the notion of *similarity measures* is introduced. Recall that $\mathcal{K}$ is the family of all nonempty compact subsets of $\mathbb{R}^3$.

Definition 4.1 Let H be a subgroup of G and $\mathcal{J} \subseteq \mathcal{K}$. A function $\sigma : \mathcal{J} \times \mathcal{J} \to [0, 1]$ is called an *H-invariant similarity measure* on $\mathcal{J}$ if

(1) $\sigma(A, B) = \sigma(B, A)$;
(2) $\sigma(A, B) = \sigma(A', B')$ if $A \equiv A'$ and $B \equiv B'$;
(3) $\sigma(A, B) = \sigma(h(A), B)$, $h \in H$;
(4) $\sigma(A, B) = 1 \iff B \equiv h(A)$ for some $h \in H$;
(5) σ is continuous in both arguments with respect to the Hausdorff metric.

When H contains only the identity mapping, then σ will be called a *similarity measure.*

Although not stated explicitly in the definition above, it is also required that $\mathcal{J}$ is invariant under H, that is, $h(A) \in \mathcal{J}$ if $A \in \mathcal{J}$ and $h \in H$.

Remark 4.2 If σ in Definition 4.1 satisfies the inequality

$$\sigma(A, C) \geq \sigma(A, B)\sigma(B, C),$$

then the function $d(A, B) = -\log(\sigma(A, B))$ constitutes a metric on $\mathcal{J}$ modulo translations and transformations $h \in H$. That is, d satisfies the triangle inequality.

The following result is needed.

Proposition 4.3 *If σ is a similarity measure on $\mathcal{J}$ and H is a $\mathcal{J}$-compact subgroup of G, then*

$$\sigma'(A,B) = \sup_{h\in H} \sigma(h(A),B)$$

defines an H-invariant similarity measure on $\mathcal{J}$.

Unfortunately, σ' is difficult to compute in many practical situations. Below, however, we consider several cases (with $\mathcal{J} = \mathcal{C}$) for which the computational complexity can be reduced if one limits oneself to convex polyhedra.

Let H be a given subgroup of G, and define

$$\sigma_1(A,B) = \sup_{h\in H} \frac{8|\det h|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A\oplus h(B))}, \tag{4.1}$$

$$\sigma_2(A,B) = \sup_{h\in H} \frac{|\det h|^{\frac{1}{3}} V(A)^{\frac{2}{3}} V(B)^{\frac{1}{3}}}{V(A,A,h(B))}, \tag{4.2}$$

$$\sigma_3(A,B) = \frac{1}{2}\sup_{h\in H}\Big(\frac{|\det h|^{\frac{1}{3}} V(A)^{\frac{2}{3}} V(B)^{\frac{1}{3}}}{V\big(A,A,h(B)\big)} + \frac{|\det h|^{\frac{2}{3}} V(A)^{\frac{1}{3}} V(B)^{\frac{2}{3}}}{V\big(A,h(B),h(B)\big)}\Big), \tag{4.3}$$

$$\sigma_4(A,B) = \sup_{h_1,h_2\in H} \frac{4S(h_1(A))^{\frac{1}{2}} S(h_2(B))^{\frac{1}{2}}}{S(h_1(A)\oplus h_2(B))}, \tag{4.4}$$

$$\sigma_5(A,B) = \sup_{h_1,h_2\in H} \frac{S(h_1(A))^{\frac{1}{2}} S(h_2(B))^{\frac{1}{2}}}{3V(h_1(A),h_2(B),\boldsymbol{\mathcal{B}})}. \tag{4.5}$$

Remark 4.4 It is easy to show that $\sigma_3(A,B) = \frac{1}{2}(\sigma_2(A,B)+\sigma_2(B,A))$. Instead, one can also define $\sigma_3'(A,B) = \min\{\sigma_2(A,B),\sigma_2(B,A)\}$.

In order to compute mixed volumes one can make use of surface areas via the following formula, which is a consequence of (3.18):

$$V(A,h(B),\boldsymbol{\mathcal{B}}) = \frac{1}{6}\Big(S(A\oplus h(B)) - S(A) - S(h(B))\Big).$$

The following proposition and its proof are very similar to Proposition 4.4 in [12].

Proposition 4.5 *If H is a $\mathcal{C}$-compact subgroup of G, then*

(a) *σ_1 is an H-invariant similarity measure on $\mathcal{C}$;*
(b) *σ_3 is an MH-invariant similarity measure on $\mathcal{C}$;*
(c) *σ_2 possesses only properties 2-5 of an MH-invariant similarity measure on $\mathcal{C}$;*
(d) *σ_4 is an H-invariant similarity measure on $\mathcal{C}$;*
(e) *σ_5 is an MH-invariant similarity measure on $\mathcal{C}$.*

Proof. We prove (a). The proof of (b), (c), (d) and (e) goes along the same lines. Conditions (1), (2) and (5) in Definition 4.1 are straightforward to verify. First let us prove (3). Using

(3.9) and (2.5) one gets

$$
\begin{aligned}
\sigma_1(h(A),B) &= \sup_{h'\in H} \frac{8|\det h'|^{\frac{1}{2}} V(h(A))^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(h(A)\oplus h'(B))} \\
&= \sup_{h'\in H} \frac{8|\det h'|^{\frac{1}{2}} |\det h|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V\Big(h(A\oplus h^{-1}h'(B))\Big)} \\
&= \sup_{h'\in H} \frac{8|\det h'|^{\frac{1}{2}} |\det h|^{-\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A\oplus h^{-1}h'(B))} \\
&= \sup_{h'\in H} \frac{8|\det h^{-1}h'|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A\oplus h^{-1}h'(B))} \qquad \text{(putting } h''=h^{-1}h'\text{)} \\
&= \sup_{h''\in H} \frac{8|\det h''|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A\oplus h''(B))} \\
&= \sigma_1(A,B).
\end{aligned}
$$

Finally we prove (4). It is easy to see that $\sigma_1(A,B)=1$ if $B\equiv h(A)$. To prove the converse, assume that $\sigma_1(A,B)=1$. Since H is $\mathcal{C}$-compact, one can conclude from Lemma 2.5 that there exists an $h\in H$ such that

$$\frac{8|\det h|^{\frac{1}{2}} V(A)^{\frac{1}{2}} V(B)^{\frac{1}{2}}}{V(A\oplus h(B))} = 1,$$

that is,

$$V(A\oplus h(B)) = 8V(A)^{\frac{1}{2}} V(h(B))^{\frac{1}{2}}.$$

In (3.14) we have seen that this implies that $A\equiv h(B)$. This concludes the proof. ∎

In the next section invariance under rotations and multiplications is investigated. Here we consider similarity measures which are invariant under the multiplication group.

Example 4.6 (Invariance under multiplications) Take $H=M$, the multiplication group. Since the determinant of the multiplication by λ equals λ^3 one has

$$\sigma_2(A,B) = \sup_{\lambda>0} \frac{\lambda V(A)^{\frac{2}{3}} V(B)^{\frac{1}{3}}}{V(A,A,\lambda B)} = \frac{V(A)^{\frac{2}{3}} V(B)^{\frac{1}{3}}}{V(A,A,B)}.$$

Using Remark 4.4 one may also find a simple expression for σ_3. For σ_5 we find

$$\sigma_5(A,B) = \sup_{\lambda,\mu>0} \frac{S(\lambda A)^{\frac{1}{2}} S(\mu B)^{\frac{1}{2}}}{3V(\lambda A,\mu B,\boldsymbol{\mathcal{B}})} = \frac{S(A)^{\frac{1}{2}} S(B)^{\frac{1}{2}}}{3V(A,B,\boldsymbol{\mathcal{B}})}.$$

The computation of $\sigma_1(A,B)$ is reduced to minimizing $V(\lambda^{-\frac{1}{2}}A\oplus\lambda^{\frac{1}{2}}B)$ for $\lambda>0$, a non-trivial task. Proposition 4.7 below presents a result that can be applied to overcome this difficulty.

Let us consider the similarity measure σ_4. One has

$$\begin{aligned}
\sigma_4(A,B) &= \sup_{\lambda_1,\lambda_2>0} \frac{4\lambda_1\lambda_2 S(A)^{\frac{1}{2}} S(B)^{\frac{1}{2}}}{S(\lambda_1 A \oplus \lambda_2 B)} \\
&= \sup_{\lambda_1,\lambda_2>0} \frac{4\lambda_1\lambda_2 S(A)^{\frac{1}{2}} S(B)^{\frac{1}{2}}}{3V(\lambda_1 A \oplus \lambda_2 B, \lambda_1 A \oplus \lambda_2 B, \boldsymbol{\mathcal{B}})} \\
&= \sup_{\lambda_1,\lambda_2>0} \frac{4\lambda_1\lambda_2 S(A)^{\frac{1}{2}} S(B)^{\frac{1}{2}}}{3\Big(\lambda_1^2 V(A,A,\boldsymbol{\mathcal{B}}) + 2\lambda_1\lambda_2 V(A,B,\boldsymbol{\mathcal{B}}) + \lambda_2^2 V(B,B,\boldsymbol{\mathcal{B}})\Big)} \\
&= \sup_{\lambda_1,\lambda_2>0} \frac{4\lambda_1\lambda_2 S(A)^{\frac{1}{2}} S(B)^{\frac{1}{2}}}{\lambda_1^2 S(A) + 6\lambda_1\lambda_2 V(A,B,\boldsymbol{\mathcal{B}}) + \lambda_2^2 S(B)} \\
&= 4\Big[\inf_{\mu>0}\{\frac{6V(A,B,\boldsymbol{\mathcal{B}})}{S(A)^{\frac{1}{2}}S(B)^{\frac{1}{2}}} + \mu\cdot\Big(\frac{S(B)}{S(A)}\Big)^{\frac{1}{2}} + \frac{1}{\mu}\cdot\Big(\frac{S(A)}{S(B)}\Big)^{\frac{1}{2}}\}\Big]^{-1},
\end{aligned}$$

where $\mu = \lambda_2/\lambda_1$. It is easy to see that the infimum is achieved at $\mu = (S(A)/S(B))^{\frac{1}{2}}$, whence

$$\sigma_4(A,B) = 2\Big[1 + \frac{3V(A,B,\boldsymbol{\mathcal{B}})}{S(A)^{\frac{1}{2}}S(B)^{\frac{1}{2}}}\Big]^{-1}.$$

In a number of cases it is possible to transform an H-invariant similarity measure into an MH-invariant similarity measure. Towards that goal the following normalization procedure can be used. Given $A \in \mathcal{K}$, define $A' = A/V(A)^{\frac{1}{3}}$. Thus A' has volume 1. Furthermore, $g' = |\det g|^{-1} g$ for $g \in G$; the normalized transform g' has determinant 1. It is obvious that

$$[g(A)]' = g'(A'), \text{ for } A \in \mathcal{K},\ g \in G.$$

The following result holds; the proof is rather straightforward.

Proposition 4.7 *Assume that $H \subseteq G$ and $\mathcal{J} \subseteq \mathcal{K}$ are such that*

$$\begin{aligned} h \in H &\Rightarrow h' \in H \\ A \in \mathcal{J} &\Rightarrow A' \in \mathcal{J}. \end{aligned}$$

If σ is an H-invariant similarity measure on $\mathcal{J}$, then σ' given by

$$\sigma'(A,B) = \sigma(A',B')$$

is an MH-invariant similarity measure on $\mathcal{J}$. Furthermore, $\sigma' = \sigma$ if and only if σ is MH-invariant.

We conclude this section with the following simple but useful result on reflections, which is similar to the Proposition 4.6 from [12]. Denote by e the reflection on $\mathbb{R}^3$ with respect to the origin.

Proposition 4.8 *Assume that $A \in \mathcal{J}$ implies $e(A) \in \mathcal{J}$. Let σ be a similarity measure on $\mathcal{J}$, and define*

$$\sigma'(A,B) = \max\{\sigma(A,B), \sigma(e(A),B)\}.$$

(a) *If σ is R-invariant, then σ' is an I-invariant similarity measure.*
(b) *If σ is G_+-invariant, then σ' is a G-invariant similarity measure.*

Proof. The proofs of (a) and (b) are almost identical. Here only (b) will be proved. The properties (1), (2), and (5) of Definition 4.1 are straightforward to verify. We prove (3) and (4).

(3): Let $g \in G$. There are two possibilities: $g \in G_+$ or $g \in G \setminus G_+$. Consider the second case. One can write $g = he$ with $h = ge$, and also $g = eh'$ with $h' = eg$; then $h, h' \in G_+$. Now

$$\begin{aligned} \sigma'(g(A), B) &= \max\{\sigma(g(A), B), \sigma(e(g(A)), B)\} \\ &= \max\{\sigma(h(e(A)), B), \sigma(h'(A), B)\} \\ &= \max\{\sigma(e(A), B), \sigma(A, B)\} \\ &= \sigma'(A, B), \end{aligned}$$

as was to be shown.

(4): Assume $\sigma'(A, B) = 1$, then either $\sigma(A, B) = 1$ or $\sigma(e(A), B) = 1$. In the first case one has $B \equiv g(A)$, for some $g \in G_+$, and in the second case $B \equiv g(e(A))$ for some $g \in G_+$. Therefore, $B \equiv g(A)$ for some $g \in G$.

∎

5 Rotations and multiplications

In this section we consider similarity measures on the space $\mathcal{P}$ of convex polyhedra which are S_+-invariant, i.e., invariant under rotations and multiplications. Towards this goal, the similarity measures will be used as defined in (4.1)–(4.5) with $H = S_+$ and $H = R$, respectively. In these expressions, the terms $V(P \oplus h(Q))$, $V(P, P, h(Q))$, $V(P, h(Q), h(Q))$ and $S(P \oplus h(Q))$ play an important role. Let the slope diagram representations of two convex polyhedra P and Q be given by $(\mathcal{V}(P), \mathcal{A}(P), \mathcal{W}(P))$ and $(\mathcal{V}(Q), \mathcal{A}(Q), \mathcal{W}(Q))$, where $\mathcal{V}(P)$ $=\{u_1, u_2, \ldots, u_{n_P}\}$ and $\mathcal{V}(Q)$=$\{v_1, v_2, \ldots, v_{n_Q}\}$ are the normal vectors to facets of polyhedra P and Q, respectively.

5.1 Representation and objective functionals

It is well known (see e.g. [21]) that every similitude transformation can be represented as a product of a homothetic transformation with prescribed center and an orthogonal transformation. Every orthogonal transformation in $\mathbb{R}^3$ with a positive determinant can be represented (up to translation) as a rotation about some axis.

Let ℓ be an axis passing through the coordinate origin and $r_{\ell,\alpha}$ be the rotation in $\mathbb{R}^3$ about ℓ by an angle α in a counter-clockwise direction. Let θ be the angle between ℓ and the z-axis, and ϕ the angle between the projection of ℓ on the xy-plane and the x-axis, see Fig. 4. The rotation $r_{\ell,\alpha}$ can be expressed as a product of 5 rotations:

$$r_{\ell,\alpha} = r_{z,\phi}\, r_{y,\theta}\, r_{z,\alpha}\, r_{y,-\theta}\, r_{z,-\phi}.$$

First the rotation axis ℓ is made to coincide with the z-axis through rotation about the z-axis by an angle $-\phi$, followed by rotation about the y-axis by an angle $-\theta$. Then the rotation by α is performed about the z-axis. Finally, the axis ℓ is rotated back to its original position.

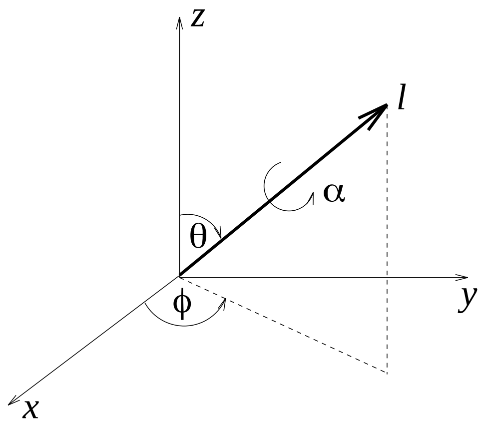


Figure 4: *Geometry of rotation by an angle α about an axis with spherical angles (θ, ϕ).*

The required matrices of these transformations are given by:

$$R_{z,\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{5.1}$$

Alternatively, the rotation $r_{\ell,\alpha}$ can be decomposed as a product of 3 rotations about the coordinate axes using Euler angles.

The slope diagram representation (SDR) of polyhedron P is assumed to have the same center as the SDR of Q by definition, that is, they are considered to be defined on the same unit sphere. Moreover the SDR of P is fixed and the SDR of Q can be rotated about any axis passing through the origin. It is easy to formulate the rotation of a polyhedron in terms of its SDR: $SDR(r(Q)) = r(SDR(Q))$, for every rotation r.

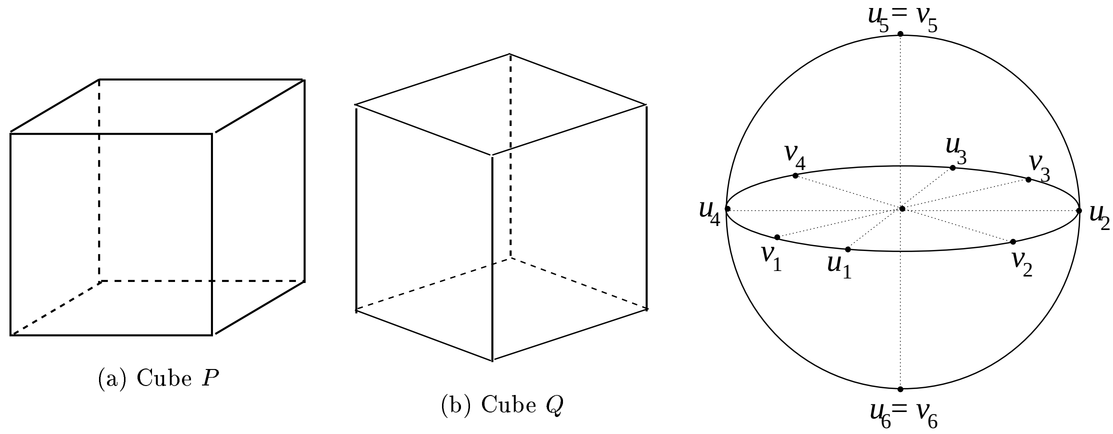
Given a fixed axis ℓ, (3.10) can be used to compute $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$ for $\alpha \in [0, 2\pi)$:

$$V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q)) = \frac{1}{3}\sum_{j=1}^{n_Q} h(P, r_{\ell,\alpha}(v_j))a(v_j). \tag{5.2}$$

The problem to be considered is the minimization of one of the functionals $V(P, P, r_{\ell,\alpha}(Q))$, $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$, and $V(P \oplus r_{\ell,\alpha}(Q))$. In the sequel we refer to these functionals as *objective functionals.*

5.2 Critical rotations

While rotating the slope diagram of polyhedron Q situations arise when spherical points of the rotated SDR of Q intersect spherical arcs or points of the SDR of P. Such relative configurations of Q w.r.t. P are *critical* in the sense that they may correspond to (local) minima of the objective functionals to be minimized.



(a) Cube P

(b) Cube Q

(c) SDR of P and Q

Figure 5: *Critical rotations of two cubes. In (c), spherical points u_1-u_6 belong to $SDR(P)$ and v_1-v_6 to $SDR(Q)$. Not all spherical arcs are indicated.*

First some definitions are needed distinguishing several situations. Instead of $r_{\ell,\alpha}$ we will also write (ℓ, α) to denote a rotation about the axis ℓ by an angle α.

Definition 5.1 Let ℓ be a fixed rotation axis. Polyhedron Q is called *ℓ-critical w.r.t. P* when there exists at least one spherical point v_j in the SDR of Q which is on the boundary of a spherical polygon in the SDR of P, i.e., for every $\epsilon > 0$ the points $r_{\ell,-\epsilon}(v_j)$ and $r_{\ell,+\epsilon}(v_j)$ belong to different spherical polygons in the SDR of polyhedron P.
Polyhedron Q is called *critical w.r.t. P* when Q is ℓ-critical w.r.t. P for at least one rotation axis ℓ.

Definition 5.2 Let ℓ be a fixed rotation axis. Polyhedra P and Q are called *mutually ℓ-critical* when Q is ℓ-critical w.r.t. P or P is ℓ-critical w.r.t. Q.

Notice that if Q is ℓ-critical w.r.t P (or vice versa), P and Q are mutually ℓ-critical.

Definition 5.3 Let ℓ be a fixed rotation axis. The *ℓ-critical angles of Q w.r.t. P for mixed volume* are the angles α'_i,

$$0 \leq \alpha'_1 < \alpha'_2 < \ldots < \alpha'_N < 2\pi$$

such that $Q' := r_{\ell,\alpha'_i}(Q)$ is ℓ-critical w.r.t. P. The rotation $h := r_{\ell,\alpha'_i}$ is called a *critical rotation of Q w.r.t. P for mixed volume.*

Let us emphasize here that the ℓ-critical angles of Q w.r.t. P for mixed volume are defined only by spherical points in the SDR of the rotating polyhedron Q and not by spherical points in the SDR of P. The ℓ-critical angles of Q w.r.t. P will in general be different from the ℓ-critical angles of P w.r.t. Q.

This motivates the following definition.

Definition 5.4 Let ℓ be a fixed rotation axis. The *ℓ-critical angles of Q w.r.t. P for volume* are the angles α_i^*,

$$0 \leq \alpha_1^* < \alpha_2^* < \ldots < \alpha_K^* < 2\pi, \tag{5.3}$$

such that P and $Q' := r_{\ell,\alpha_i^*}(Q)$ are mutually ℓ-critical.

Note that the angles α_i^* are obtained by merging into one ordered sequence the angles α_i' such that Q' is ℓ-critical w.r.t. P and the angles α_j'' such that $P' := r_{\ell,\alpha_j''}^{-1}(Q)$ is ℓ-critical w.r.t. Q.

Also, a classification of critical angles is needed. To this end, we introduce the index of criticality of two polyhedra.

Definition 5.5 The *index of criticality $n(Q,P)$ of Q w.r.t. P* is the number of spherical points v_j in the SDR of Q which are on the boundary of spherical polygons in the SDR of P.

Definition 5.6 Polyhedron Q is called *simply (doubly, multiply) critical w.r.t. P* when $n(Q,P)$ equals *one (two, more than two)*. Polyhedra P and Q are called *simply (doubly, multiply) mutually critical* when $n(P,Q) + n(Q,P)$ equals *one (two, more than two)*.

Definition 5.7 When P and Q are doubly mutually critical, and at least one spherical point of P coincides with a spherical point of Q, this critical configuration is called *point-double.*

Definition 5.8 Polyhedron Q is called *strongly critical w.r.t. P* if Q is multiply critical w.r.t. P, or doubly critical of type point-double. Polyhedra P and Q are called *strongly mutually critical* if P and Q are multiply mutually critical, or doubly critical of type point-double.

We will also say that (ℓ, α) is a simply (doubly, multiply, strongly) critical rotation of Q w.r.t. P for *mixed volume* if $Q' := r_{\ell,\alpha}(Q)$ is simply (doubly, multiply, strongly) critical w.r.t. P. Similarly, (ℓ, α) is called a simply (doubly, multiply, strongly) critical rotation of Q w.r.t. P for *volume* if $Q' := r_{\ell,\alpha}(Q)$ and P are simply (doubly, multiply, strongly) mutually critical.

Example 5.9 Take for P a cube, whose sides are parallel to the coordinate axes, and for Q a cube identical to P except for a rotation of $\pi/4$ w.r.t. the vertical axis, so that the spherical points of P and Q on the equator are distinct, cf. Fig. 5. Q is multiply critical w.r.t. P because all spherical points of Q are on arcs of P. Q is not z-critical w.r.t. P, where z is the vertical axis, because an infinitesimal rotation about this axis does not move points from one spherical region to another. However, the angle $\pi/4$ is z-critical, because after rotating Q by $\pi/4$ spherical points of Q hit spherical points of P; continuing the rotation, they move along arcs of P from one spherical region of P to another (remember that spherical arcs are included in spherical regions by definition, cf. Section 2.2).

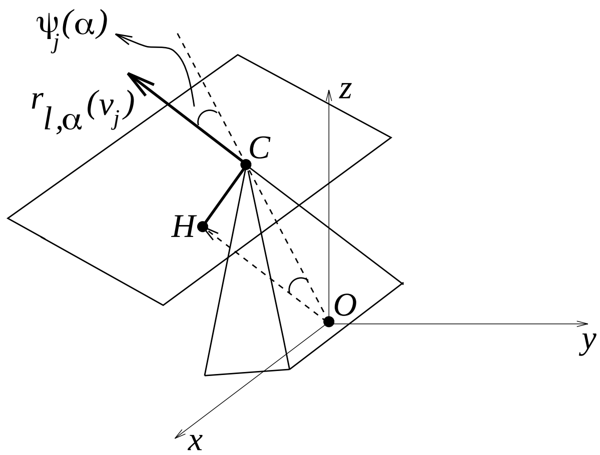


Figure 6: *The value $|OH|$ of the support function equals $|OC|\cos(\psi_j(\alpha))$ for the normal vector $r_{\ell,\alpha}(v_j)$.*

5.3 Minimization for fixed rotation axis

Let ℓ be a fixed rotation axis and $\alpha \in (\alpha'_k, \alpha'_{k+1})$ for some k, where $\{\alpha'_j\}$ are the ℓ-critical angles of Q w.r.t. P for mixed volume, cf. Definition 5.3. Then, for every spherical point v_j of Q, the value of the support function $h(P, r_{\ell,\alpha}(v_j))$ is defined by some vertex of polyhedron P, say by vertex C as in Fig. 6. (The support plane of P with normal $r_{\ell,\alpha}(v_j)$ may also hit P in an *edge*; in that case one takes for C a vertex adjacent to this edge.) Let $\psi_j(\alpha)$ be the angle between the vector $r_{\ell,\alpha}(v_j)$ (translated to the point C) and the vector OC. Then $h(P, r_{\ell,\alpha}(v_j)) = d_j \cos(\psi_j(\alpha))$, where $d_j = |OC|$. Thus

$$V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q)) = \frac{1}{3}\sum_{j=1}^{n_Q} d_j a(v_j)\cos(\psi_j(\alpha)).$$

If the origin is chosen inside polyhedron P, then $|\psi_j(\alpha)| < \frac{\pi}{2}$. Now $\cos(\psi_j(\alpha))$ is a concave function of α for every j, as follows from Lemma A.1 which is proven in Appendix A. Thus one gets that $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$ is a concave function of $\alpha \in (\alpha'_k, \alpha'_{k+1})$, since it is the sum of concave functions. Thus we arrive at the following result.

Proposition 5.10 *Given an axis of rotation ℓ, the mixed volume $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$ of the convex polyhedra P and Q is a function of α which is piecewise concave on $[0, 2\pi)$, i.e., concave on every interval $(\alpha'_k, \alpha'_{k+1})$, for $k = 1, 2, \ldots, N$ and $\alpha'_{N+1} = \alpha'_1$. Here $0 \le \alpha'_1 < \alpha'_2 < \ldots < \alpha'_N < 2\pi$ are the ℓ-critical angles of Q with respect to P for mixed volume.*

It is clear that the proposition is true for the mixed volume $V(P, P, r_{\ell,\alpha}(Q)) = V(Q, Q, r^{-1}_{\ell,\alpha}(P))$ as well, provided the ℓ-critical angles of P with respect to Q are used, i.e., polyhedron Q is considered to be fixed and polyhedron P is rotated about the axis ℓ in a clockwise direction.

Next consider the volume $V(P \oplus r_{\ell,\alpha}(Q))$. Now the ℓ-critical angles of polyhedron Q w.r.t. P for volume, as introduced in Definition 5.4, play a decisive role. From (3.1) and Proposition 5.10 one derives the following result.

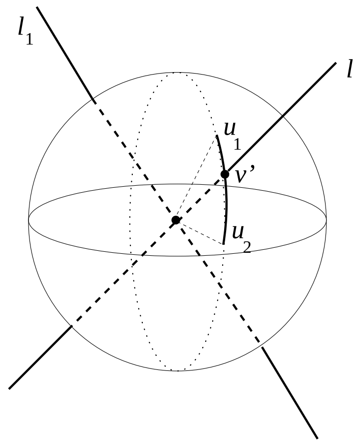


Figure 7: *Polyhedron $Q' = r_{\ell_1,\alpha_1}(Q)$ is ℓ_1-critical, but not ℓ'-critical, w.r.t. P, where the axis ℓ' passes through the point v' on the spherical arc (u_1, u_2).*

Proposition 5.11 *Given an axis of rotation ℓ, the volume $V(P \oplus r_{\ell,\alpha}(Q))$ of the convex polyhedra P and Q is a function of α which is piecewise concave on $[0, 2\pi)$, i.e., concave on every interval $(\alpha_k^*, \alpha_{k+1}^*)$, for $k = 1, 2, \ldots K$ and $\alpha_{K+1}^* = \alpha_1^*$. Here $0 \leq \alpha_1^* < \alpha_2^* < \ldots < \alpha_K^* < 2\pi$ are the ℓ-critical angles of Q w.r.t. P for volume.*

It follows from Propositions 5.10 and 5.11 that in order to minimize one of the functionals $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$, $V(P, P, r_{\ell,\alpha}(Q))$ and $V(P \oplus r_{\ell,\alpha}(Q))$ *for any fixed axis of rotation* ℓ it is enough to compute this functional only at a finite number of ℓ-critical angles.

5.4 Minimization for varying rotation axis

Since our interest is to find the minimum of objective functionals for all possible axes of rotation, we have to know which axes have to be checked. If for a fixed position of polyhedron $Q' = r_{\ell_1,\alpha_1}(Q)$ there exists an axis ℓ' such that Q' is not ℓ'-critical w.r.t. P, then the mixed volume $V(P, Q', Q')$ is not a minimum of the mixed volume functional $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$, because a smaller value of the functional can be found by rotating polyhedron Q' about the axis ℓ'. This property, which is true for other objective functionals as well, will be used to reduce the set of axes to be checked.

Whenever we speak of critical rotations of Q w.r.t. P below, we mean critical rotations of Q w.r.t. P for *mixed volume* or *volume*, respectively, depending on the objective functional under consideration.

5.4.1 Simply critical rotations

Lemma 5.12 *If (ℓ_1, α_1) is a simply critical rotation of Q w.r.t. P, then the objective functionals do not have a minimum at the relative configuration of polyhedra P and Q determined by (ℓ_1, α_1).*

Proof. Let a spherical point $v' = r_{\ell_1,\alpha_1}(v)$ in the SDR of $r_{\ell_1,\alpha_1}(Q)$ intersect an arc (u_1, u_2) in the SDR of P for a simple ℓ_1-critical angle α_1. Denote the polyhedron $r_{\ell_1,\alpha_1}(Q)$ by Q' and the axis which passes through the point v' by ℓ', cf. Fig. 7. Then rotation of Q' about the axis ℓ' allows us to find a smaller functional value, since Q' is not ℓ'-critical w.r.t. P. ∎

Therefore the conclusion is that only doubly or multiply critical rotations have to be checked.

5.4.2 Doubly critical rotations

For the case of doubly critical rotations, we next show that only the ones of type point-double may correspond to minima of the objective functionals.

Lemma 5.13 *If (ℓ_1, α_1) is a doubly critical rotation then the objective functionals may have a minimum at the relative configuration of polyhedra P and Q determined by (ℓ_1, α_1) only if the critical rotation is of type point-double.*

Proof. We only consider the case when two spherical points of polyhedron $r_{\ell_1,\alpha_1}(Q)$ intersect two spherical arcs of polyhedron P, as is appropriate for $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$. For other possible functionals the proof goes along the same lines. We show that this relative configuration does not correspond to a minimum of the objective functional.

Let two spherical points ${v_1}' = r_{\ell_1,\alpha_1}(v_1)$ and ${v_2}' = r_{\ell_1,\alpha_1}(v_2)$ in the SDR of polyhedron $Q' = r_{\ell_1,\alpha_1}(Q)$ intersect spherical arcs (u_1, u_2) and (u_3, u_4) in the SDR of P, respectively, see Fig. 8. Denote by s_1 and s_2 the lines through the origin which are orthogonal to the planes of the great circles containing spherical arcs (u_1, u_2) and (u_3, u_4), respectively. First assume $s_1 \neq s_2$. Consider now two planes through v_1', s_1 and v_2', s_2, respectively, intersecting the sphere in two great circles. These circles either intersect in two points or coincide. Let s be a point of their intersection and ℓ' be the axis through the coordinate origin and s. Rotating polyhedron Q' about the axis ℓ' allows us find a smaller value of the objective functional since Q' is not ℓ'-critical w.r.t. P. The trajectories of spherical points v_1' and v_2' under this rotation are small circles which, although touching the arcs (u_1, u_2) and (u_3, u_4), do not intersect them. If $s_1 = s_2$, then (u_1, u_2) and (u_3, u_4) are on the same great circle; taking $\ell' = s_1 = s_2$, the trajectories under rotation about ℓ' are on this great circle, so that also in this case Q' is not ℓ'-critical w.r.t. P. ∎

Remark 5.14 In the doubly critical case of type point-double, there are two further situations where objective functionals, say mixed volume, will not have a minimum. The first case occurs when the two spherical points v_1' and v_2' are antipodes on the sphere, because rotation about the axis through these two spherical points allows one to find a smaller value of the functionals, cf. the proof of Lemma 5.12. The second case requires that one of the points, say v_1', coincides with a spherical point of P and the other point v_2' is in the interior of a spherical arc (u_1, u_2) of P, with the additional condition that the axis ℓ' through the origin and the point v_1' lies in the plane through the origin and v_2' which is orthogonal to the plane of the great circle containing the spherical arc (u_1, u_2) on which v_2' is located. Looking back at the proof of Lemma 5.13 it is clear that $Q' = r_{\ell_1,\alpha_1}(Q)$ is not ℓ'-critical w.r.t. P in this case as well.

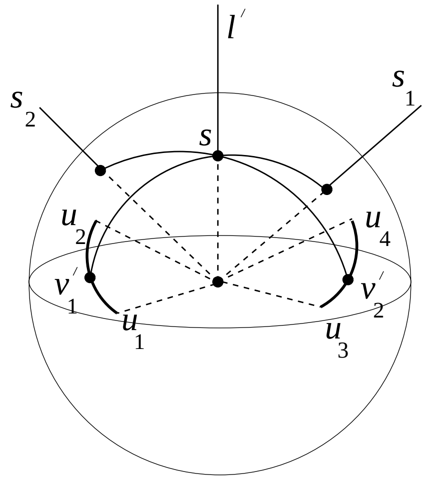


Figure 8: *The line s_1, c.q. s_2, is orthogonal to the plane of the great circle containing (u_1, u_2), c.q. (u_3, u_4). Axis ℓ' is the intersection of the planes through v_1', s_1 and v_2', s_2, respectively.*

5.4.3 Strongly critical rotations

The results so far imply that candidate minima of objective functionals only have to be searched among the *strongly critical* rotations of Definition 5.8 (note however the exceptions for doubly critical rotations in Remark 5.14; similar exceptions for the multiply critical case can easily be constructed).

Let us examine the problem of minimization of $V(P, r_{\ell,\alpha}(Q), r_{\ell,\alpha}(Q))$. Let u_i, v_i, $i = 1, 2, \ldots$ be spherical points in the SDR of polyhedra P and Q, respectively. To find all strongly critical rotations the following procedure can be applied.

Let u' be a spherical point belonging to the boundary of a spherical polygon in the SDR of P, i.e., u' is either a spherical point corresponding to a facet of P or an internal point of a spherical arc in the SDR of P. Let ℓ be the axis through the origin and u'. There exists a rotation h' of polyhedron Q, such that $u' = h'(v_i)$ for a chosen spherical point v_i of the SDR of Q. Now one can find all ℓ-critical angles of $h'(Q)$ w.r.t. P. If u' is a spherical point corresponding to a facet of P then every ℓ-critical angle will be at least point-double. So by performing the above procedure for all spherical points of Q and P we find all critical rotations where at least one spherical point of Q coincides with a spherical point of P.

What remains is to find those multiply critical rotations where three spherical points intersect the interior of three spherical arcs. That corresponds to the case that the point u' defined above is an internal point of an arc, and we have to find ℓ-critical rotations with two spherical points different from v_i intersecting spherical arcs of P. This also has to be performed for all points u' from the boundary of spherical polygons in the SDR of P, and all spherical points v_i in the SDR of Q.

Remark 5.15 Note that if the objective functional is the volume $V(P \oplus r_{\ell,\alpha}(Q))$, then the strongly critical rotations of Q w.r.t. P for volume have to be considered, which implies that the set of critical rotations to be checked is larger than when minimizing mixed volume.

5.5 Finiteness of the set of critical rotations

There remains the question of how many critical rotations exist: is their number finite or infinite? The number of critical rotations of type point-double is certainly finite: for the number of axes to be checked and the number of critical angles per axis ℓ a bound can be given depending on the number of vertices of P and Q.

For the multiply critical rotations the answer depends on the question in how many ways a given triple of spherical points in the SDR of Q can be made to coincide with three edges of the SDR of P by rotation: that is, how many solutions exist for the system of conditions

$$r_{\ell,\alpha}(v_1) \in (u_1, u_2),\ r_{\ell,\alpha}(v_2) \in (u_3, u_4),\ r_{\ell,\alpha}(v_3) \in (u_5, u_6), \tag{5.4}$$

for given spherical points v_1, v_2, v_3 in the SDR of Q and spherical arcs (u_1, u_2), (u_3, u_4), (u_5, u_6) in the SDR of P. It is clear that the system of conditions (5.4) may have no solutions at all, only one solution, or two solutions (it is easy to find an example).

The question can be formulated in an equivalent way as follows. Suppose that a spherical triangle $\triangle\, v_1\, v_2\, v_3$ is inscribed in a spherical triangle $\triangle\, u_1\, u_2\, u_3$. The question is how many other positions of the triangle $\triangle\, v_1\, v_2\, v_3$ (denoted by $\triangle\, v_1'\, v_2'\, v_3'$, see Fig. 9) exist, such that it is inscribed into $\triangle\, u_1\, u_2\, u_3$ and $v_1' \in (u_1, u_2),\ v_2' \in (u_2, u_3),\ v_2' \in (u_3, u_2)$.

For the case of planar triangles there exists no more than one other position [4]. It is natural to suppose that the same should be true for the case of spherical triangles, although we have no proof of this at the moment. However, what we can show is that, given $\triangle\, u_1\, u_2\, u_3$ and $\triangle\, v_1\, v_2\, v_3$, there is only a finite number of axes of rotations through the center of the sphere, carrying $\triangle\, v_1\, v_2\, v_3$ to another inscribed triangle $\triangle\, v_1'\, v_2'\, v_3'$ of $\triangle\, u_1\, u_2\, u_3$, cf. Theorem B.1 in Appendix B. Clearly, when $\triangle\, u_1\, u_2\, u_3$, $\triangle\, v_1\, v_2\, v_3$ and a rotation axis ℓ are given, $\triangle\, v_1'\, v_2'\, v_3'$ is uniquely determined. Therefore the number of triangles, obtained from $\triangle\, v_1\, v_2\, v_3$ by rotation and inscribed in $\triangle\, u_1\, u_2\, u_3$, is finite as well.

Hence, the following result has been established.

Theorem 5.16 *The number of strongly critical rotations, to be checked in computing the minimum of the objective functionals, is finite.*

5.6 Similarity measures

Suppose one wants to compute σ_3 for $H = S_+$, the group comprising all rotations and multiplications. One easily derives that

$$\sigma_3(P,Q) = \frac{1}{2} \sup_{h \in R^*} \Big(\frac{V(P)^{\frac{2}{3}} V(Q)^{\frac{1}{3}}}{V\big(P, P, h(Q)\big)} + \frac{V(P)^{\frac{1}{3}} V(Q)^{\frac{2}{3}}}{V\big(P, h(Q), h(Q)\big)} \Big),$$

where R^* are the strongly critical rotations. A similar observation applies to σ_5. For σ_4 one gets

$$\sigma_4(P,Q) \quad = \sup_{\lambda_1,\lambda_2>0,\ h_1,h_2 \in R} \frac{4\lambda_1\lambda_2 S(P)^{\frac{1}{2}} S(Q)^{\frac{1}{2}}}{S(\lambda_1 h_1(P) \oplus \lambda_2 h_2(Q))} \quad \text{(putting } \lambda = \lambda_2/\lambda_1,\ h = h_1^{-1} h_2)$$

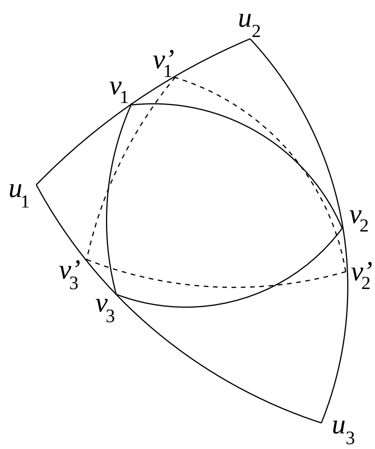


Figure 9: *Different positions of the inscribed spherical triangle.*

$$
\begin{aligned}
&= \sup_{\lambda>0,\ h\in R} \frac{4\lambda S(P)^{\frac{1}{2}} S(Q)^{\frac{1}{2}}}{S(P\oplus \lambda h(Q))} \\
&= \sup_{\lambda>0,\ h\in R} \frac{4\lambda S(P)^{\frac{1}{2}} S(Q)^{\frac{1}{2}}}{3V(P\oplus \lambda h(Q), P\oplus \lambda h(Q), \boldsymbol{\mathcal{B}})} \\
&= \sup_{\lambda>0,\ h\in R} \frac{4\lambda S(P)^{\frac{1}{2}} S(Q)^{\frac{1}{2}}}{3\Big(V(P,P,\boldsymbol{\mathcal{B}}) + 2\lambda V(P,h(Q),\boldsymbol{\mathcal{B}}) + \lambda^2 V(h(Q),h(Q),\boldsymbol{\mathcal{B}})\Big)} \\
&= \sup_{\lambda>0,\ h\in R} \frac{4\lambda S(P)^{\frac{1}{2}} S(Q)^{\frac{1}{2}}}{S(P) + 6\lambda V(P,h(Q),\boldsymbol{\mathcal{B}}) + \lambda^2 S(Q)} \\
&= 4\Big[\inf_{h\in R} \frac{6V(P,h(Q),\boldsymbol{\mathcal{B}})}{S(P)^{\frac{1}{2}} S(Q)^{\frac{1}{2}}} + \inf_{\lambda>0}\{\lambda\cdot\Big(\frac{S(Q)}{S(P)}\Big)^{\frac{1}{2}} + \frac{1}{\lambda}\cdot\Big(\frac{S(P)}{S(Q)}\Big)^{\frac{1}{2}}\}\Big]^{-1}.
\end{aligned}
$$

Since $V(P,h(Q),\boldsymbol{\mathcal{B}}) = \frac{1}{6}\Big(S(P\oplus h(Q)) - S(P) - S(Q)\Big)$, the first expression in σ_4 achieves a minimum for $h \in R^*$, where R^* is the set of *critical rotations for surface area* (cf. the next section), and the second expression for $\lambda = (S(P)/S(Q))^{\frac{1}{2}}$. Thus,

$$
\sigma_4(A,B) = 2\Big[1 + \inf_{h\in R^*} \frac{3V(P,h(Q),\boldsymbol{\mathcal{B}})}{S(P)^{\frac{1}{2}} S(Q)^{\frac{1}{2}}}\Big]^{-1}.
$$

6 Discussion

In this paper, we have discussed similarity measures for convex polyhedra based on Minkowski addition and the Brunn-Minkowski inequality, using the slope diagram representation of convex polyhedra. All measures considered are invariant under translations; furthermore, they may also be invariant under rotations, multiplications, reflections, or the class of all affine transformations. For the case of rotation invariance, we proved that to obtain the measures bases on (mixed) volumes it is sufficient to compute objective functionals only for a finite number of critical rotations.

Let us conclude by some comments on possible future work. A first remark concerns the similarity measures defined in Section 4 depending on surface area $S(P \oplus r_{\ell,\alpha}(Q))$. For this case one has to introduce additional critical angles as follows. Let u_i and v_j be unit vectors orthogonal to great circles containing spherical arcs of polyhedra P and Q. Given an axis of rotation ℓ, find for all u_i and v_j all angles α_k such that ℓ, u_i and $r_{\ell,\alpha_k}(v_j)$ are coplanar and the spherical arcs corresponding to u_i and $r_{\ell,\alpha_k}(v_j)$ in the SDR of polyhedra P and $r_{\ell,\alpha_k}(Q)$ intersect. These angles have to be merged with the critical angles (5.3) into one ordered sequence, called the *ℓ-critical angles of Q w.r.t. P for surface area*:

$$0 \le \alpha_1'' < \alpha_2'' < \ldots < \alpha_L'' < 2\pi. \tag{6.1}$$

Now, just as for the case of (mixed) volume, one may prove that the surface area is a concave function of α at every interval $(\alpha_k'', \alpha_{k+1}'')$. Also for this case a classification of critical rotations has to be made.

Another question which is the subject of current research is to develop (numerical) methods for the efficient computation of all critical rotations.

Acknowledgements

A. Tuzikov was supported by the Netherlands Organization for Scientific Research (NWO) through a visitor's grant. He is grateful to the Centre for Mathematics and Computer Science (CWI, Amsterdam) and the Department of Mathematics and Computing Science (University of Groningen) for their hospitality.

References

[1] Allgower, E. L., and Schmidt, P. H. Computing volumes of polyhedra. *Mathematics of Computation 46* (1986), 171–174.

[2] Betke, U. Mixed volumes of polytopes. *Arch. Math. 58* (1992), 388–391.

[3] Brou, P. Using the Gaussian image to find the orientation of objects. *International Journal of Robotics Research 3*, 4 (1984), 89–125.

[4] Coolidge, J. L. *A Treatise on the Circle and the Sphere.* Clarendon press, Oxford, 1916.

[5] Gardner, R. *Geometric Tomography.* Cambridge University Press, 1995.

[6] Ghosh, P. K. A unified computational framework for Minkowski operations. *Computers and Graphics 17* (1993), 357–378.

[7] Ghosh, P. K., and Haralick, R. M. Mathematical morphological operations of boundary-represented geometric objects. *Journal of Mathematical Imaging and Vision 6*, 2/3 (1996), 199–222.

[8] Gritzmann, P., and Hufnagel, A. A polynomial time algorithm for Minkowski reconstruction. In *Proc. 11th Ann. ACM Sympos. Comput. Geom.* (1995), pp. 1–9.

[9] Grünbaum, B. *Convex Polytopes.* Interscience Publishers, 1967.

[10] Hadwiger, H. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie.* Springer, Berlin, 1957.

[11] Hebert, M., Ikeuchi, K., and Delingette, H. A spherical representation for recognition of free-form surfaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence 17*, 7 (1995), 681–690.

[12] Heijmans, H. J. A. M., and Tuzikov, A. Similarity and symmetry measures for convex sets based on Minkowski addition. Research report BS-R9610, CWI, 1996.

[13] Horn, B. K. P. *Robot Vision.* MIT Press, Cambridge, 1986.

[14] Kang, S. B., and Ikeuchi, K. The complex EGI: a new representation for 3-D pose determination. *IEEE Transactions on Pattern Analysis and Machine Intelligence 15*, 7 (July 1993), 707–721.

[15] Leichtweiss, K. *Konvexe Mengen.* VEB Deutscher Verlag der Wissenschaften, Berlin, 1980.

[16] Li, Y., and Woodham, R. J. Orientation-based representation of 3D shape. In *Proc. IEEE Computer Society Conf. on Computer Vision and Pattern Recognition* (Seattle, Washington, June 21–23 1994), pp. 182–187.

[17] Little, J. J. An iterative method for reconstructing convex polyhedra from extended Gaussian images. In *Proc. of the National Conf. on Artificial Intelligence* (Washington, D.C., August 22–26 1983), pp. 247–250.

[18] Little, J. J. Determining object attitude from extended Gaussian images. In *Proc. of the Ninth Int. Joint Conf. on Artificial Intel.* (California, August 18–23 1985), vol. 2, pp. 960–963.

[19] Lyusternik, L. A. *Convex Figures and Polyhedra.* Dover Publications, New York, 1963.

[20] Minkowski, H. Volumen and Oberfläche. *Math. Ann. 57* (1903), 447–495.

[21] Modenov, P. S., and Parkhomenko, A. S. *Geometric Transformations. Volume 1: Euclidean and Affine Transformations.* Academic Press, New York, 1965.

[22] Moni, S. A closed-form solution for the reconstruction of a convex polyhedron from its extended Gaussian image. In *Int. Conf. on Pattern Recognition* (1990), pp. 223–226.

[23] Nalwa, V. S. Representing oriented piecewise C^2 surfaces. In *Proc. of the Second Int. Conf. on Computer Vision* (Florida, Dec. 5–8 1988), pp. 40–51.

[24] Sangwine-Yager, J. R. Mixed volumes. In *Handbook of Convex Geometry. Vol. A, B.*, P. M. Gruber and J. M. Wills, Eds. North-Holland, Amsterdam, 1993, pp. 43–71.

[25] SCHNEIDER, R. *Convex Bodies. The Brunn-Minkowski Theory.* Cambridge University Press, Cambridge, 1993.

[26] SNAPPER, E., AND TROYER, R. J. *Metric Affine Geometry.* Dover Publications, Inc, New York, 1989.

A Appendix: Concavity property

Consider two unit vectors

$$\begin{aligned} a &= (a_1, a_2, a_3) = \Big(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\Big) \\ b &= (b_1, b_2, b_3) = \Big(\sin\theta_1, 0, \cos\theta_1\Big) \end{aligned}$$

in $\mathbb{R}^3$, with $0 \leq \theta \leq \pi$, $0 \leq \theta_1 \leq \pi$, $0 \leq \phi \leq 2\pi$. Here the projection of vector b on the xy-plane is directed along the x-axis. Denote by $b(\alpha)$ the rotation of vector b about the z-axis by an angle α, $0 \leq \alpha \leq \alpha_1$, i.e., $b(\alpha) = r_{z,\alpha}(b)$. Here $b(0) = b$. Denote by $\psi(\alpha)$ the angle between vectors a and $b(\alpha)$; see Fig. 10.

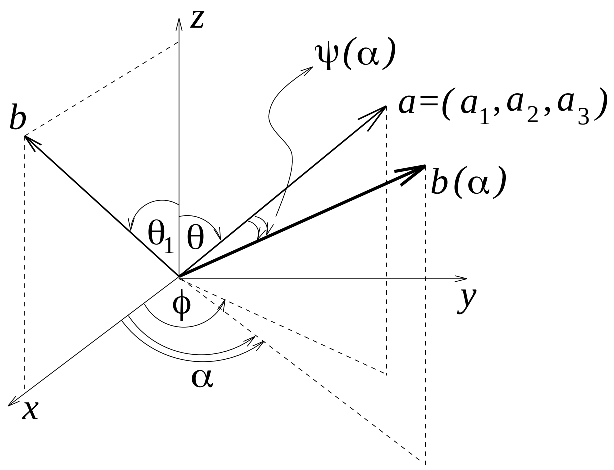


Figure 10: *Definition of the angle $\psi(\alpha)$ between the vector $b(\alpha) = r_{z,\alpha}(b)$ and an axis a with spherical angles (θ, ϕ).*

Lemma A.1 *Suppose that $|\psi(\alpha)| < \frac{\pi}{2}$ for all α, $0 \leq \alpha \leq \alpha_1$. Then $\cos(\psi(\alpha))$ is a concave function of $\alpha \in [0, \alpha_1]$.*

Proof. The vector $b(\alpha)$ equals $r_{z,\alpha}(b)$, i.e.,

$$b(\alpha) = \Big(\sin\theta_1\cos\alpha, \sin\theta_1\sin\alpha, \cos\theta_1\Big).$$

So

$$\cos(\psi(\alpha)) = \langle a, b(\alpha)\rangle$$

$$
\begin{aligned}
&= \sin\theta\sin\theta_1\Big(\cos\phi\cos\alpha+\sin\phi\sin\alpha\Big)+\cos\theta\cos\theta_1\\
&= \sin\theta\sin\theta_1\cos(\phi-\alpha)+\cos\theta\cos\theta_1.
\end{aligned}
$$

Now, the angle between the projection of two vectors is not larger than the angle between the original vectors, so $|\phi-\alpha| \le |\psi(\alpha)| < \frac{\pi}{2}$. Since also $\sin\theta\sin\theta_1 \ge 0$, we get that $\cos(\psi(\alpha))$ is a concave function of α. ∎

B Appendix: Inscribed spherical triangles

Consider the following problem. On a sphere, a triangle with vertices A, B, C is given. In this triangle are inscribed two spherical triangles with vertices D, E, F and D_1, E_1, F_1, respectively, which can be transformed into one another by rotation about an axis through the center of the sphere. The question is in how many ways this can be achieved.

We prove the following theorem.

Theorem B.1 *Let two spherical triangles $\triangle\, D\, E\, F$ and $\triangle\, D_1\, E_1\, F_1$ be inscribed in a spherical triangle $\triangle\, A\, B\, C$, such that D_1 and D are on the arc AC, E and E_1 are on the arc AB, F and F_1 are on the arc BC. Let $\triangle\, D\, E\, F$ be fixed and assume $\triangle\, D_1\, E_1\, F_1$ is the result of a 3D rotation of $\triangle\, D\, E\, F$ about an axis through the center of the sphere. Then the number of possible orientations of the rotation axis is finite.*

Proof. In the proof we make use of the following result for planar triangles [4, Ch. 1, §6]. When two planar triangles $\triangle\, D'\, E'\, F'$ and $\triangle\, D_1'\, E_1'\, F_1'$ are inscribed in a triangle $\triangle\, A'\, B'\, C'$, such that D_1' and D' are on the side $A'C'$, E' and E_1' are on the side $A'B'$, F' and F_1' are on the side $B'C'$, with $\triangle\, D_1'\, E_1'\, F_1'$ resulting from a 2D rotation of $\triangle\, D'\, E'\, F'$, then the center of rotation O must be the similarity point of $\triangle\, D'\, E'\, F'$ with respect to $\triangle\, A'\, B'\, C'$. That is, O is the intersection of the circumscribed circles of the triangles $\triangle\, A'\, D'\, E'$, $\triangle\, C'\, D'\, F'$ and $\triangle\, B'\, E'\, F'$. The limiting case when the circles are tangent to one another corresponds to the situation that $\triangle\, D'\, E'\, F'$ and $\triangle\, D_1'\, E_1'\, F_1'$ are identical.

Now consider the case of spherical triangles. If A, B, C are on a great circle, then all involved triangles are in the same plane, so that the result for planar triangles applies. Therefore assume that A, B, C are not on a great circle, meaning that A, B, C are all on one side of some equatorial plane of the sphere.

Let $\mathbf{n}$ be the axis of the 3D rotation carrying $\triangle\, D\, E\, F$ to $\triangle\, D_1\, E_1\, F_1$. Let O be a point on this axis, not equal to the center T of the sphere. Consider the plane V through O orthogonal to the axis $\mathbf{n}$. Now from the center T of the sphere carry out a central projection of the spherical triangles on the plane V, resulting in planar triangles $\triangle\, A'\, B'\, C'$, $\triangle\, D'\, E'\, F'$ and $\triangle\, D_1'\, E_1'\, F_1'$. Then, since V is orthogonal to the axis $\mathbf{n}$, $\triangle\, D_1'\, E_1'\, F_1'$ is the result of a rotation of $\triangle\, D'\, E'\, F'$ around the point O. Hence, from the case of planar triangles we know that O is the intersection of the circumscribed circles of the triangles $\triangle\, A'\, D'\, E'$, $\triangle\, C'\, D'\, F'$ and $\triangle\, B'\, E'\, F'$, cf. Fig. 11. (We may assume that $\triangle\, D'\, E'\, F'$ and $\triangle\, D_1'\, E_1'\, F_1'$ are distinct.)

So we know that the points A, D, E are on a cone $\mathcal{K}_{ADE}$ whose axis $\mathbf{n}$ is orthogonal to the plane of its base circle. Similarly, the sets C, D, F and B, E, F are on cones $\mathcal{K}_{CDF}$ and $\mathcal{K}_{BEF}$, respectively, with the same axis $\mathbf{n}$, orthogonal to the planes of the base circles of $\mathcal{K}_{CDF}$ and $\mathcal{K}_{BEF}$. The question now is in how many ways the axis $\mathbf{n}$ can be chosen.

First we consider a single cone $\mathcal{K}_{ABC}$ through three points A, B, C (not on the same great circle) on a sphere with radius 1. The axis of the cone is defined by a unit vector $\mathbf{n}$, and the ray through the center of the base circle by $\mathbf{m}$, satisfying the orthogonality relation $\langle \mathbf{n}, \mathbf{m} \rangle = 1$, see Fig. 12. If $\mathbf{r}$ is an arbitrary point on the cone, it has to satisfy the equation

$$\|\mathbf{r} - \langle \mathbf{r}, \mathbf{n} \rangle \mathbf{m}\|^2 = \|\langle \mathbf{r}, \mathbf{n} \rangle \mathbf{m} - \langle \mathbf{r}, \mathbf{n} \rangle \mathbf{n}\|^2 ,$$

which after some simplification reduces to

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{n} \rangle \langle \mathbf{r}, \mathbf{m} \rangle + (\langle \mathbf{r}, \mathbf{n} \rangle)^2 = 0 \tag{B.1}$$

This equation is subject to the conditions

$$\langle \mathbf{n}, \mathbf{m} \rangle = 1, \qquad \langle \mathbf{n}, \mathbf{n} \rangle = 1. \tag{B.2}$$

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the three linearly independent unit vectors corresponding to the three points A, B, C on the sphere. Then these satisfy (B.1), yielding three equations linear in $\mathbf{m}$. In matrix form this system of equations reads:

$$\begin{pmatrix} \langle \mathbf{n}, \mathbf{a} \rangle \mathbf{a}^T \\ \langle \mathbf{n}, \mathbf{b} \rangle \mathbf{b}^T \\ \langle \mathbf{n}, \mathbf{c} \rangle \mathbf{c}^T \end{pmatrix} \mathbf{m} = \frac{1}{2} \begin{pmatrix} 1 + (\langle \mathbf{n}, \mathbf{a} \rangle)^2 \\ 1 + (\langle \mathbf{n}, \mathbf{b} \rangle)^2 \\ 1 + (\langle \mathbf{n}, \mathbf{c} \rangle)^2 \end{pmatrix}$$

The solution for $\mathbf{m}$ is:

$$\mathbf{m} = \frac{1}{2\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle} \left\{ \mathbf{b}\times\mathbf{c} \, \frac{1 + (\langle \mathbf{n}, \mathbf{a} \rangle)^2}{\langle \mathbf{n}, \mathbf{a} \rangle} + \mathbf{c}\times\mathbf{a} \, \frac{1 + (\langle \mathbf{n}, \mathbf{b} \rangle)^2}{\langle \mathbf{n}, \mathbf{b} \rangle} + \mathbf{a}\times\mathbf{b} \, \frac{1 + (\langle \mathbf{n}, \mathbf{c} \rangle)^2}{\langle \mathbf{n}, \mathbf{c} \rangle} \right\},$$

assuming that $\langle \mathbf{n}, \mathbf{a} \rangle, \langle \mathbf{n}, \mathbf{b} \rangle, \langle \mathbf{n}, \mathbf{c} \rangle$ are not zero. Imposing the condition $\langle \mathbf{n}, \mathbf{m} \rangle = 1$ yields

$$1 = \frac{1}{2\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle} \left\{ \langle \mathbf{n}, \mathbf{b}\times\mathbf{c} \rangle \, \frac{1 + (\langle \mathbf{n}, \mathbf{a} \rangle)^2}{\langle \mathbf{n}, \mathbf{a} \rangle} + \langle \mathbf{n}, \mathbf{c}\times\mathbf{a} \rangle \, \frac{1 + (\langle \mathbf{n}, \mathbf{b} \rangle)^2}{\langle \mathbf{n}, \mathbf{b} \rangle} + \langle \mathbf{n}, \mathbf{a}\times\mathbf{b} \rangle \, \frac{1 + (\langle \mathbf{n}, \mathbf{c} \rangle)^2}{\langle \mathbf{n}, \mathbf{c} \rangle} \right\} \tag{B.3}$$

By construction, the system of three vectors $\mathbf{v}_1 = \mathbf{a}$, $\mathbf{v}_2 = \mathbf{b}$, $\mathbf{v}_3 = \mathbf{c}$ is biorthogonal to the system $\mathbf{w}_1 = (\mathbf{b}\times\mathbf{c})/\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle$, $\mathbf{w}_2 = (\mathbf{c}\times\mathbf{a})/\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle$, $\mathbf{w}_3 = (\mathbf{a}\times\mathbf{b})/\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle$: $\langle \mathbf{v}_i, \mathbf{w}_j \rangle = \delta_{i,j}$. Therefore any vector $\mathbf{r}$ has the expansion

$$\mathbf{r} = \frac{1}{\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle} \{ (\mathbf{b}\times\mathbf{c}) \langle \mathbf{r}, \mathbf{a} \rangle + (\mathbf{c}\times\mathbf{a}) \langle \mathbf{r}, \mathbf{b} \rangle + (\mathbf{a}\times\mathbf{b}) \langle \mathbf{r}, \mathbf{c} \rangle \}$$

Applying this formula to the vector $\mathbf{n}$, and using the normalization condition $\langle \mathbf{n}, \mathbf{n} \rangle = 1$, one finds the identity

$$1 = \frac{1}{\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle} \{ \langle \mathbf{n}, \mathbf{b}\times\mathbf{c} \rangle \langle \mathbf{n}, \mathbf{a} \rangle + \langle \mathbf{n}, \mathbf{c}\times\mathbf{a} \rangle \langle \mathbf{n}, \mathbf{b} \rangle + \langle \mathbf{n}, \mathbf{a}\times\mathbf{b} \rangle \langle \mathbf{n}, \mathbf{c} \rangle \} \tag{B.4}$$

Combining (B.3) with (B.4) gives

$$1 = \frac{1}{\langle \mathbf{a}, \mathbf{b}\times\mathbf{c} \rangle} \left\{ \frac{\langle \mathbf{n}, \mathbf{b}\times\mathbf{c} \rangle}{\langle \mathbf{n}, \mathbf{a} \rangle} + \frac{\langle \mathbf{n}, \mathbf{c}\times\mathbf{a} \rangle}{\langle \mathbf{n}, \mathbf{b} \rangle} + \frac{\langle \mathbf{n}, \mathbf{a}\times\mathbf{b} \rangle}{\langle \mathbf{n}, \mathbf{c} \rangle} \right\},$$

or,

$$\langle \mathbf{n},\mathbf{a}\rangle\langle \mathbf{n},\mathbf{b}\rangle\langle \mathbf{n},\mathbf{c}\rangle = \frac{\langle \mathbf{n},\mathbf{b}\times\mathbf{c}\rangle\langle \mathbf{n},\mathbf{b}\rangle\langle \mathbf{n},\mathbf{c}\rangle + \langle \mathbf{n},\mathbf{c}\times\mathbf{a}\rangle\langle \mathbf{n},\mathbf{c}\rangle\langle \mathbf{n},\mathbf{a}\rangle + \langle \mathbf{n},\mathbf{a}\times\mathbf{b}\rangle\langle \mathbf{n},\mathbf{a}\rangle\langle \mathbf{n},\mathbf{b}\rangle}{\langle \mathbf{a},\mathbf{b}\times\mathbf{c}\rangle}. \tag{B.5}$$

Note that if $\mathbf{n}$ is a solution of (B.5), then $\lambda\mathbf{n}$ is also a solution for any λ. Therefore Eq. B.5 represents a cubic cone. The cross section of this cone with any plane not through the origin will be a polynomial curve of degree three.

Returning now to the original problem, we have to find three cones $\mathcal{K}_{ADE}$, $\mathcal{K}_{CDF}$ and $\mathcal{K}_{BEF}$ with a common axis $\mathbf{n}$ orthogonal to the planes of their base circles. The axis has to satisfy three equations of the form (B.5). For completeness we give them here explicitly:

$$\begin{aligned}
\langle \mathbf{n},\mathbf{a}\rangle\langle \mathbf{n},\mathbf{d}\rangle\langle \mathbf{n},\mathbf{e}\rangle &= \frac{\langle \mathbf{n},\mathbf{d}\times\mathbf{e}\rangle\langle \mathbf{n},\mathbf{d}\rangle\langle \mathbf{n},\mathbf{e}\rangle + \langle \mathbf{n},\mathbf{e}\times\mathbf{a}\rangle\langle \mathbf{n},\mathbf{e}\rangle\langle \mathbf{n},\mathbf{a}\rangle + \langle \mathbf{n},\mathbf{a}\times\mathbf{d}\rangle\langle \mathbf{n},\mathbf{a}\rangle\langle \mathbf{n},\mathbf{d}\rangle}{\langle \mathbf{a},\mathbf{d}\times\mathbf{e}\rangle}\\
\langle \mathbf{n},\mathbf{c}\rangle\langle \mathbf{n},\mathbf{d}\rangle\langle \mathbf{n},\mathbf{f}\rangle &= \frac{\langle \mathbf{n},\mathbf{d}\times\mathbf{f}\rangle\langle \mathbf{n},\mathbf{d}\rangle\langle \mathbf{n},\mathbf{f}\rangle + \langle \mathbf{n},\mathbf{f}\times\mathbf{c}\rangle\langle \mathbf{n},\mathbf{f}\rangle\langle \mathbf{n},\mathbf{c}\rangle + \langle \mathbf{n},\mathbf{c}\times\mathbf{d}\rangle\langle \mathbf{n},\mathbf{c}\rangle\langle \mathbf{n},\mathbf{d}\rangle}{\langle \mathbf{c},\mathbf{d}\times\mathbf{f}\rangle}\\
\langle \mathbf{n},\mathbf{b}\rangle\langle \mathbf{n},\mathbf{e}\rangle\langle \mathbf{n},\mathbf{f}\rangle &= \frac{\langle \mathbf{n},\mathbf{e}\times\mathbf{f}\rangle\langle \mathbf{n},\mathbf{e}\rangle\langle \mathbf{n},\mathbf{f}\rangle + \langle \mathbf{n},\mathbf{f}\times\mathbf{b}\rangle\langle \mathbf{n},\mathbf{f}\rangle\langle \mathbf{n},\mathbf{b}\rangle + \langle \mathbf{n},\mathbf{b}\times\mathbf{e}\rangle\langle \mathbf{n},\mathbf{b}\rangle\langle \mathbf{n},\mathbf{e}\rangle}{\langle \mathbf{b},\mathbf{e}\times\mathbf{f}\rangle}.
\end{aligned}$$

Also, the normalization condition $\|\mathbf{n}\| = 1$ has to be imposed. If solutions exist, they can be found by intersecting the algebraic surfaces corresponding to the three cubic cones with a unit sphere, and looking for common intersection points of the resulting three (non-identical) algebraic curves. The number of such intersections is finite, therefore the number of solutions for the axis $\mathbf{n}$ is finite as well. ∎

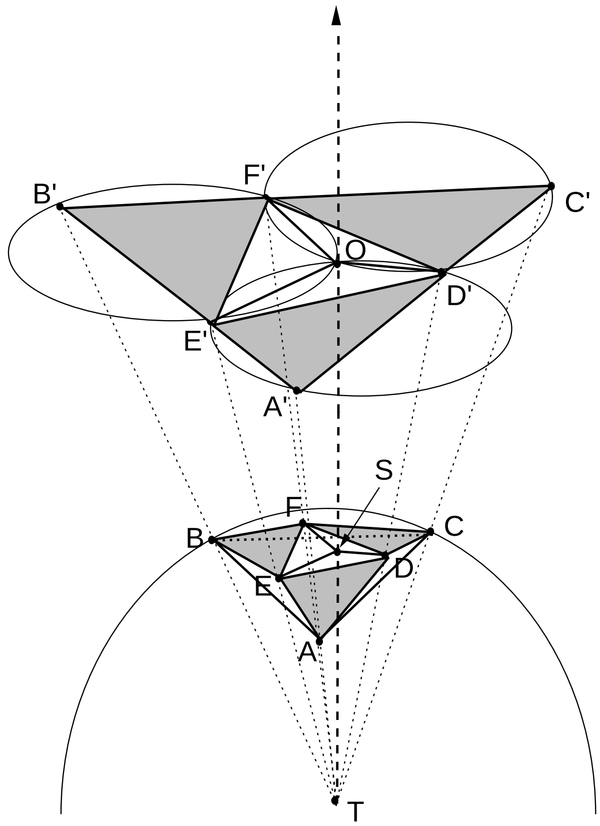


Figure 11: *Spherical triangle $\triangle\, A\, B\, C$ with inscribed triangle $\triangle\, D\, E\, F$ centrally projected from T on a plane orthogonal to the rotation axis $T\,O$.*

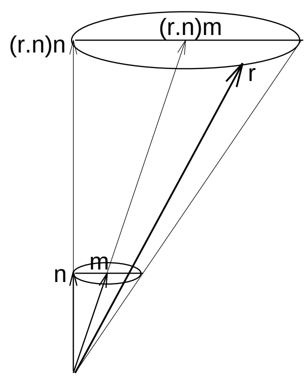


Figure 12: *Cone with axis* $\mathbf{n}$ *orthogonal to the plane of its base circle with center* $\mathbf{m}$.