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A CUMULANT EXPANSION FOR THE TIME CORRELATION FUNCTIONS OF SOLUTIONS TO LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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It is shown that the cumulant expansion for linear stochastic differential equations, hitherto used to compute one-time averages of the solution process, is also capable of yielding the two-time correlation and probability density functions. The general case with a coefficient matrix, an inhomogeneous part and an initial condition which are all random and mutually correlated, is discussed. Two examples are given, the latter of which treats the harmonic oscillator with stochastic frequency and driving term studied before. Finally we investigate the relation of our method with the so-called smoothing method.

1. Introduction

This article is concerned with linear stochastic differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = A(t, \omega)u(t) + f(t, \omega), \tag{1.1}$$

where $u(t)$ is a vector, $A(t, \omega)$ a random coefficient matrix or linear operator and $f(t, \omega)$ a random vector*. The random nature of these quantities is indicated by the parameter ω which will often be omitted in the following. The initial condition $u(t_0)$ may be taken as fixed or in general as a random quantity $u_0(\omega)$.

In a previous article[1]), hereafter referred to as I, we considered the case in which $A(t, \omega)$, $f(t, \omega)$ and $u_0(\omega)$ are mutually correlated. It was shown that the average of $u(t)$ obeys itself a differential equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle u(t)\rangle = K(t/t_0)\langle u(t)\rangle + F(t/t_0) + I(t/t_0), \tag{1.2}$$

provided that $\alpha\tau_c$ is small, where α is a measure for the strength of the

* Although the variable t in (1.1) in this article is interpreted as denoting a physical time, it could be any one-dimensional physical variable.

fluctuations in $A(t)$ and τ_c is the largest of three correlation times: the autocorrelation time of $A(t)$, the crosscorrelation time of $A(t)$ with $f(t)$ and that of $A(t)$ with u_0. Here the angular brackets denote an average over the probability measure $P(\omega)$ which determines the prescribed statistics of all the random quantities involved. Both the matrix $K(t/t_0)$ (involving the ordered cumulants of $A(t)$ alone) and the vectors $F(t/t_0)$ and $I(t/t_0)$ (involving the joint cumulants of $A(t)$ with $f(t)$ and u_0 respectively) were obtained as expansions in the parameter $\alpha\tau_c$. Moreover, after a transient time of order τ_c, $K(t/t_0)$ and $F(t/t_0)$ become independent of t_0 while $I(t/t_0)$ vanishes.

In this paper we will be concerned with the problem of obtaining the time-correlation function $\langle u(t) \otimes u(t')\rangle$ of $u(t)$, hereafter denoted by $C_u(t, t')$. Here the $\otimes$ symbol denotes a Kronecker product. The essential step in our method is to derive from the cumulant expansion for $\langle u(t)\rangle$ first an expansion for the characteristic functional of $A(t)$. First we consider the homogeneous case with $A(t)$ of the form $A(t) = A_0 + \alpha A_1(t, \omega)$, with A_0 non-random. It is shown that if $\alpha\tau_c \ll 1$, the correlation function of u, the latter satisfying (1.1) with $f(t) \equiv 0$ and *fixed* initial condition, obeys differential equations of the form

$$\frac{\partial}{\partial\tau} C_u(t, t+\tau) = M_A(t+\tau; t/t_0) C_u(t, t+\tau) \tag{1.3}$$

and

$$\frac{\partial}{\partial t} C_u(t, t+\tau) = [M_A(t+\tau; t/t_0) + N_A(t+\tau; t/t_0)] C_u(t, t+\tau), \tag{1.4}$$

where $\tau \geq 0$, $t \geq t_0$. To second order in α, the matrices M_A and N_A are given by

$$\begin{aligned} M_A = A_0'' + \alpha\langle A_1''(t+\tau)\rangle &+ \alpha^2 \int_{t_0}^{t} \mathrm{d}s\ \langle\langle A_1''(t+\tau)\, \mathrm{e}^{(t-s)A_0'} A_1'(s)\rangle\rangle\, \mathrm{e}^{-(t-s)A_0'} \\ &+ \alpha^2 \int_{t_0}^{t+\tau} \mathrm{d}s\ \langle\langle A_1''(t+\tau)\, \mathrm{e}^{(t+\tau-s)A_0''} A_1''(s)\rangle\rangle\, \mathrm{e}^{-(t+\tau-s)A_0''}, \end{aligned} \tag{1.5}$$

$$\begin{aligned} N_A = A_0' + \alpha\langle A_1'(t)\rangle &+ \alpha^2 \int_{t_0}^{t} \mathrm{d}s\ \langle\langle A_1'(t)\, \mathrm{e}^{(t-s)A_0'} A_1'(s)\rangle\rangle\, \mathrm{e}^{-(t-s)A_0'} \\ &+ \alpha^2 \int_{t_0}^{t+\tau} \mathrm{d}s\ \langle\langle A_1'(t)\, \mathrm{e}^{(t+\tau-s)A_0''} A_1''(s)\rangle\rangle\, \mathrm{e}^{-(t+\tau-s)A_0''}, \end{aligned} \tag{1.6}$$

where the brackets $\langle\langle\ldots\rangle\rangle$ denote ordinary (second order) cumulants and for any matrix C we have defined

$$C' = C \otimes \hat{1}, \quad C'' = \hat{1} \otimes C, \tag{1.7}$$

where $\hat{1}$ is the unit matrix of the same dimension as C. In (1.3) the initial condition $C_u(t, t)$ is the equal-time second moment $\langle u(t) \otimes u(t) \rangle$, while in (1.4) it is $C_u(t_0, t_0 + \tau) = u(t_0) \otimes \langle u(t_0 + \tau) \rangle$ ($u(t_0)$ is not random). Both initial conditions can in their turn be calculated from the one-time expansion as in (1.2). The matrices M_A and N_A, involving the ordered cumulants of A, are again expansions in $\alpha\tau_c$. Both are independent of the initial time t_0 if $t - t_0 \gg \tau_c$.

It is also shown how one can deal with the general case (1.1), where $A(t)$, $f(t)$ and u_0 are all random and mutually correlated. Moreover, analogs of the eqs. (1.3) and (1.4) are derived for the two-time *probability density* functions of u.

Among previous approaches to obtain the correlation functions we mention the two-time method of Papanicolau and Keller[2]), diagram methods[3,4]), and that of Morrison and McKenna[5]), which is an extension of the "smoothing method" of Bourret[6]) and Keller[7]). The latter method, which leads to an integro-differential equation for the correlation function, will be discussed below in more detail. Related projection operator methods were recently employed by Agarwal[8]). Still other methods, as that of Keller[9]) and McCoy[10]), lead to complicated partial differential equations. Finally we would like to mention that in the case where the coefficient matrix $A(t)$ is a Markov chain, the problem can be reduced to solving a linear (matrix) differential equation for the correlation function[3,5]) (without the need of assuming a small correlation time).

The organization of the paper is as follows: first we study the homogeneous case and derive eqs. (1.3) and (1.4) (section 2) (the generalization to multi-time averages is given in an appendix). Then we investigate in section 3 the behaviour of the matrices M_A and N_A for times exceeding the transient time which is of order τ_c. The general case, i.e. including random inhomogeneous and initial value terms, is considered in section 4. Next the method is illustrated by two examples, one for the homogeneous case (section 5) and another for the inhomogeneous case (section 6). The second example concerns the harmonic oscillator with random frequency and driving term for which we previously derived the differential equations for the equal time first and second order moments[1]). In section 7 we show that the method of section 2 can also yield the equations satisfied by the two-time probability distributions of u themselves*. Finally an exact integral equation for the cor-

* In this paper the expression "probability distribution" is synonymous with "probability density".

relation function is derived which is compared with the results of the smoothing method (section 8).

2. The correlation functions in the homogeneous case

In this section we derive the differential equations (1.3) and (1.4). First we review some results from I and introduce a convenient notation. By considering the cumulant expansion for the characteristic functional of $A(t)$ we next derive a formal expression for $C_u(t, t+\tau)$. From this the eqs. (1.3) and (1.4) are found by differentiation.

The method presented here can easily be extended to derive differential equations for multi-time averages (see appendix A).

2.1. *The characteristic functional of* $A(t)$

Consider the stochastic differential equation

$$\dot{u}(t) = A(t)u(t) \tag{2.1}$$

with $A(t)$ a random matrix and the initial condition $u(t_0) \equiv u_0$ fixed. The formal solution of (2.1) is

$$u(t) = Q_A(t/t_0)u_0, \tag{2.2}$$

where we define for any matrix $A(t)$

$$Q_A(t/t_0) = \overleftarrow{T}\left[\exp \int_{t_0}^{t} \mathrm{d}s\, A(s)\right], \quad (t \geq t_0) \tag{2.3}$$

with $\overleftarrow{T}$ denoting the time-ordering operator (latest times to the left).

As shown by Van Kampen[11]) the average of (2.3) can be expressed as

$$\langle Q_A(t/t_0)\rangle = \overleftarrow{T}\left[\exp \int_{t_0}^{t} \mathrm{d}s\, K_A(s/t_0)\right], \tag{2.4}$$

where the time-ordering operator $\overleftarrow{T}$ in (2.4) now acts with respect to the *first* time variable in $K_A(\cdot/t_0)$. The matrix K_A is given by the following expansion

$$K_A(t/t_0) = \langle A(t)\rangle + \sum_{m=1}^{\infty} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \ldots \int_{t_0}^{t_{m-1}} \mathrm{d}t_m\, \langle A(t)A(t_1)\ldots A(t_m)\rangle_{\mathrm{p}}. \tag{2.5}$$

Here the brackets $\langle\ldots\rangle_\mathrm{p}$ denote a partially time-ordered cumulant* (as defined in I) which is a combination of moments of A with a prescribed order of the time variables. If $[A(t), A(t')] = 0$, for all t, t', it reduces to an ordinary cumulant.

For convenience we introduce the following short-hand notation

$$K_A(t/t_0) = \langle A(t) : Q_A(t/t_0) : \rangle_\mathrm{p}, \tag{2.6}$$

meaning that to compute K_A one should first expand the matrix Q_A between the colons in powers of A, take the p-ordered cumulant of each term with $A(t)$, carry out the integrals and finally sum all the terms, as in (2.5).

Now we replace $A(s)$ in (2.3) by $k(s)A(s)$ where $k(s)$ is a scalar test function with finite support. Then we can define the functional

$$\psi_A[k] \equiv Q_{kA}(\infty/t_0) = \bar{T}\left[\exp \int_{t_0}^{\infty} \mathrm{d}s\, k(s)A(s)\right]. \tag{2.7}$$

If $k(s) = \theta(t-s)$† we find again (2.3). Because of the finite support of k, we can take $t \to \infty$ in (2.4) with A replaced by kA. Then we have the following expansion for the characteristic functional $G_A[k]$ of A (compared with the usual definition there is an imaginary unit i missing in the exponent of (2.7))

$$G_A[k] \equiv \langle \psi_A[k] \rangle = \bar{T} \exp \int_{t_0}^{\infty} \mathrm{d}s\; K_{kA}(s/t_0) \tag{2.8}$$

with K_{kA} the same as (2.6) with A replaced by kA.

2.2. *The formal expression for the correlation function*

Consider the following Kronecker product ($t \geqslant t_0$, $\tau \geqslant 0$)

$$\begin{aligned} u(t) \otimes u(t+\tau) &= [Q_A(t/t_0) \otimes Q_A(t+\tau/t_0)]u_0 \otimes u_0 \\ &= [Q_{A''}(t+\tau/t_0)Q_{A'}(t/t_0)]u_0 \otimes u_0, \end{aligned} \tag{2.9}$$

where for any matrix (or vector) A we define

$$A' = A \otimes \hat{1}; \quad A'' = \hat{1} \otimes A; \quad \tilde{A} = A' + A''. \tag{2.10}$$

Here $\hat{1}$ is the unit matrix of the same dimension as A. $Q_{A'}$ and $Q_{A''}$ are again defined by (2.3).

Now notice the important property of the commutator

$$[A'(t), A''(t')] = 0, \quad \text{all } t, t'. \tag{2.11}$$

* Often we will write "p-ordered cumulant" or "p-cumulant".

† θ denotes the Heaviside-stepfunction.

This implies that we can write

$$Q_{A''}(t+\tau/t_0)Q_{A'}(t/t_0)=\tilde{T}\left[\exp\left\{\int_{t_0}^{t+\tau} \mathrm{d}s\, A''(s)+\int_{t_0}^{t} \mathrm{d}s\, A'(s)\right\}\right]. \tag{2.12}$$

Compared to (2.9) only the mutual order of A'-and A''-matrices has been altered, but this allowed because they commute. Next we put

$$B(s)=\theta(t-s)A'(s)+\theta(t+\tau-s)A''(s). \tag{2.13}$$

Then we can write

$$u(t)\otimes u(t+\tau)=\psi_B(u_0\otimes u_0), \tag{2.14}$$

where

$$\psi_B=Q_B(\infty/t_0) \tag{2.15}$$

since B is a matrix function of s with finite support. Now we use (2.8) with the matrix B instead of kA

$$\langle\psi_B\rangle=\tilde{T}\left[\exp\int_{t_0}^{\infty} \mathrm{d}s\, K_B(s/t_0)\right], \tag{2.16}$$

where K_B is defined as in (2.6).

From (2.13) one deduces

$$B(s)=\begin{cases}\tilde{A}(s), & t_0\leqslant s\leqslant t\\ A''(s), & t<s\leqslant t+\tau\\ 0, & \text{otherwise}\end{cases}, \tag{2.17}$$

where $\tilde{A}$ is defined in (2.10), and therefore

$$K_B(s/t_0)=\begin{cases}L_A(s/t_0), & t_0\leqslant s\leqslant t\\ M_A(s\,;t/t_0), & t<s\leqslant t+\tau\\ 0 & \text{otherwise.}\end{cases} \tag{2.18}$$

Here we define (for any matrix A) L_A and M_A as

$$L_A(s/t_0)=\langle\tilde{A}(s):Q_{\tilde{A}}(s/t_0):\rangle_{\mathrm{p}}\quad(s\geqslant t_0), \tag{2.19}$$

$$M_A(s\ ;t/t_0)=\langle A''(s):Q_{A''}(s/t)Q_{\tilde{A}}(t/t_0):\rangle_{\mathrm{p}}\quad(s\geqslant t\geqslant t_0), \tag{2.20}$$

where the colons have the same meaning as in (2.6). Inserting (2.18) in (2.16) one has

$$\begin{aligned}\langle\psi_B\rangle &= \left\{\bar{T}\exp\int_t^{t+\tau} ds\, K_B(s/t_0)\right\}\left\{\bar{T}\exp\int_{t_0}^{t} ds\, K_B(s/t_0)\right\}\\ &= \left\{\bar{T}\exp\int_t^{t+\tau} ds\, M_A(s\,;t/t_0)\right\}\left\{\bar{T}\exp\int_{t_0}^{t} ds\, L_A(s/t_0)\right\}\\ &\equiv Q_{M_A}(t+\tau/t)Q_{L_A}(t/t_0).\end{aligned} \tag{2.21}$$

Here it is essential to note that the time ordering operators in (2.21) *act with respect to the first time variable* s of the quantities K_B, M_A and L_A.

Summarizing, we have found that the correlation function of u is given by

$$C_u(t, t+\tau) = \langle\psi_B\rangle(u_0 \otimes u_0) \tag{2.22}$$

with $\langle\psi_B\rangle$ as expressed in (2.21).

2.3. *The differential equations for* C_u

Differentiating (2.21) with respect to the variable τ for fixed t one immediately gets

$$\frac{\partial}{\partial\tau}\langle\psi_B\rangle = M_A(t+\tau\,;t/t_0)\langle\psi_B\rangle, \tag{2.23}$$

so in view of (2.22) we have as the first central result

$$\frac{\partial}{\partial\tau}C_u(t,t+\tau) = M_A(t+\tau\,;t/t_0)C_u(t,t+\tau) \tag{2.24}$$

with $M_A(t+\tau\,;t/t_0)$ given by (2.20) with $s = t+\tau$.

Next differentiate (2.21) with respect to t with fixed τ:

$$\begin{aligned}\frac{\partial}{\partial t}\langle\psi_B\rangle = {}& M_A(t+\tau; t/t_0)Q_{M_A}Q_{L_A} - Q_{M_A}M_A(t\,;t/t_0)Q_{L_A}\\ &+ \bar{T}\left[\left\{\exp\int_t^{t+\tau} ds\, M_A(s\,;t/t_0)\right\}\int_t^{t+\tau} ds'\,\frac{\partial M_A}{\partial t}(s';t/t_0)\right]Q_{L_A}.\\ &+ Q_{M_A}L_A(t/t_0)Q_{L_A}.\end{aligned} \tag{2.25}$$

Here we omitted the time variables in the expressions for Q_{L_A} etc. Using the fact that

$$A''(s')Q_{A''}(s'/t) = \frac{\partial}{\partial s'}Q_{A''}(s'/t),$$

we find

$$\int_t^{t+\tau} \mathrm{d}s\, \frac{\partial M_A}{\partial t}(s\,;\, t/t_0) = \int_t^{t+\tau} \mathrm{d}s\, \langle A''(s) : Q_{A''}(s/t) A'(t) Q_{\tilde{A}}(t/t_0) : \rangle_p$$
$$= N_A(t+\tau\,;\, t/t_0) - N_A(t\,;\, t/t_0), \tag{2.26}$$

where

$$N_A(s\,;\, t/t_0) = \langle : Q_{A''}(s/t) A'(t) Q_{\tilde{A}}(t/t_0) : \rangle_p \quad (s \geqslant t \geqslant t_0). \tag{2.27}$$

Now we insert the result (2.26) in (2.25). Because of the time ordering operator in (2.25) the factor $N_A(t+\tau\,;\, t/t_0)$ can be shifted in front of the $\bar{T}$-operator (which acts on the first variable of the expressions $N_A(\cdot\,;\, t/t_0)$!), while the factor $N_A(t\,;\, t/t_0)$ can be shifted backwards outside the brackets $\bar{T}[\ldots]$. Thus (2.25) becomes

$$\frac{\partial}{\partial t}\langle \psi_B \rangle = [M_A(t+\tau\,;\, t/t_0) + N_A(t+\tau\,;\, t/t_0)] \langle \psi_B \rangle$$
$$+ Q_{M_A}[L_A(t/t_0) - M_A(t\,;\, t/t_0) - N_A(t\,;\, t/t_0)] Q_{L_A} \tag{2.28}$$

and in view of the relation

$$L_A(t/t_0) - M_A(t\,;\, t/t_0) - N_A(t\,;\, t/t_0)$$
$$= \langle \tilde{A}(t) : Q_{\tilde{A}}(t/t_0) : \rangle_p - \langle A''(t) : Q_{\tilde{A}}(t/t_0) : \rangle_p - \langle A'(t) : Q_{\tilde{A}}(t/t_0) : \rangle_p = 0, \tag{2.29}$$

we finally arrive by (2.22) at the following equation for the correlation function

$$\frac{\partial}{\partial t} C_u(t, t+\tau) = [M_A(t+\tau\,;\, t/t_0) + N_A(t+\tau\,;\, t/t_0)]\, C_u(t, t+\tau), \tag{2.30a}$$

where

$$M_A(t+\tau\,;\, t/t_0) = \langle A''(t+\tau) : Q_{A''}(t+\tau/t) Q_{\tilde{A}}(t/t_0) : \rangle_p, \tag{2.30b}$$

$$N_A(t+\tau\,;\, t/t_0) = \langle : Q_{A''}(t+\tau/t) A'(t) Q_{\tilde{A}}(t/t_0) : \rangle_p. \tag{2.30c}$$

Note that the operator M_A in (2.30a) is the same as in (2.24). To compute M_A and N_A one should again use the prescription below eq. (2.6).

The initial conditions $\langle u(t) \otimes u(t) \rangle$ and $u(t_0) \otimes \langle u(t_0+\tau) \rangle$ corresponding to (2.24) and (2.30a) can be determined from the cumulant expansion (2.4) for equal-time averages (see I).

If A is of the form $A(t) = A_0 + \alpha A_1(t)$, with A_0 a sure matrix and A_1 random, one can apply (2.24) and (2.30) to the eq. (2.1) in the interaction representation:

$$\dot{v}(t) = \alpha A_1^{(1)}(t) v(t),$$

where

$$v(t) = e^{-(t-t_0)A_0} u(t), \quad A_1^{(1)}(t) = e^{-(t-t_0)A_0} A_1(t)\, e^{(t-t_0)A_0}$$

and transform the result back to the original representation. Here we give only the result to order α^2:

$$\frac{\partial}{\partial t} C_u(t, t+\tau) = \Big[\tilde{A}_0 + \alpha \langle A_1'(t) \rangle + \alpha \langle A_1''(t+\tau) \rangle$$

$$+ \alpha^2 \int_{t_0}^{t} ds\, \langle\langle (A_1'(t) + A_1''(t+\tau))\, e^{(t-s)A_0'} A_1'(s) \rangle\rangle\, e^{-(t-s)A_0'}$$

$$+ \alpha^2 \int_{t_0}^{t+\tau} ds\, \langle\langle (A_1'(t) + A_1''(t+\tau))\, e^{(t+\tau-s)A_0''} A_1''(s) \rangle\rangle\, e^{-(t+\tau-s)A_0''} \Big] C_u(t, t+\tau), \tag{2.31}$$

where the brackets $\langle\langle \ldots \rangle\rangle$ denote ordinary cumulants. The cumulants in (2.31) are of order two, i.e. the evolution operators occurring within them have to be considered as forming one operator with the operator $A_1'(s)$ or $A_1''(s)$ succeeding them. The expression for $\partial C_u / \partial t$ to order α^2 is the same as (2.31) with A_0' and all terms which contain $A_1'(t)$ omitted (so from the second line of (2.31) one should omit the first term, but not the second). If $A_1(t)$ is Gaussian and $[A_1(t), A_1(t')] = 0$ (for all $t \neq t'$), such as for a scalar or delta-correlated vectorial Gaussian process, the second order approximation (2.31) is *exact*.

Remark. If one wants to calculate higher order corrections to the result (2.31) one should always keep all the operators A' and A'' in (2.30b, c) within the cumulant brackets $\langle \ldots \rangle_p$ in decreasing time order (even though they commute). That the order of the operators within the time ordered cumulants is important follows from the prescription to be used for expressing the cumulants in terms of the moments[1]). For example

$$\langle A''(t+\tau) A'(t) A'(s) \rangle_p \neq \langle A'(t) A''(t+\tau) A'(s) \rangle_p \quad \text{if } [A'(t), A'(s)] \neq 0.$$

In the result (2.31) to order α^2 the order of the A_1'- and A_1''-operators doesn't matter, because only ordinary cumulants are involved.

3. Estimates for times exceeding the transient time

In I we showed that if $t - t_0 \gg \tau_c$ the expression (2.5) approaches $K(t/-\infty)$. Using completely analogous arguments we will now show that

$$M_A(t + \tau\,;\, t/t_0) \simeq M_A(t + \tau\,;\, t/-\infty), \quad t - t_0 \gg \tau_c \tag{3.1a}$$

$$\simeq M_A(t + \tau\,;\, -\infty/-\infty), \quad t - t_0 \gg \tau_c\,; \quad \tau \gg \tau_c \tag{3.1b}$$

and

$$N_A(t + \tau\,;\, t/t_0) \simeq N_A(t + \tau\,;\, t/-\infty), \quad t - t_0 \gg \tau_c \tag{3.2a}$$

$$\simeq N_A(\infty\,;\, t/-\infty), \quad t - t_0 \gg \tau_c\,; \quad \tau \gg \tau_c. \tag{3.2b}$$

This implies in particular that in (2.31) we can replace t_0 by $-\infty$ if $t - t_0 \gg \tau_c$.

To show (3.1) we consider a typical term in the expansion of $M_A(t + \tau\,;\, t/t_0)$:

$$M_A^{(n,m)} = \int_t^{t+\tau} \mathrm{d}s_1 \ldots \int_t^{s_{n-1}} \mathrm{d}s_n \int_{t_0}^{t} \mathrm{d}t_1 \ldots \int_{t_0}^{t_{m-1}} \mathrm{d}t_m \, \langle A''(t + \tau)A''(s_1) \ldots A''(s_n)\tilde{A}(t_1) \ldots \tilde{A}(t_m)\rangle_p \tag{3.3}$$

Due to the finite correlation time τ_c of A and the cluster property of the ordered cumulant[1]), subsequent time variables in (3.3) are at most a distance of order τ_c apart, otherwise the ordered cumulant vanishes. So we can imagine the time variables as points being interconnected by flexible strings with maximum length τ_c. If the time increases in (3.3) all time-points are carried along by the first one, $t + \tau$. Thus if $t - t_0 \geqslant m\tau_c$ (and therefore certainly $t + \tau - t_0 \geqslant m\tau_c$) $M_A^{(n,m)}$ becomes independent of t_0 (there is no string between t_m and t_0) and we may as well put $t_0 \to -\infty$ in (3.3) (see fig. 1a).

If in addition $\tau \geqslant n\tau_c$ the cumulant vanishes altogether because the maximal distance between $t + \tau$ and t_1 is $(n + 1)\tau_c$, thus certainly that between $t + \tau$ and t (fig. 1b). Therefore if $\tau \gg \tau_c$ only the terms $M_A^{(n,0)}$ remain and in these remaining terms we can put $t \to -\infty$ because all the s_i time-variables are carried along by $t + \tau$ and therefore become eventually independent of t (fig. 1c). This completes the proof of (3.1).

Now consider a typical term of $N_A(t + \tau\,;\, t/t_0)$:

$$N_A^{(n,m)} = \int_t^{t+\tau} \mathrm{d}s_1 \ldots \int_t^{s_{n-1}} \mathrm{d}s_n \int_{t_0}^{t} \mathrm{d}t_1 \ldots \int_{t_0}^{t_{m-1}} \mathrm{d}t_m \langle A''(s_1) \ldots A''(s_n)A'(t)\tilde{A}(t_1) \ldots \tilde{A}(t_m)\rangle_p. \tag{3.4}$$

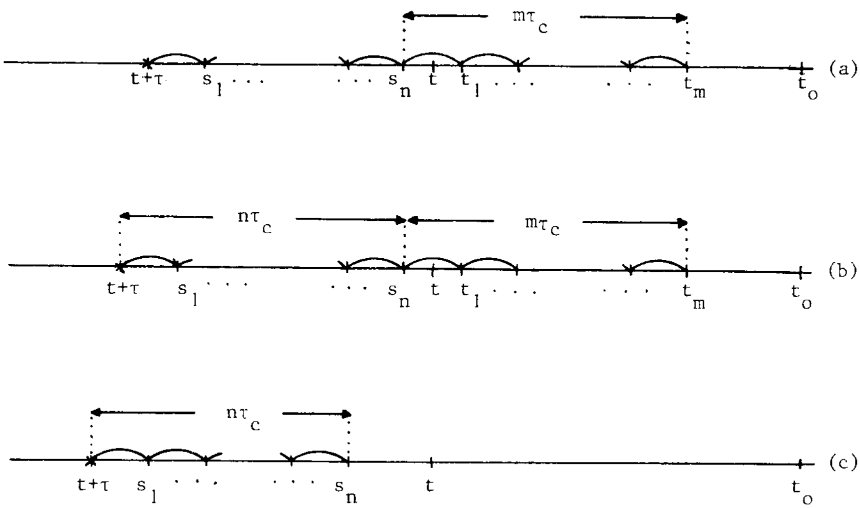

Fig. 1. Extension of the integration domains in (3.3) (time running from right to left).

In this case the variable t carries the subsequent t_i-variables along. So if $t - t_0 \gtrsim m\tau_c$ the expression (3.4) becomes independent of t_0 (fig. 2a). On the other hand, s_1 can be at most a distance $n\tau_c$ apart from t, so if $\tau \gtrsim n\tau_c$ we can extend the s_1 integration to ∞ because the cumulant is zero anyway if $s_1 \gtrsim n\tau_c$ (fig. 2b). Hence also (3.2) follows.

From the above considerations we can deduce the following estimates, taking A of order α and $t - t_0 \gg \tau_c$, $\tau \gg \tau_c$:

$$M_A^{(n,0)} \simeq \alpha(\alpha\tau_c)^n, \quad N_A^{(n,m)} \simeq \alpha(\alpha\tau_c)^{n+m}. \tag{3.5}$$

Hence we find the condition $\alpha\tau_c \ll 1$ for the convergence of the expansions (2.24) and (2.30), which is the same condition as found previously for the validity of the one-time expansion[1,11]).

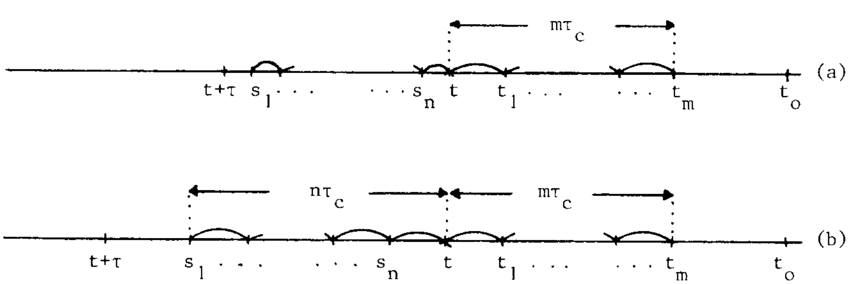

Fig. 2. Extension of the integration domains in (3.4).

4. The general case

In this section we construct the differential equations for the correlation functions in the inhomogeneous case (1.1). The situation, where also the initial condition is random, is handled by reducing the problem to the inhomogeneous case with sure initial condition (section 4.2).

4.1. *The inhomogeneous case*

We start again with the equation

$$\dot{u}(t) = A(t, \omega)u(t) + f(t, \omega) \tag{4.1}$$

with $u(t_0) = u_0$ (fixed). The random matrix $A(t, \omega)$ and vector $f(t, \omega)$ may be statistically dependent. First we reduce (4.1) to a homogeneous equation by the following trick. Define a new vector $w(t)$ by

$$w(t) = \begin{pmatrix} u(t) \\ z(t) \end{pmatrix}, \tag{4.2}$$

where $z(t)$ is a scalar function with $z(t) = 1$. Then $w(t)$ obeys the equation

$$\dot{w}(t) = B(t)w(t) \tag{4.3}$$

with*

$$B(t) = \left(\begin{array}{c|c} A(t) & f(t) \\ \hline \emptyset & 0 \end{array}\right) \tag{4.4a}$$

and initial condition

$$w(t_0) = \begin{pmatrix} u(t_0) \\ 1 \end{pmatrix}. \tag{4.4b}$$

Applying the results (2.24) and (2.30) to (4.3) one finds equations for the correlation function $C_w(t, t+\tau) \equiv \langle w(t) \otimes w(t+\tau)\rangle$, from which equations for $C_u(t, t+\tau)$ can be extracted. We will carry out this scheme only for eq. (2.30a). One finds

$$\frac{\partial}{\partial t} C_w(t, t+\tau) = [M_B(t+\tau\,;\, t/t_0) + N_B(t+\tau\,;\, t/t_0)]C_w(t, t+\tau). \tag{4.5}$$

Here M_B and N_B are defined as in (2.30b, c) with A replaced by B (and consequently A' by B' etc.) and B given by (4.4a).

Now suppose u is a vector with n components. Then we are only interested in the first n components of $w(t)$ and $w(t+\tau)$ in (4.5), for which we now establish the equation. To this end we first note that the quantities M_B and N_B

* The symbol $\emptyset$ indicates a matrix (or vector) with all elements zero.

in (4.5) can be written as

$$M_B(t+\tau\,;t/t_0) = \langle : E(t) \otimes D(t+\tau) : \rangle_{\mathrm{p}}, \tag{4.6a}$$

$$N_B(t+\tau\,;t/t_0) = \langle : D(t) \otimes E(t+\tau) : \rangle_{\mathrm{p}} \tag{4.6b}$$

with

$$E(t) = B(t)Q_B(t/t_0), \quad D(t) = Q_B(t/t_0). \tag{4.6c}$$

Here we made use of identities as

$$Q_{A''}(t+\tau/t)Q_{\tilde{A}}(t/t_0) = Q_{A''}(t+\tau/t_0)Q_{A'}(t/t_0)$$

and substituted them into the expressions (2.30b, c). However, it has to be kept in mind that if one explicitly wants to compute the ordered cumulants one should first place in (4.6a, b) all quantities within the ordered cumulant-brackets $\langle\ldots\rangle_{\mathrm{p}}$ in decreasing time order (see "remark" at the end of section 2).

From (4.5) and (4.6) we find for the components

$$\frac{\partial}{\partial t}\langle w_i(t)w_k(t+\tau)\rangle$$
$$= \sum_{j,l=1}^{n+1} [\langle : E_{ij}(t)D_{kl}(t+\tau) : \rangle_{\mathrm{p}} + \langle : D_{ij}(t)E_{kl}(t+\tau) : \rangle_{\mathrm{p}}]\,\langle w_j(t)w_l(t+\tau)\rangle. \tag{4.7}$$

The matrices E and D have the following structure

$$E = \left(\begin{array}{c:c} E^{(1)} & E^{(2)} \\ \hdashline \emptyset & 0 \end{array}\right) \quad D = \left(\begin{array}{c:c} D^{(1)} & D^{(2)} \\ \hdashline \emptyset & 0 \end{array}\right), \tag{4.8}$$

where

$$E^{(1)}(t) = A(t)D^{(1)}(t), \quad E^{(2)}(t) = f(t) + A(t)D^{(2)}(t) \tag{4.9a}$$

and

$$D^{(1)}(t) = Q_A(t/t_0), \quad D^{(2)}(t) = \int_{t_0}^{t} \mathrm{d}s\, Q_A(t/s)f(s). \tag{4.9b}$$

$E^{(1)}$ and $D^{(1)}$ are $n \times n$-matrices, and $E^{(2)}$ and $D^{(2)}$ are n-dimensional (column) vectors.

Now we restrict ourselves in (4.7) to $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant n$. The summation over j and l can be split up in four regions: $\{1 \leqslant j \leqslant n; 1 \leqslant l \leqslant n\}$; $\{1 \leqslant l \leqslant n;$

$j = n + 1\}$; $\{1 \leq j \leq n;\ l = n + 1\}$; $\{j = n + 1 = l\}$. We find

$$\begin{aligned}
&\frac{\partial}{\partial t}\langle u_i(t)u_k(t+\tau)\rangle \\
&\quad = \sum_{j,l=1}^{n} [\langle : E_{ij}^{(1)}(t)D_{kl}^{(1)}(t+\tau) : \rangle_{\mathrm{p}} + \langle : D_{ij}^{(1)}(t)E_{kl}^{(1)}(t+\tau) : \rangle_{\mathrm{p}}]\langle u_j(t)u_l(t+\tau)\rangle \\
&\quad + \sum_{l=1}^{n} [\langle : E_i^{(2)}(t)D_{kl}^{(1)}(t+\tau) : \rangle_{\mathrm{p}} + \langle : D_i^{(2)}(t)E_{kl}^{(1)}(t+\tau) : \rangle_{\mathrm{p}}]\langle u_l(t+\tau)\rangle \\
&\quad + \sum_{j=1}^{n} [\langle : E_{ij}^{(1)}(t)D_k^{(2)}(t+\tau) : \rangle_{\mathrm{p}} + \langle : D_{ij}^{(1)}(t)E_k^{(2)}(t+\tau) : \rangle_{\mathrm{p}}]\langle u_j(t)\rangle \\
&\quad + \langle : E_i^{(2)}(t)D_k^{(2)}(t+\tau) : \rangle_{\mathrm{p}} + \langle : D_i^{(2)}(t)E_k^{(2)}(t+\tau) : \rangle_{\mathrm{p}}
\end{aligned} \tag{4.10}$$

or in vector notation

$$\begin{aligned}
\frac{\partial}{\partial t} C_u(t, t+\tau) &= G^{(1,1)}(t, t+\tau)C_u(t, t+\tau) + G^{(2,1)}(t, t+\tau)\langle u(t+\tau)\rangle \\
&\quad + G^{(1,2)}(t, t+\tau)\langle u(t)\rangle + G^{(2,2)}(t, t+\tau),
\end{aligned} \tag{4.11a}$$

where we define

$$\begin{aligned}
G^{(i,j)}(t, t+\tau) &= \langle : E^{(i)}(t) \otimes D^{(j)}(t+\tau) : \rangle_{\mathrm{p}} \\
&\quad + \langle : D^{(i)}(t) \otimes E^{(j)}(t+\tau) : \rangle_{\mathrm{p}} \quad (i = 1, 2).
\end{aligned} \tag{4.11b}$$

the $E^{(\cdot)}$ and $D^{(\cdot)}$ matrices being given by (4.9) (remember the remark succeeding (4.6)). Eqs. (4.11) constitute the final result for the correlation function in the inhomogeneous case. If C_u is differentiated with respect to τ we obtain the same expression (4.11a) where however only the *second* part of (4.11b) contributes. The matrix $G^{(1,1)}$ contains the ordered cumulants of A alone, $G^{(2,1)}$ and $G^{(1,2)}$ those of A with *one* f and $G^{(2,2)}$ those of A with *two* f's. If we formally regard A and f to be of the same order of magnitude, the result (4.11b) to second order yields

$$\begin{aligned}
G^{(1,1)} &= \langle A'(t)\rangle + \langle A''(t+\tau)\rangle + \int_{t_0}^{t} \mathrm{d}s\ \langle\langle\{A'(t) + A''(t+\tau)\}A'(s)\rangle\rangle \\
&\quad + \int_{t_0}^{t+\tau} \mathrm{d}s\ \langle\langle\{A'(t) + A''(t+\tau)\}A''(s)\rangle\rangle,
\end{aligned} \tag{4.12a}$$

$$G^{(2,1)} = \langle f'(t)\rangle + \int_{t_0}^{t+\tau} \mathrm{d}s\ \ \langle\langle A''(s)f'(t)\rangle\rangle + \int_{t_0}^{t} \mathrm{d}s\ \langle\langle\{A'(t) + A''(t+\tau)\}f'(s)\rangle\rangle, \tag{4.12b}$$

$$G^{(1,2)} = \langle f''(t+\tau)\rangle + \int_{t_0}^{t} \mathrm{d}s\ \langle\langle A'(s) f''(t+\tau)\rangle\rangle$$

$$+ \int_{t_0}^{t+\tau} \mathrm{d}s\ \langle\langle \{A'(t) + A''(t+\tau)\} f''(s)\rangle\rangle, \tag{4.12c}$$

$$G^{(2,2)} = \int_{t_0}^{t+\tau} \mathrm{d}s\ \ \langle\langle f'(t) f(s)\rangle\rangle + \int_{t_0}^{t} \mathrm{d}s\ \langle\langle f''(t+\tau) f(s)\rangle\rangle, \tag{4.12d}$$

where $\langle\langle\ldots\rangle\rangle$ again denotes ordinary cumulants.

In the special case that A and f in (4.1) are *statistically independent* the general result (4.11) reads

$$\frac{\partial}{\partial t} C_u(t, t+\tau) = [M_A(t+\tau\ ;\ t/t_0) + N_A(t+\tau\ ;\ t/t_0)]\ C_u(t, t+\tau)$$
$$+ \langle f'(t)\rangle \langle u(t+\tau)\rangle + \langle f''(t+\tau)\rangle \langle u(t)\rangle$$
$$+ \int_{t_0}^{t+\tau} \mathrm{d}s \langle\langle f'(t) f(s)\rangle\rangle + \int_{t_0}^{t} \mathrm{d}s\ \langle\langle f''(t+\tau) f(s)\rangle\rangle \tag{4.13}$$

with M_A and N_A given by (2.30b, c).

4.2. *Random initial conditions*

Consider finally (4.1) with random initial condition $u(t_0) = u_0(\omega)$, where the random initial condition can be correlated with both A and f. Then the quantity

$$v(t) = u(t) - u(t_0)$$

satisfies

$$\dot{v}(t) = A(t, \omega) v(t) + g(t, \omega), \quad v(t_0) = 0 \quad \text{(non-random!)} \tag{4.14}$$

with

$$g(t, \omega) = A(t, \omega) u_0(\omega) + f(t, \omega). \tag{4.15}$$

Equation (4.14) is again of the type considered in section 4.1 so that (4.11) immediately leads to an equation satisfied by $C_v(t, t+\tau) = \langle v(t) \otimes v(t+\tau)\rangle$, from which $C_v(t, t+\tau)$ has to be determined with initial condition $C_v(t_0, t_0 + \tau) = 0$.

The correlation of u itself can be expressed as

$$C_u(t, t+\tau) = C_v(t, t+\tau) + \langle u(t) \otimes u_0\rangle + \langle u_0 \otimes u(t+\tau)\rangle - \langle u_0 \otimes u_0\rangle. \tag{4.16}$$

So what remains to be done is to find the quantities $\langle u(t) \otimes u_0\rangle$ and $\langle u_0 \otimes u(t+\tau)\rangle$ in (4.16). This can be achieved in the following way. It was shown in I that the *average* $\langle u(t)\rangle$ of (4.1) with random initial condition u_0 obeys the equation

$$\langle \dot{u}(t)\rangle = K_A(t/t_0)\langle u(t)\rangle + F_A(t/t_0) + I_A(t/t_0), \tag{4.17}$$

where

$$K_A(t/t_0) = \langle A(t) : Q_A(t/t_0) : \rangle_p, \tag{4.18a}$$

$$F_A(t/t_0) = \langle f(t)\rangle + \int_{t_0}^{t} ds\, \langle A(t) : Q_A(t/s) : f(s)\rangle_p, \tag{4.18b}$$

$$I_A(t/t_0) = \langle A(t)(u_0 - \langle u_0\rangle)\rangle + \int_{t_0}^{t} ds\, \langle A(t) : Q_A(t/s) : [A(s)(u_0 - \langle u_0\rangle)]\rangle_p. \tag{4.18c}$$

Here we used the shorthand notation introduced in this paper also for F_A and I_A.

By integrating (4.17) we have

$$\begin{aligned}\langle u(t)\rangle &\equiv \langle : Q_A(t/t_0) : u_0\rangle + \int_{t_0}^{t} ds\, \langle : Q_A(t/s) : f(s)\rangle \\ &= \left[\vec{T} \exp \int_{t_0}^{t} ds\, K_A(s/t_0)\right]\langle u_0\rangle \\ &\quad + \int_{t_0}^{t} ds \left[\vec{T} \exp \int_{s}^{t} ds'\, K_A(s'/t_0)\right][F_A(s/t_0) + I_A(s/t_0)].\end{aligned} \tag{4.19}$$

Now the quantities

$$\langle u(t) \otimes u_0\rangle = \langle : Q_{A'}(t/t_0) : \{u_0 \otimes u_0\}\rangle + \int_{t_0}^{t} ds\, \langle : Q_{A'}(t/s) : \{f(s) \otimes u_0\}\rangle \tag{4.20}$$

and

$$\begin{aligned}\langle u_0 \otimes u(t+\tau)\rangle &= \langle : Q_{A''}(t+\tau/t_0) : \{u_0 \otimes u_0\}\rangle \\ &\quad + \int_{t_0}^{t+\tau} ds\, \langle : Q_{A''}(t+\tau/s) : \{u_0 \otimes f(s)\}\rangle\end{aligned} \tag{4.21}$$

obey the same identity (4.19) with the replacements $A \to A'$, $u_0 \to u_0 \otimes u_0$,

$f \to f \otimes u_0$ in the case of (4.20) and $A \to A''$, $u_0 \to u_0 \otimes u_0$, $f \to u_0 \otimes f$, $t \to t + \tau$ in case of (4.21). Therefore also $\langle u(t) \otimes u_0 \rangle$ and $\langle u_0 \otimes u(t+\tau) \rangle$ obey (4.17) with the replacements just mentioned in K_A, F_A and I_A. From these equations the quantities (4.20) and (4.21) can be determined with initial condition $\langle u_0 \otimes u_0 \rangle$.

This completes the solution in the most general case (4.1). The correlation times involved now are not only those of A with itself and with f and u_0 (which have to be finite for the above expansions to be valid) but also those of f with itself and with u_0.

5. First example: the homogeneous case

As a first illustration we consider the following homogeneous equation (already introduced in I)

$$\dot{u}(t) = \{\sigma_z + \alpha\xi(t)\sigma_x\}u(t) \tag{5.1}$$

with u a two-component vector, σ_z and σ_x Pauli matrices and $\xi(t)$ a stationary dichotomic Markov process with values ± 1. In this case we take a fixed initial condition u_0.

By application of the approximate result (2.31) (with $t_0 \equiv 0$) one gets from (5.1)

$$\frac{\partial}{\partial t} C_u(t+\tau, t) = B(t, \tau) C_u(t+\tau, t), \tag{5.2}$$

where

$$B(t,\tau) = \tilde{\sigma}_z + \alpha^2 \int_0^t \mathrm{d}s\, \{\mathrm{e}^{-2\gamma|t-s|}\sigma''_x + \mathrm{e}^{-2\gamma|t+\tau-s|}\sigma'_x\}\sigma''_x(t-s)$$
$$+ \alpha^2 \int_0^{t+\tau} \mathrm{d}s\, \{\mathrm{e}^{-2\gamma|t-s|}\sigma''_x + \mathrm{e}^{-2\gamma|t+\tau-s|}\sigma'_x\}\sigma'_x(t+\tau-s), \tag{5.3}$$

where

$$\sigma_x(t) \equiv \mathrm{e}^{t\sigma_z}\sigma_x\,\mathrm{e}^{-t\sigma_z} \tag{5.4}$$

and we used

$$\langle \xi(t) \rangle = 0, \quad \langle\langle \xi(t)\xi(t') \rangle\rangle = \mathrm{e}^{-2\gamma|t-t'|}. \tag{5.5}$$

Note that we calculated $C_u(t+\tau, t)$ instead of $C_u(t, t+\tau)$ as in (2.31). Carrying out the integrals in (5.3) and considering times $t \gg \tau_c \sim (\gamma \pm 1)^{-1} (\gamma > 1)$, we find

$$B(t,\tau) = \tilde{\sigma}_z + \frac{\alpha^2}{2(\gamma^2-1)}[2\gamma - \tilde{\sigma}_z + \mathrm{i}\,\mathrm{e}^{-2\gamma\tau}(\sigma'_x\sigma''_y + \sigma'_y\sigma''_x)$$
$$+ 2\gamma(\sigma'_x\sigma''_x \cosh 2\tau + \mathrm{i}\,\sigma'_y\sigma''_x \sinh 2\tau)]. \tag{5.6}$$

Introducing the vector

$$U(t,\tau)=\mathrm{Col}\{\langle u_1(t+\tau)u_1(t)\rangle,\ \langle u_2(t+\tau)u_2(t)\rangle,\ \langle u_1(t+\tau)u_2(t)\rangle,\ \langle u_2(t+\tau)u_1(t)\rangle\}, \tag{5.7}$$

we derive from (5.2) and (5.6) the equation

$$\frac{\partial}{\partial t}U(t,\tau)=\left(\begin{array}{c|c}2\sigma_2 & \emptyset\\ \hline \emptyset & \emptyset\end{array}\right)+\frac{\alpha^2}{\gamma^2-1}\left(\begin{array}{c|c}B_1 & \emptyset\\ \hline \emptyset & B_2\end{array}\right)\quad (t\gg\tau_c), \tag{5.8}$$

where

$$B_1=\begin{pmatrix}\gamma-1 & \gamma\,\mathrm{e}^{2\tau}+\mathrm{e}^{-2\gamma\tau}\\ \gamma\,\mathrm{e}^{-2\tau}-\mathrm{e}^{-2\gamma\tau} & \gamma+1\end{pmatrix};\quad B_2=\begin{pmatrix}\gamma & \gamma\,\mathrm{e}^{2\tau}\\ \gamma\,\mathrm{e}^{-2\tau} & \gamma\end{pmatrix}. \tag{5.9}$$

So it turns out that (at least to order α^2) the first and last two components of U are decoupled.

In appendix B we briefly indicate how the results (5.8) and (5.9) can be checked*. There it also turns out that the decoupling found above is in fact exact to all orders in α (at least if $t\gg\tau_c$).

6. Second example: harmonic oscillator with stochastic frequency

In this section we will consider again the harmonic oscillator problem of West et al.[12]) as an example of the inhomogeneous case discussed in section 4.1. In I we already derived the equations for the equal-time second moments.

The model is defined by the equation

$$\dot{u}(t)=[A_0+A_1(t)]u(t)+f(t) \tag{6.1a}$$

with $u(t_0)$ fixed and

$$A_0=\begin{pmatrix}\cdot & 1\\ -\Omega^2 & -2\lambda\end{pmatrix};\quad A_1(t)=\gamma(t)B,\quad B=\begin{pmatrix}\cdot & \cdot\\ -1 & \cdot\end{pmatrix}, \tag{6.1b}$$

$$f(t)=f_2(t)f_0,\quad f_0=\begin{pmatrix}\cdot\\ 1\end{pmatrix}. \tag{6.1c}$$

Here $u(t)=\mathrm{Col}\{x(t),\,p(t)\}$ where x and p are the displacement and momentum of the oscillator; $f_2(t)$ and $\gamma(t)$ are two statistically independent scalar random processes. The process $\gamma(t)$ has zero mean and delta-correlated cumulants

$$\langle\langle\gamma(t_1)\gamma(t_2)\ldots\gamma(t_m)\rangle\rangle=m!D_m\,\delta(t_1-t_2)\ldots\delta(t_{m-1}-t_m)\quad(m\geq 2). \tag{6.2}$$

* See also ref. 5, §5.

The process $f_2(t)$ is stationary with zero mean and its correlation function (which is not yet specified) is denoted by

$$\langle f_2(t) f_2(t-\tau)\rangle = 2\tilde{D}\phi(\tau). \tag{6.3}$$

Since $A_1(t)$ and $f(t)$ in (6.1a) are statistically independent, the appropriate equation for the correlation function in this case is (4.13), which however is best applied first to the equation for the interaction representation $v(t) = \mathrm{e}^{-tA_0}$ $u(t)$ (take $t_0 \equiv 0$) and subsequently transformed back to the original representation u. We find

$$\begin{aligned}\frac{\partial}{\partial t} C_u(t, t+\tau) = {} & [\tilde{A}_0 + \mathrm{e}^{tA_0'+(t+\tau)A_0''}\{M_{\hat{A}_1}(t+\tau\,;\, t/t_0) \\ & + N_{\hat{A}_1}(t+\tau\,;\, t/t_0)\}\mathrm{e}^{-(tA_0'+(t+\tau)A_0'')}]C_u(t, t+\tau) \\ & + \mathrm{e}^{tA_0'+(t+\tau)A_0''}\Big[\int_0^t \mathrm{d}s\ \langle\langle \hat{f}''(t+\tau)\hat{f}(s)\rangle\rangle + \int_0^{t+\tau} \mathrm{d}s\ \langle\langle \hat{f}'(t)\hat{f}(s)\rangle\rangle\Big],\end{aligned} \tag{6.4}$$

where the hat symbol "^" defines the interaction representation

$$\hat{A}_1(t) = \mathrm{e}^{-tA_0} A_1(t)\, \mathrm{e}^{tA_0}; \quad \hat{f}(t) = \mathrm{e}^{-tA_0} f(t). \tag{6.5}$$

Note that $Q_{\hat{A}'} = (Q_{\hat{A}})'$ etc.

Now consider the quantities $M_{\hat{A}_1}$ and $N_{\hat{A}_1}$ in (6.4). Let us first look at $M_{\hat{A}_1}$:

$$M_{\hat{A}_1}(t+\tau; t/t_0) = \langle \hat{A}_1''(t+\tau) : Q_{\hat{A}_1}{}''(t+\tau/t) Q_{\hat{\hat{A}}_1}(t/t_0) : \rangle_{\mathrm{p}}. \tag{6.6}$$

Since the process $\gamma(t)$, and therefore also $A_1(t)$, is delta-correlated, there is only a contribution to[*] (6.6) if after expanding all the Q-operators all time variables of $A_1''(\cdot)$ and $\hat{\hat{A}}_1(\cdot)$ are equal to $(t+\tau)$. If $\tau > 0$ this cannot be true for any of the $\hat{\hat{A}}_1$-matrices since the upper time-limit in $Q_{\hat{A}_1}$ is t^*. So only A_1''-matrices remain. The m^{th} order term of $Q''_{\hat{A}_1}$ in (6.6) then yields a contribution

$$D_{m+1}\, \mathrm{e}^{-(t+\tau)A_0''}(B'')^{m+1}\, \mathrm{e}^{(t+\tau)A_0''},$$

which vanishes for all $m \geqslant 1$ since $(B'')^{m+1} = \hat{1} \otimes B^{m+1}$ and $B^{m+1} = 0$, $m \geqslant 1$. Also the term with $m = 0$ vanishes ($\gamma(t)$ has zero average) so

$$M_{\hat{A}_1}(t+\tau\,;\, t/t_0) = 0. \tag{6.7}$$

Next turn to $N_{\hat{A}_1}$:

$$N_{\hat{A}_1}(t+\tau\,;\, t/t_0) = \langle : Q''_{\hat{A}_1}(t+\tau/t)\hat{A}_1'(t) Q_{\hat{A}_1}(t/t_0) : \rangle_{\mathrm{p}}. \tag{6.8}$$

[*] If $\tau = 0$ both (6.6) and (6.8) contribute. However, one easily checks that the result (6.10), derived for $\tau > 0$, is in fact also correct for $\tau = 0$.

By similar arguments as above one finds that the expression (6.8) is of the form

$$\mathrm{e}^{-tA_0'} B' \mathrm{e}^{tA_0'} \sum_{m=1}^{\infty} (m+1) D_{m+1} \mathrm{e}^{-tA_0''} B''^m \mathrm{e}^{tA_0''}$$

of which again only the term with $m = 1$ survives, so

$$N_{\hat{A}_1}(t+\tau\,;\,t/t_0) = 2D_2\, \mathrm{e}^{-tA_0'} B' \mathrm{e}^{tA_0'} \mathrm{e}^{-tA_0''} B''^m \mathrm{e}^{tA_0''} \tag{6.9}$$

Inserting (6.1b, c), (6.3), (6.7) and (6.9) in (6.4) we get

$$\begin{aligned} \frac{\partial}{\partial t} C_u(t, t+\tau) = {} & [\tilde{A}_0 + 2D_2 B \otimes \{\mathrm{e}^{\tau A_0} B\, \mathrm{e}^{-\tau A_0}\}] C_u(t, t+\tau) \\ & + 2\tilde{D}\Big[\int_0^t \mathrm{d}s\, \phi(t+\tau-s)\{\mathrm{e}^{(t-s)A_0} \otimes f_0\} f_0 \\ & + \int_0^{t+\tau} \mathrm{d}s\, \phi(t-s)\{f_0 \otimes \mathrm{e}^{(t+\tau-s)A_0}\} f_0\Big]. \end{aligned} \tag{6.10}$$

The matrix $\mathrm{e}^{\tau A_0}$ is of the form[1])

$$\mathrm{e}^{\tau A_0} = \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}, \tag{6.11}$$

where

$$a(\tau) = \Big(\cos \tau\omega_1 + \frac{\lambda}{\omega_1} \sin \tau\omega_1\Big) \mathrm{e}^{-\tau\lambda}; \quad b(\tau) = \Big(\frac{1}{\omega_1} \sin \tau\omega_1\Big) \mathrm{e}^{-\tau\lambda}, \tag{6.12a}$$

$$c(\tau) = -\Big(\frac{\Omega^2}{\omega_1} \sin \tau\omega_1\Big) \mathrm{e}^{-\tau\lambda}; \quad d(\tau) = \Big(\cos \tau\omega_1 - \frac{\lambda}{\omega_1} \sin \tau\omega_1\Big) \mathrm{e}^{-\tau\lambda} \tag{6.12b}$$

and

$$\omega_1 = (\Omega^2 - \lambda^2)^{1/2}.$$

From (6.1b, c) and (6.10) – (6.12) we find the following equations for the components of $C_u(t, t+\tau)$:

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} \langle x^2 \rangle_\tau \\ \langle xp \rangle_\tau \\ \langle px \rangle_\tau \\ \langle p^2 \rangle_\tau \end{pmatrix} = {} & \left[\begin{pmatrix} \cdot & 1 & 1 & \cdot \\ -\Omega^2 & -2\lambda & \cdot & 1 \\ -\Omega^2 & \cdot & -2\lambda & 1 \\ \cdot & -\Omega^2 & -\Omega^2 & -4\lambda \end{pmatrix} + 2D_2 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1 & \alpha_2 & \cdot & \cdot \\ \alpha_3 & \alpha_4 & \cdot & \cdot \end{pmatrix} \right] \begin{pmatrix} \langle x^2 \rangle_\tau \\ \langle xp \rangle_\tau \\ \langle px \rangle_\tau \\ \langle p^2 \rangle_\tau \end{pmatrix} \\ & + \int_0^t \mathrm{d}s\, 2\tilde{D}\, \phi(s+\tau) \begin{pmatrix} \cdot \\ b(s) \\ \cdot \\ d(s) \end{pmatrix} + \int_0^{t+\tau} \mathrm{d}s\, 2\tilde{D}\, \phi(s-\tau) \begin{pmatrix} \cdot \\ \cdot \\ b(s) \\ d(s) \end{pmatrix}, \end{aligned} \tag{6.13}$$

where

$$\alpha_1 = b(\tau)a(-\tau); \quad \alpha_2 = b(\tau)b(-\tau); \quad \alpha_3 = d(\tau)a(-\tau); \quad \alpha_4 = d(\tau)b(-\tau) \tag{6.14}$$

and

$$\langle xp\rangle_\tau = \langle x(t)p(t+\tau)\rangle, \quad \text{etc.}$$

Now we can calculate the equilibrium correlation functions (which certainly exist if the equal time second moments exist, i.e. if $D_2 < 2\lambda\Omega^2$) by putting the derivative in (6.13) equal to zero and solving the resulting matrix equation for $t \to \infty$, τ fixed. We find

$$\langle x^2\rangle_\tau^{eq} = \{p(A_+ - A_-) + 4\lambda A_+ + C\}\{4\lambda\Omega^2 - 2D_2\alpha_3 + 2D_2\alpha_1 p\}^{-1}, \tag{6.15a}$$

$$\langle xp\rangle_\tau^{eq} = -\langle px\rangle_\tau^{eq} = \{A_+ - A_- - 2D_2\alpha_1\langle x^2\rangle_\tau^{eq}\}\{2D_2\alpha_2 + 4\lambda\}^{-1}, \tag{6.15b}$$

$$\langle p^2\rangle_\tau^{eq} = (\Omega^2 - q)\langle x^2\rangle_\tau^{eq} + [2\lambda\{2D_2\alpha_2 + 4\lambda\}^{-1}(A_+ - A_-)] - A_+, \tag{6.15c}$$

where

$$p = (2D_2\alpha_4 - 8\lambda^2)(2D_2\alpha_2 + 4\lambda)^{-1}; \quad q = 4\lambda D_2\alpha_1(2D_2\alpha_2 + 4\lambda)^{-1},$$

$$A_\pm = 2\tilde{D}\left\{\int_0^\infty ds\, \phi(s \pm \tau)b(s)\right\},$$

$$C = 2\tilde{D}\left\{\int_0^\infty ds\, (\phi(s-\tau) + \phi(s+\tau))d(s)\right\}.$$

It is not difficult to show that for $\tau = 0$ the results (6.15) reduce to the equilibrium moments (6.25) of I. If $f_2(t)$ is delta-correlated, i.e. $\phi(s) = \delta(s)$, it follows that $(\tau > 0)$

$$A_+ = 0, \quad A_- = 2\tilde{D}b(\tau), \quad C = 2\tilde{D}d(\tau)$$

and we find

$$\langle x^2\rangle_\tau^{eq} = E\left(\cos\tau\omega_1 + \frac{\lambda}{\omega_1}\sin\tau\omega_1\right)e^{-\lambda\tau},$$

$$\langle xp\rangle_\tau^{eq} = -\langle px\rangle_\tau^{eq} = -E\left(\frac{\Omega^2}{\omega_1}\sin\tau\omega_1\right)e^{-\lambda\tau}, \tag{6.16}$$

$$\langle p^2\rangle_\tau^{eq} = E\Omega^2\left(\cos\tau\omega_1 - \frac{\lambda}{\omega_1}\sin\tau\omega_1\right)e^{-\lambda\tau}, \quad E = \tilde{D}(2\lambda\Omega^2 - D_2)^{-1}$$

in agreement with the result (4.24) of West et al.[12]). The equilibrium correlations (6.16) define a correlation matrix C_τ

$$C_\tau = \begin{pmatrix} \langle x^2\rangle_\tau^{\rm eq} & \langle xp\rangle_\tau^{\rm eq} \\ \langle px\rangle_\tau^{\rm eq} & \langle p^2\rangle_\tau^{\rm eq} \end{pmatrix}, \tag{6.17}$$

which satisfies

$$C_\tau = C_0(\mathrm{e}^{\tau A_0})^+. \tag{6.18}$$

Here C_0 is the matrix of equal-time equilibrium second moments. In other words, the fluctuations satisfy the regression theorem[13]), hence a quantity as $\langle x^2\rangle_\tau^{\rm eq}$ can be calculated as follows

$$\langle x^2\rangle_\tau^{\rm eq} = \langle x_0\langle x(\tau)\rangle_{u_0}\rangle^{\rm eq}, \tag{6.19}$$

meaning that one should first evaluate $\langle x(\tau)\rangle$ from (6.1) with fixed initial condition u_0 ($\langle x(\tau)\rangle_{u_0}$ is not affected by the fluctuations but only involves the unperturbed matrix A_0 [1,12])) and subsequently average $x_0\langle x(\tau)\rangle_{u_0}$ over initial values u_0 distributed according to the equilibrium distribution. However, that this is a correct procedure in this case is due to the fact that the process (x, p) is Markovian if both γ and f in (6.1) are delta-correlated. It is no longer true (at least in general) if the process is not Markovian. Let us take an explicit example. From the exact result (6.13) one can conclude that in this case an infinite autocorrelation time of the additive noise term $f(t)$ in (6.1a) is allowed* since the system still equilibrates as $\lambda > 0$ (however, a finite correlation time of $A_1(t)$ in (6.1a) remains essential*). So if we take $\phi(s) = 1$, we find from (6.15)

$$\langle x^2\rangle_\tau^{\rm eq} = \frac{E}{\Omega^2}\left\{-\frac{2D_2}{\omega_1^2}\sin^2\tau\omega_1 + 4\lambda\right\}, \tag{6.20}$$

while

$$\langle x_0\langle x(\tau)\rangle_{u_0}\rangle^{\rm eq} = E\left(\frac{4\lambda}{\Omega^2}\right)\left\{\mathrm{e}^{-\lambda\tau}\left(\cos\omega_1\tau + \frac{\lambda}{\omega_1}\sin\omega_1\tau\right)\right\}. \tag{6.21}$$

To remedy this, one would have to replace (6.1a) by an equation containing a memory kernel such that the fluctuation–dissipation relation is restored[12,14]).

7. The two-time distribution function

The method of section 2 can also be exploited to derive expansions for the *two-time probability-density function* $P_{2/1}(ut\,;u't'/u_0t_0)(t \geq t' \geq t_0)$ of u.

* Our general assumptions made previously concerned only those auto- and crosscorrelation times, where the multiplicative process $A_1(t)$ was involved (see section 4.2).

To this end consider (1.1) with a sure initial condition u_0. The quantity

$$\rho(u, t) = \delta(u(t) - u) \tag{7.1}$$

with initial condition $\rho(u, t_0) = \delta(u - u_0)$, satisfies

$$\frac{\partial \rho(u, t)}{\partial t} = L_u(t)\rho(u, t), \tag{7.2}$$

where the operator $L_u(t)$ acts on the u-dependence of ρ and is given by

$$L_u(t)\ldots = -\frac{\partial}{\partial u} \cdot [\{A(t)u + f(t)\}\ldots]. \tag{7.3}$$

The two-point distribution function is now calculated as

$$P_{2/1}(ut\,;u't'/u_0t_0) = \langle \rho(u, t)\rho(u', t')\rangle$$

$$= \left\langle \left[\overleftarrow{T} \exp \int_{t_0}^{t} \mathrm{d}s\, L_u(s)\right]\left[\overleftarrow{T} \exp \int_{t_0}^{t'} \mathrm{d}s'\, L_{u'}(s')\right]\right\rangle. \tag{7.4}$$

Using the fact that the crucial identity (2.11) is in this case replaced by

$$[L_u(t), L_{u'}(t')] = 0 \quad (\text{all } t, t'), \tag{7.5}$$

where u and u' are regarded as independent variables, we can write on the analogy of (2.21)

$$P_{2/1} = \left[\overleftarrow{T} \exp \int_{t'}^{t} \mathrm{d}s\, \langle L_u(s) : Q_{L_u}(s/t')Q_{L_u+L_{u'}}(t'/t_0) : \rangle_{\mathrm{p}}\right]$$

$$\times \left[\overleftarrow{T} \exp \int_{t_0}^{t'} \mathrm{d}s'\, \langle \{L_u(s') + L_{u'}(s')\} : Q_{L_u+L_{u'}}(s'/t_0) : \rangle_{\mathrm{p}}\right]. \tag{7.6}$$

From (7.6) one can again derive the following "forward" and "backward" equations

$$\frac{\partial P_{2/1}}{\partial t} = \langle L_u(t) : Q_{L_u}(t/t')Q_{L_u+L_{u'}}(t'/t_0) : \rangle_{\mathrm{p}}\, P_{2/1}, \tag{7.7a}$$

$$\frac{\partial P_{2/1}}{\partial t'} = \langle : Q_{L_u}(t/t')L_u(t')Q_{L_u+L_{u'}}(t'/t_0) : \rangle_{\mathrm{p}}\, P_{2/1}. \tag{7.7b}$$

If $u(t)$ is a Markov process, i.e. if both $A(t)$ and $f(t)$ in (7.3) have delta-correlated cumulants, the operator acting on $P_{2/1}$ in the RHS of (7.7a) depends only on u and t, while that in (7.7b) depends on u, u' and t'. In this case (7.7a) can also be written as

$$\frac{\partial P_{2/1}(ut\,;\,u't'/u_0t_0)}{\partial t} = \int du''\, W_t(u/u'')P_{2/1}(u''t\,;\,u't'/u_0t_0), \tag{7.8}$$

where W_t is the Master operator and (7.7a) is just the Kramers–Moyal expansion of the equation (7.8) (for example in the case of the harmonic oscillator problem in section 6, W_t is the same operator as that occurring in eqs. (7.17) and (7.18) of I). Of course one could also derive an expansion for $\partial P_{2/1}(u,\, t+\tau\,;\, u't/u_0t_0)/\partial t$ which will yield the analogue of (2.30) with A' and A'' replaced by L_u and $L_{u'}$ respectively.

Finally we remark that in principle the method of this section can also be applied to nonlinear equations, although in that case one cannot extract closed equations for the correlation functions from (7.7).

8. Comparison with the smoothing method

In this section we will discuss the integro-differential equation for the correlation function $C_u(t, t+\tau)$ which was derived by Morrison and McKenna[5]) using the so-called "smoothing method" and by Agarwal[8]) via projection operator methods. It will be shown first that by considering the characteristic functional of $A(t)$ again, as we did in section 2, a formally exact integral equation for the correlation function can be derived. By differentiating and retaining only the lowest two orders in $A(t)$ we arrive at the same result as obtained by the smoothing method. Moreover, it can be shown in general from the exact result that this approximate equation is exact if $A(t) = \xi(t)B(t)$, where $B(t)$ is a sure matrix and $\xi(t)$ a stationary dichotomic Markov process (also called random telegraph process; in the following abbreviated as D.M.P.). This is in agreement with the results of Agarwal and those of Morrison and McKenna who proved this statement by comparison with the results of the phase space method[5]).

8.1. *The integral equation for the correlation function*

Consider the homogeneous equation

$$\dot{u}(t) = A(t)u(t) \tag{8.1}$$

with a sure initial condition u_0. In I we showed that the average of u obeys the integro-differential equation

$$\langle \dot{u}(t)\rangle = \int_{t_0}^{t} ds\, \Gamma_A(t/s)\langle u(s)\rangle, \tag{8.2}$$

where

$$\Gamma_A(t/s) = \langle A(s)\rangle \delta_+(t-s) + \int_{t_0}^{t} \mathrm{d}s\, \langle A(t) : Q_A(t/s) : A(s)\rangle_t \tag{8.3}$$

and for any continuous test function f: $\int_0^t \mathrm{d}s\, f(s)\delta_+(t-s) = f(t)$. In (8.3) the symbol $\langle \ldots \rangle_t$ indicates the so-called "totally time ordered cumulant", defined in I as

$$\langle A(t)A(t_1)\ldots A(t_m)\rangle_t = \langle A(t)(1-\mathscr{P})A(t_1)(1-\mathscr{P})\ldots(1-\mathscr{P})A(t_m)\rangle, \tag{8.4}$$

where $\mathscr{P}$ denotes the averaging operator: $\mathscr{P}\ldots = \langle\ldots\rangle$. Before taking the t-ordered cumulant in (8.3) one should first expand $Q_A(t/s)$ in powers of A (indicated by the colons) and calculate the cumulants for each separate term.

Integrating (8.2) one finds the integral equation ("Dyson-equation"[3])

$$\langle u(t)\rangle = u_0 + \int_{t_0}^{t} \mathrm{d}s \int_{t_0}^{s} \mathrm{d}s'\, \Gamma_A(s/s')\langle u(s')\rangle \tag{8.5}$$

and since $\langle u(t)\rangle = \langle Q_A(t/t_0)\rangle u_0$ we infer from (8.5)

$$\langle Q_A(t/t_0)\rangle = 1 + \int_{t_0}^{t} \mathrm{d}s \int_{t_0}^{s} \mathrm{d}s'\, \Gamma_A(s/s')\langle Q_A(s'/t_0)\rangle. \tag{8.6}$$

In exactly the same way as done in section 2 we can generalize (8.6) to the characteristic functional of a process $B(t)$ with compact support

$$\langle Q_B(\infty/t_0)\rangle = 1 + \int_{t_0}^{\infty} \mathrm{d}s \int_{t_0}^{s} \mathrm{d}s'\, \Gamma_B(s/s')\langle Q_B(s'/t_0)\rangle. \tag{8.7}$$

For the matrix B we take again (2.13), so

$$\langle Q_B(\infty/t_0)\rangle = \langle Q_{A''}(t+\tau/t_0)Q_{A'}(t/t_0)\rangle = \langle Q_{A''}(t+\tau/t)Q_{\bar{A}}(t/t_0)\rangle \tag{8.8}$$

and we find from (8.7)

$$\begin{aligned}\langle Q_B(\infty/t_0)\rangle = 1 &+ \int_{t_0}^{t} \mathrm{d}s \int_{t_0}^{s} \mathrm{d}s'\, \Gamma_{\bar{A}}(s/s')\langle Q_{\bar{A}}(s'/t_0)\rangle \\ &+ \int_{t}^{t+\tau} \mathrm{d}s \int_{t_0}^{t} \mathrm{d}s'\, \Delta_A(s, t/s')\langle Q_{\bar{A}}(s'/t_0)\rangle \\ &+ \int_{t}^{t+\tau} \mathrm{d}s \int_{t}^{s} \mathrm{d}s'\, \Gamma_{A''}(s/s')\langle Q_{A''}(s'/t)Q_{\bar{A}}(t/t_0)\rangle,\end{aligned} \tag{8.9}$$

where

$$\Delta_A(s, t/s') = \langle A''(s) : Q_{A'}(s/t)Q_{\tilde{A}}(t/s') : \tilde{A}(s')\rangle_t. \tag{8.10}$$

Remembering that

$$C_u(t, t+\tau) \equiv \langle u(t) \otimes u(t+\tau)\rangle = \langle Q_{A'}(t+\tau/t_0)Q_{A'}(t/t_0)\rangle u_0 \otimes u_0, \tag{8.11}$$

one has from (8.9) the following integral equation ("Bethe–Salpeter equation"[3])) for C_u:

$$C_u(t, t+\tau) = u_0 \otimes u_0 + \int_{t_0}^{t} ds \int_{t_0}^{s} ds'\, \Gamma_{\tilde{A}}(s/s')C_u(s', s')$$

$$+ \int_{t}^{t+\tau} ds \int_{t_0}^{t} ds' \Delta_A(s, t/s')C_u(s', s') + \int_{t}^{t+\tau} ds \int_{t}^{s} ds'\, \Gamma_{A'}(s/s')C_u(t, s'). \tag{8.12}$$

This integral equation is the central result of this section, from which earlier results of Morrison and McKenna[5]) and Agarwal[8]) immediately follow. The equal-time second moments $C_u(s', s')$ in (8.12) can be calculated from an equation like (8.2).

Although (8.12) is formally exact, it is in general not allowed to cut off the expansions of Γ_A and Δ_A unless $\alpha\tau_c$ is small, where α and τ_c are again the magnitude and correlation time of the fluctuations in $A(t)$[3,15]).

Of course one can study the case with inhomogeneous and/or initial value random terms by starting from (4.3) or (4.14) instead of (8.1), and applying the result (8.12) to the new equations.

8.2 *The integro-differential equation for C_u*

Simple differentiation of (8.12) with respect to the variable τ yields*

$$\frac{\partial}{\partial\tau} C_u(t, t+\tau) = \int_{t_0}^{t} ds\, \Delta_A(t+\tau, t/s)C_u(s, s) + \int_{t}^{t+\tau} ds\Gamma_{A'}(t+\tau/s)C_u(t, s). \tag{8.13}$$

This equation should be supplemented with the initial condition $C_u(t, t)$. Taking A of order α and retaining terms up to order α^2 one finds from (8.13),

* Compare also eq. (2.21) of ref. 8.

(8.3) and (8.10)

$$\frac{\partial}{\partial\tau} C_u(t, t+\tau) = \int_{t_0}^{t} ds\, \alpha^2 \langle\langle A''(t+\tau)\tilde{A}(s)\rangle\rangle C_u(s, s)$$

$$+ \alpha\langle A''(t+\tau)\rangle C_u(t, t+\tau) + \int_{t}^{t+\tau} ds\, \alpha^2 \langle\langle A''(t+\tau)A''(s)\rangle\rangle C_u(t, s). \tag{8.14}$$

The equation (8.14) is completely equivalent (apart from the absence of an unperturbed matrix) with the result (9.26) of Morrison and McKenna[5]) obtained by the smoothing method. It is now also clear from (8.13) how one can proceed to higher orders in α. However, to find the correlation function one has to solve an integro-differential equation which is of course far more difficult than solving the differential equations (2.24) or (2.30). Moreover, if $\alpha\tau_c \ll 1$, the solution of (8.14) will be no more accurate to the order considered here than that of (2.24)[15]), except in special cases. One such special case occurs when $A(t) = \xi(t)B(t)$, $B(t)$ a sure matrix and $\xi(t)$ a D.M.P. Namely, all higher orders in (8.13) contain t-cumulants of the form $(m+n \geq 1)$

$$\langle A''(t+\tau)A''(t_1)\ldots A''(t_m)\tilde{A}(s_1)\ldots\tilde{A}(s_n)\tilde{A}(s)\rangle_t$$
$$= \langle\xi(t+\tau)\xi(t_1)\ldots\xi(t_m)\xi(s_1)\ldots\xi(s_n)\xi(s)\rangle_t B''(t+\tau)\ldots\tilde{B}(s) \tag{8.15a}$$

or $(m \geq 1)$

$$\langle A''(t+\tau)A''(t_1)\ldots A''(t_m)A''(s)\rangle_t$$
$$= \langle\xi(t+\tau)\xi(t_1)\ldots\xi(t_m)\xi(s)\rangle_t B''(t+\tau)\ldots B''(s) \tag{8.15b}$$

which all vanish since third and higher t-cumulants of a D.M.P. are zero[1,16]).

Eq. (8.14) can be solved by Laplace transforms since the correlation functions of $A(t) = \xi(t)B(t)$ depend only on time differences. In fact one obtains a solution in the form of a sum of exponentials[16]). The exponents must be determined as roots of a secular equation, i.e. poles of the Laplace transform. If these roots are computed to order α^2 the result is just what one would get by solving (2.24) with M_A approximated to order $\alpha^2\tau_c$ (where $\tau_c \sim \gamma^{-1}$, γ being the same as in (5.5)).

Finally we remark that the smoothing method does not seem to give the exact answer for the correlation function in case of the D.M.P., if this method is used to derive an integro-differential equation for $C_u(t, t+\tau)$, considered as a function of t[17]). And indeed, although (8.12) with $\Gamma_{\tilde{A}}$, $\Gamma_{A''}$ and Δ_A expanded up to order α^2 is exact for a D.M.P., one does *not* obtain a *closed* integro-differential equation from it by differentiation with respect to the variable t (the last term in the RHS of (8.12) is responsible for this).

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Appendix A

Multi-time averages

In this appendix we will generalize the results of section 2 to the calculation of multi-time averages. Since the derivation is completely analogous to the two-time case we will omit the details. The inhomogeneous case (1.1) can be handled with the method of section 4.

So consider the n-time correlation function of the solution of (2.1):

$$C_u(t_1, t_2, \dots, t_n) = \langle u(t_1) \otimes u(t_2) \otimes \dots \otimes u(t_n)\rangle$$
$$= \langle \psi(t_1, t_2, \dots, t_n)\rangle u_0 \otimes u_0 \otimes \dots \otimes u_0 \quad (t_1 \geqslant t_2 \geqslant \dots \geqslant t_n), \tag{A.1}$$

where

$$\psi(t_1, t_2, \dots, t_n) = \prod_{i=1}^{n} Q_{A^{(i)}}(t_i) \tag{A.2}$$

and

$$A^{(i)}(t_i) = \hat{1} \otimes \hat{1} \otimes \dots \otimes \overset{\overset{i}{\downarrow}}{A}(t_i) \otimes \hat{1} \dots \otimes \hat{1}. \tag{A.3}$$

So in (A.3) the i^{th} operator is A while the others are unit matrices of the same dimension as A.

Instead of (2.11) we now have

$$[A^{(i)}(t), A^{(j)}(t')] = 0, \quad i \neq j, \text{ all } t, t' \tag{A.4}$$

and

$$\psi(t_1, t_2, \dots, t_n) = Q_B(\infty/t_0) \tag{A.5}$$

with

$$B(s) = \sum_{i=1}^{n} \theta(t_i - s)\, A^{(i)}(s). \tag{A.6}$$

Taking the average of (A.5) we have

$$\langle\psi(t_1, t_2, \ldots, t_n)\rangle = \bar{T} \exp \int_{t_0}^{\infty} \mathrm{d}s\, K_B(s/t_0) \tag{A.7}$$

with

$$K_B(s/t_0) = \langle B(s) : Q_B(s/t_0) : \rangle_{\mathrm{p}}. \tag{A.8}$$

Symbolically (A.7) can be written as

$$\langle\psi(t_1, t_2, \ldots, t_n)\rangle = \bar{T} \exp\left[\langle : Q_B(\infty/t_0) : \rangle_{\mathrm{p}} - 1\right], \tag{A.9}$$

where it has always to be understood that the $\bar{T}$ operator acts on the first time variable of every expression in the exponent.

Eq. (2.21) is now generalized to

$$\langle\psi(t_1, t_2, \ldots, t_n)\rangle = \bar{T} \exp\left[\left\langle \prod_{l=1}^{n} : Q_{A_l}(t_l/t_{l+1}) : \right\rangle_{\mathrm{p}} - 1\right], \tag{A.10}$$

where $t_{n+1} \equiv t_0$ and

$$A_l = \sum_{i=1}^{l} A^{(i)}. \tag{A.11}$$

Differentiation with respect to t_κ yields, using (A.1)

$$\frac{\partial}{\partial t_\kappa} C_u(t_1, t_2, \ldots, t_n) = M_A^{(\kappa)}(t_1, t_2, \ldots, t_n/t_0)\, C_u(t_1, t_2, \ldots, t_n), \tag{A.12}$$

where

$$M_A^{(\kappa)}(t_1, t_2, \ldots, t_n/t_0) = \left\langle \prod_{l=1}^{\kappa-1} : Q_{A_l}(t_l/t_{l+1}) : A^{(\kappa)}(t_\kappa) \prod_{l=\kappa}^{n} : Q_{A_l}(t_l/t_{l+1}) : \right\rangle_{\mathrm{p}}. \tag{A.13}$$

Putting $t_l = t + \tau_l$ in (A.10) with $\tau_l \geq \tau_{l+1}$, and differentiating with respect to t one finds

$$\frac{\partial}{\partial t} C_u(t+\tau_1, t+\tau_2, \ldots, t+\tau_n) = \sum_{\kappa=1}^{n} M_A^{(\kappa)}(t+\tau_1, t+\tau_2, \ldots, t+\tau_n/t_0)\, C_u(t+\tau_1, \ldots, t+\tau_n) \tag{A.14}$$

with all $M_A^{(\kappa)}$ as defined in (A.13). To calculate these to a given order one should again use the prescription given in section 2.

Again this method can also be used to derive the equations satisfied by the *n-time distribution functions* $P_{n/1}(u_1t_1, \ldots, u_nt_n/u_0t_0)$ (just replace $A^{(i)}$ by L_{u_i}, see also section 7).

Appendix B

In this appendix we will briefly indicate how the result (5.8) can be checked. Equation (5.1) defines a composite stochastic process (u, ξ) which is Markovian. So the following phase–space equation for its two-time distribution holds $(t > t')$

$$\frac{\partial}{\partial t} P_2(u\xi t\ ; u'\xi' t') = -\frac{\partial}{\partial u} \cdot [(\sigma_z u + \alpha\xi\sigma_x u) P_2(u\xi t ; u'\xi' t')] + \sum_{\xi''=\pm 1} W_{\xi\xi''} P_2(u\xi'' t\ ; u'\xi' t'), \tag{B.1}$$

where

$$W_{\xi\xi''} = \gamma - 2\gamma\delta_{\xi,\xi''}. \tag{B.2}$$

Defining the "marginal correlation functions" by

$$\langle u(t) \otimes u(t')\rangle_\xi = \sum_{\xi'} \int \mathrm{d}u \int \mathrm{d}u'\, (u \otimes u') P_2(u\xi t\ ; u'\xi' t') \tag{B.3}$$

one finds for the vector $W(t, t') = \mathrm{Col}\{W^+_{11}, W^-_{11}, W^+_{21}, W^-_{21}, W^+_{12}, W^-_{12}, W^+_{22}, W^-_{22}\}$ where $W^{\pm}_{ij} = \langle u_i(t) u_j(t')\rangle_{\pm}$ the following matrix equation

$$\frac{\partial}{\partial t} W(t, t') = M W(t, t'), \tag{B.4}$$

where M is the following 8×8-matrix

$$M = \left(\begin{array}{c|c} M' & \emptyset \\ \hline \emptyset & M' \end{array}\right), \quad M' = \left(\begin{array}{c|c} 1 - \gamma + \gamma\sigma_x & \alpha\sigma_z \\ \hline \alpha\sigma_z & -1 - \gamma + \gamma\sigma_x \end{array}\right). \tag{B.5}$$

From (B.4) one has

$$W(t + \tau, t) = \mathrm{e}^{\tau M} W(t, t) \tag{B.6}$$

and we need $W(t, t)$. From (B.1) also follows an equation for

$$\langle u(t) \otimes u(t)\rangle_\xi = \int \mathrm{d}u\, (u \otimes u) P_1(u\xi t) = \langle u(t) \otimes u(t')\rangle_\xi \Big|_{t'=t}. \tag{B.7}$$

In components

$$\frac{\mathrm{d}}{\mathrm{d}t} W(t, t) = N W(t, t), \tag{B.8}$$

where

$$N = -\gamma \hat{1} + \left(\begin{array}{c|c} N_1^+ & N_2 \\ \hline N_2 & N_1^- \end{array}\right) \tag{B.9}$$

and

$$N_1^{\pm} = \left(\begin{array}{c|c} \pm 2 + \gamma\sigma_x & \alpha\sigma_z \\ \hline \alpha\sigma_z & \gamma\sigma_x \end{array}\right); \quad N_2 = \left(\begin{array}{c|c} \alpha\sigma_z & \emptyset \\ \hline \emptyset & \alpha\sigma_z \end{array}\right). \tag{B.10}$$

In (B.9) $\hat{1}$ denotes the 8×8 unit-matrix.

Now $W(t+\tau, t)$ can be calculated as follows

$$W(t+\tau, t) = \mathrm{e}^{\tau M}\, \mathrm{e}^{tN}\, W(0,0), \tag{B.11}$$

where $W(0,0)$ is computed from the initial distribution

$$P(u, \xi, 0) = \delta(u - u_0)\,\{\tfrac{1}{2}\delta_{\xi,+} + \tfrac{1}{2}\delta_{\xi,-}\}. \tag{B.12}$$

From the solution (B.11) the desired correlation functions are obtained as

$$\langle u_i(t+\tau) u_j(t)\rangle = \sum_{\xi=\pm 1} \langle u_i(t+\tau) u_j(t)\rangle_\xi \quad (i, j = 1, 2). \tag{B.13}$$

When these are again arranged in a vector U as in (5.7) they can be shown to satisfy a differential equation as in (5.8) where the decoupling of first and last two components is now exact to all orders ($t \gg \tau_c \sim \gamma^{-1}$). Retaining only terms up to order α^2 one recovers after a rather tedious calculation the results (5.8) and (5.9).

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