INHOMOGENEOUS LINEAR RANDOM DIFFERENTIAL EQUATIONS WITH MUTUAL CORRELATIONS BETWEEN MULTIPLICATIVE, ADDITIVE AND INITIAL-VALUE TERMS

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The cumulant expansion for linear stochastic differential equations is extended to the general case in which the coefficient matrix, the inhomogeneous part and the initial condition are all random and, moreover, statistically interdependent. The expansion now involves not only the autocorrelation functions of the coefficient matrix (as in the homogeneous case) but also crosscorrelation functions of the coefficient matrix with the inhomogeneous part and with the initial value term. As a first illustration we consider an exactly solvable stochastic differential equation with initial correlations and compare the exact solution with that of the cumulant expansion. Secondly we show in general how the method can be used for the calculation of second moments, and treat the harmonic oscillator with random frequency and driving term as an example.

1. Introduction

Consider a system of linear differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = A(t, \omega)u(t) + f(t, \omega),$$ (1.1a)

$$u(t_0) = u_0(\omega),$$ (1.1b)

where $u(t)$ is a vector. The coefficient matrix (or operator) $A(t, \omega)$, the inhomogeneous vector $f(t, \omega)$ and the initial vector $u_0(\omega)$ are all regarded as random quantities of which the joint probability distribution (or other statistical characteristics such as joint moments, cumulants etc.) are prescribed. The random nature of these quantities is indicated by the parameter ω, which is an element of a set Ω which, together with a σ-algebra Σ of subsets of Ω and a probability measure P on Σ, constitutes a probability space. In the following we will often omit the parameter ω for brevity.

The elements of the matrix $A(t)$ and the vector $f(t)$ are random processes, i.e. they describe fluctuations in time. They are not necessarily stationary. The random matrix process $A(t)$ and the vector process $f(t)$ have been called *multiplicative* and *additive* noise respectively[1], because they enter (1.1a) in a multiplicative, resp. additive way. This distinction is only meaningful for

linear equations, because in nonlinear equations also the noise $f(t)$ will come in nonadditively in the solution $u(t)$.

The solution $u(t, \omega)$ of (1.1) is again a random process. The problem is to find the statistical properties of $u(t)$ as for example its average $\langle u(t) \rangle$, when the statistical characteristics of $A(t)$, $f(t)$ and u_0 are given. Here the angular brackets $\langle \ldots \rangle$ denote an average with respect to the probability measure $P: \langle u(t) \rangle = \int_\Omega u(t, \omega) P(\omega) \, d\omega$.

The case in which $A(t)$, $f(t)$ and u_0 are *statistically independent* has been studied before[2-4]. The result, which will be described in more detail in the next section, was that in the case of small and rapid fluctuations in $A(t)$, a linear differential equation for the average $\langle u(t) \rangle$ exists for fixed initial condition $u(t_0) = u_0$,

$$\frac{d}{dt} \langle u(t) \rangle = K(t/t_0) \langle u(t) \rangle + \langle f(t) \rangle. \tag{1.2}$$

Here $K(t/t_0)$ is a non-random matrix, which is obtained as an infinite series of terms in successive powers of the parameter $\alpha \tau_c$ (sometimes called the Kubo number), where τ_c is the (short, but non-zero) autocorrelation time of the fluctuations in $A(t)$ and α a measure for their strength. Moreover $K(t/t_0)$ becomes independent of the initial time t_0 as soon as $|t - t_0| \gtrsim \tau_c$.

In this article we are concerned with the case in which A, f and u_0 are *mutually correlated*. We find again that $\langle u(t) \rangle$ obeys a linear differential equation, provided that $\alpha \tau_c$ is small,

$$\frac{d}{dt} \langle u(t) \rangle = K(t/t_0) \langle u(t) \rangle + F(t/t_0) + I(t/t_0), \tag{1.3}$$

where both the matrix $K(t/t_0)$ and the vectors $F(t/t_0)$ and $I(t/t_0)$ are found in successive powers of $\alpha \tau_c$. K involves the moments of $A(t)$ alone, and is the same as in (1.2); F involves the joint moments of A and f and I those of A and u_0. In this case there are three correlation times involved: the autocorrelation time of $A(t)$, the crosscorrelation time of $A(t)$ with $f(t)$, and that of $A(t)$ with u_0. All of them assumed to be finite, and the largest is denoted by τ_c. Then we find that after a transient time of order τ_c, K and F become independent of t_0 while I vanishes, so (1.3) becomes

$$\frac{d}{dt} \langle u(t) \rangle = \bar{K}(t) \langle u(t) \rangle + \bar{F}(t) \quad (t \gg \tau_c). \tag{1.4}$$

If this equation is to be of any value, the transient time should be short compared to the duration of the process $u(t)$ itself.

A model of type (1.1) with correlated multiplicative and additive noise recently arose within the context of meteorological investigations[5]. The first

two terms of F were derived by a method different from ours, as discussed by Soong[6]).

Apart from specific models as the one mentioned above, another motivation for admitting a correlation between A and f in (1.1) lies in the following. If one considers the second moments of $u(t)$, satisfying (1.1) with A and f *uncorrelated*, then these second moments can again be studied by an equation of this type, but now with a new A' and f' which *are* correlated.

In the following we first review some basic results for the homogeneous case (section 2). Then we derive the results for the inhomogeneous case (section 3), which are summarized in section 4. In section 5 an example is given for which one can explicitly show the decay of initial correlations between A and u_0. In section 6 we give the general result for the second moments of $u(t)$, if A and f in the original equation for $u(t)$ are uncorrelated. As an application we show that the expansion (1.3) reproduces the results of West et al. for the damped harmonic oscillator with stochastic frequency[7,8]). In the final section an alternative derivation of some of the results of section 3 is given via the so called "stochastic Liouville equation".

2. Summary of previous results for the homogeneous case

We briefly review here the results of Van Kampen[2,3]) for the homogeneous case $(f(t) \equiv 0)$ with non-random initial conditions.

2.1. *The cumulant expansion*

The solution of

$$\frac{d}{dt} u(t) = \alpha A(t)u(t), \tag{2.1}$$

with initial condition $u(t_0) = u_0$ (fixed) is given by

$$u(t) = \tilde{T}\left[\exp\left\{\alpha \int_{t_0}^{t} ds A(s)\right\}\right]u_0, \tag{2.2}$$

where $\tilde{T}$ denotes time-ordering (latest times to the left) with respect to the operators $A(.)$. From (2.2) one obtains the *moment expansion*

$$\langle u(t)\rangle = \left\{1 + \alpha \int_{t_0}^{t} dt_1 \langle A(t_1)\rangle + \alpha^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \langle A(t_1)A(t_2)\rangle + \cdots\right\}u_0. \tag{2.3}$$

The series (2.3) in general converges very slowly and in particular, finite approximations are only valid for a limited time of order α^{-1}: the expansion (2.3) is nonuniform as $t \to \infty$. To get a more uniform expansion, which avoids secular terms (although it may be asymptotic) one expands the average of (2.2) in *cumulants*, rather than in moments.

To this end, operators $K_1, K_2, \ldots$ are constructed in successive steps such that the average $\langle u(t) \rangle$ can be written as

$$\langle u(t) \rangle = \hat{T} \left[\exp \left\{ \sum_{m=1}^{\infty} \alpha^m \int_{t_0}^{t} ds K_m(s/t_0) \right\} \right] u_0, \tag{2.4}$$

where the time-ordering $\hat{T}$ now acts with respect to the operators $K_m(./t_0)$. The remarkable implication of (2.4) is that the average $\langle u(t) \rangle$ itself again obeys a differential equation

$$\frac{d}{dt} \langle u(t) \rangle = K(t/t_0) \langle u(t) \rangle, \tag{2.5}$$

where

$$K(t/t_0) = \sum_{m=1}^{\infty} \alpha^m K_m(t/t_0). \tag{2.6}$$

It turns out that the operators K_m are given by

$$K_m(t/t_0) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{m-2}} dt_{m-1} C_m(t, t_1, \ldots, t_{m-1}), \tag{2.7a}$$

with

$$C_m(t, t_1, \ldots, t_{m-1}) = \langle A(t) A(t_1) \ldots A(t_{m-1}) \rangle_p, \tag{2.7b}$$

where by definition $K_1(t/t_0) = C_1(t) \equiv \langle A(t) \rangle$. The expression (2.7b) is the so-called "*time-ordered cumulant*", which is a certain combination of moments of $A(.)$ with a specific ordering of the time-variables. As first pointed out by Kubo, various orderings are possible[10]). The one meant in (2.7b) is the so-called "partial time-ordering"[11,12]). Consequently, the ordered cumulant (2.7b) will be called "*partially time-ordered cumulant*" (p-(ordered) cumulant for short) and it is indicated by $\langle \ldots \rangle_p$. In section 3 another type of ordered cumulant will be encountered, namely the "*totally time-ordered cumulant*" corresponding to "total time-ordering"[11]) (or "chronological time-ordering"[12])). This one will be denoted as t-(ordered) cumulant and indicated by $\langle \ldots \rangle_t$. For a relation between these two types of ordered cumulants, see appendix A. If not explicitly stated, the term "ordered cumulant" is to be understood as "p-ordered cumulant".

The connection between the moments and p-ordered cumulants of A can be

found by comparing equal powers of α in the expansion of (2.3) and (2.4)[1]. The rules for constructing the p-ordered cumulants from the moments were first given by Van Kampen[2]. For later use we give them here in a slightly more general form. Consider m (non-commuting) time-dependent quantities (operators, matrices, vectors) $A_0(t), A_1(t), \ldots, A_{m-1}(t)$. The ordered cumulant

$$\langle A_0(t)A_1(t_1) \ldots A_{m-1}(t_{m-1})\rangle_p, \quad t \geq t_1 \geq \cdots \geq t_{m-1}, \tag{2.8}$$

is obtained in the following way:

i) Write a sequence of m-dots.

ii) Partition them into subsequences $\langle \ldots \rangle$ (denoting moments) by inserting angular brackets in all possible ways (excluding empty subsequences).

iii) For each partition consisting of p subsequences supply a factor $(-)^{p-1}$.

iv) For each partition write a zero on the first dot, and any permutation of the numerals $1, 2, \ldots, m-1$ on the remaining dots, subject to the condition that in each subsequence they must not decrease.

v) Replace each numeral n written on the kth dot by the quantity $A_{k-1}(t_n)$ (the numeral 0 stands for $t_0 \equiv t$).

For example (for the interpretation of the numerals see rule (v)):

$$\langle 0\rangle_p = \langle 0\rangle, \quad \langle 01\rangle_p = \langle 01\rangle - \langle 0\rangle\langle 1\rangle, \tag{2.9a}$$

$$\langle 012\rangle_p = \langle 012\rangle - \langle 0\rangle\langle 12\rangle - \langle 01\rangle\langle 2\rangle - \langle 02\rangle\langle 1\rangle + \langle 0\rangle\langle 1\rangle\langle 2\rangle + \langle 0\rangle\langle 2\rangle\langle 1\rangle. \tag{2.9b}$$

The reverse transformation from cumulants to moments is given by the rules (i)–(v) with the following modifications:

– in (ii) replace $\langle \ldots \rangle$ by $\langle \ldots \rangle_p$
– omit (iii)
– in (iv) add the condition that also the first numerals in successive subsequences do not decrease.

For example

$$\langle 0\rangle = \langle 0\rangle_p, \quad \langle 01\rangle = \langle 01\rangle_p + \langle 0\rangle_p\langle 1\rangle_p, \tag{2.10a}$$

$$\langle 012\rangle = \langle 012\rangle_p + \langle 0\rangle_p\langle 12\rangle_p + \langle 01\rangle_p\langle 2\rangle_p + \langle 02\rangle_p\langle 1\rangle_p + \langle 0\rangle_p\langle 1\rangle_p\langle 2\rangle_p. \tag{2.10b}$$

The p-ordered cumulant (2.8) reduces to that of Van Kampen's definition if $A_0 = A_1 = \cdots = A_{m-1} = A$. If all quantities commute at different times it is identical to the ordinary cumulant in the many variable case[13]. For a scalar Gaussian process, all cumulants beyond the second vanish (the same holds for a delta-correlated vectorial Gaussian process[1]).

2.2. Cluster property and large time estimates

An important property of the ordered cumulant is the "cluster property", meaning that the cumulants vanish as soon as the moments factorize[3,14]). More precisely, suppose that $A(t)$ has a finite auto-correlation time τ_c, such that all matrix elements of $A(t)$ are statistically independent of those of $A(t')$ if $|t - t'| \gtrsim \tau_c$. Then the cumulants $\langle A(t)A(t_1) \dots A(t_{m-1})\rangle_p$, with $t \geq t_1 \geq \dots \geq t_{m-1}$, vanish if there is a gap between two successive times which is large compared to τ_c (in general this is strictly valid only asymptotically).

This cluster property implies that for large time the contribution to the integrals in (2.7a) comes essentially from the region

$$t - \tau_c \lesssim t_1 \lesssim t; \dots ; t_{m-2} - \tau_c \lesssim t_{m-1} \lesssim t_{m-2}.$$

So t and t_{m-1} are at most a distance $(m-1)\tau_c$ apart. Therefore if $|t - t_0| \gtrsim (m-1)\tau_c$ the expression (2.7a) approaches

$$\bar{K}_m(t) = \int\limits_{-\infty}^{t} dt_1 \int\limits_{-\infty}^{t_1} dt_2 \dots \int\limits_{-\infty}^{t_{m-2}} dt_{m-1} C_m(t, t_1, \dots, t_{m-1}), \tag{2.11}$$

which is *independent of the initial time* t_0. From this we also deduce that for large time the mth order term in (2.6) is of order $\alpha^m \tau_c^{m-1}$ (assuming that A is of order unity). The resulting equation

$$\frac{d}{dt} \langle u(t)\rangle = \bar{K}(t)\langle u(t)\rangle \tag{2.12}$$

is now satisfied by all solutions of (2.1) after a transient time of order τ_c, independent of the time at which the initial value u_0 is fixed. If in addition the process $A(t)$ is stationary or becomes stationary on the same time scale τ_c as the correlation functions decay (as is the case for so-called "switched-on processes"[15]), then $K(t/t_0)$ approaches a value $\bar{K}$ as $t - t_0 \gg \tau_c$ which is independent of t_0 and t.

3. The derivation of (1.3)

In this section the central result (1.3) will be derived by constructing first an integro–differential equation for the average $\langle u(t)\rangle$ which subsequently is turned into a differential equation. This method was used by Terwiel[4]) who gave an alternative derivation of the results of the previous section for the homogeneous case.

So consider (1.1) with $A(t)$ of the form

$$A(t) = A_0(t) + \alpha A_1(t),$$

where $A_0(t)$ is a sure matrix, and $A_1(t)$ random (not necessarily with zero mean). First we transform $u(t)$ to the interaction representation $v(t)$ by

$$u(t) = U(t/t_0)v(t),$$

where

$$U(t/t_0) = \bar{T}\left[\exp \int_{t_0}^{t} ds A_0(s)\right].$$

Then (1.1) transforms to

$$\frac{d}{dt} v(t) = \alpha V(t)v(t) + \alpha g(t), \qquad (3.1a)$$

with

$$A_1(t) = U(t/t_0)V(t)U^{-1}(t/t_0), \quad f(t) = U(t/t_0)\alpha g(t). \qquad (3.1b)$$

For convenience we took the magnitude of f to be also of order α. If we finally transform back to the original representation this α will disappear again. The eq. (3.1a) will be the starting point of the derivation.

3.1. The integro–differential equation for $\langle v(t)\rangle$

By well-known projection operator techniques[4]) one can derive from (3.1a) an integro–differential equation for the average $\langle v(t)\rangle$ with random initial condition $v(t_0) = u_0$. That is, defining the averaging-operator $\mathscr{P}$ by

$$\mathscr{P}v(t) = \langle v(t)\rangle,$$

one finds for the average $\langle v(t)\rangle$ the following equation*), where $\mathscr{Q} = 1 - \mathscr{P}$:

$$\frac{d}{dt}\langle v(t)\rangle = \alpha\langle V(t)\rangle\langle v(t)\rangle$$

$$+ \int_{t_0}^{t} ds\, \alpha^2 \left\langle V(t)\bar{T}\left[\exp \alpha \int_{s}^{t} ds'\mathscr{Q}V(s')\right]\mathscr{Q}V(s)\right\rangle\langle v(s)\rangle$$

$$+ \alpha\langle g(t)\rangle + \int_{t_0}^{t} ds\, \alpha^2\left\langle V(t)\bar{T}\left[\exp \alpha \int_{s}^{t} ds'\mathscr{Q}V(s')\right]\mathscr{Q}g(s)\right\rangle$$

$$+ \alpha\left\langle V(t)\bar{T}\left[\exp \alpha \int_{t_0}^{t} ds\mathscr{Q}V(s)\right]\mathscr{Q}u_0\right\rangle. \qquad (3.2)$$

* The first two lines of (3.2) to second order in α are identical with Bourret's integral equation[16,2–4]).

Usually one takes a fixed initial condition u_0 which implies that the last line of (3.2) vanishes, since then $\mathcal{Q}u_0 = 0$. We will *not* make this assumption here. To condense the notation somewhat, we first note that the last line of (3.2) can also be written as

$$\alpha\langle V(t)\mathcal{Q}u_0\rangle + \int_{t_0}^{t} ds\alpha^2\Big\langle V(t)\tilde{T}\Big[\exp\alpha\int_{s}^{t} ds' \mathcal{Q}V(s')\Big]\mathcal{Q}V(s)\mathcal{Q}u_0\Big\rangle.$$

Therefore, if we define a "vector" $X(t)$ with components

$$X_1(t) = V(t), \quad X_2(t) = g(t), \quad X_3(t) = V(t)\mathcal{Q}u_0, \tag{3.3}$$

(3.2) can be written as

$$\frac{d}{dt}\langle v(t)\rangle = \int_{t_0}^{t} ds\Gamma^{(1)}(t/s)\langle v(s)\rangle + \int_{t_0}^{t} ds\{\Gamma^{(2)}(t/s) + \Gamma^{(3)}(t/s)\}, \tag{3.4}$$

where $\Gamma^{(i)}$ $(i = 1, 2, 3)$ is given by

$$\Gamma^{(i)}(t/s) = \alpha\langle X_i(s)\rangle\delta_+(t - s) + \alpha^2\Big\langle V(t)\tilde{T}\Big[\exp\alpha\int_{s}^{t} ds' \mathcal{Q}V(s')\Big]\mathcal{Q}X_i(s)\Big\rangle \tag{3.5}$$

and $\int_0^\epsilon dt\delta_+(t) = 1$ $(\epsilon > 0)$. For later use we expand the $\Gamma^{(i)}$ in powers of α

$$\Gamma^{(i)}(t/s) = \sum_{k=1}^{\infty} \alpha^k\Gamma_k^{(i)}(t/s), \tag{3.6}$$

where

$$\Gamma_1^{(i)}(t/s) = \langle X_i(s)\rangle\delta_+(t - s), \tag{3.7a}$$

$$\Gamma_2^{(i)}(t/s) = \langle V(t)\mathcal{Q}X_i(s)\rangle, \tag{3.7b}$$

and $\Gamma_{m+2}^{(i)}$ with $m \geq 1$ is given by

$$\Gamma_{m+2}^{(i)}(t/s) = \int_{s}^{t} dt_1 \int_{s}^{t_1} dt_2 \ldots \int_{s}^{t_{m-1}} dt_m C^{(i)}(t, t_1, \ldots, t_m, s), \tag{3.7c}$$

with

$$C^{(i)}(t, t_1, \ldots, t_m, s) = \langle V(t)\mathcal{Q}V(t_1)\ldots\mathcal{Q}V(t_m)\mathcal{Q}X_i(s)\rangle. \tag{3.7d}$$

The average (3.7d) is the so-called "totally time-ordered cumulant"[11,12]. For m non-commuting quantities $A_0(t)$, $A_1(t), \ldots, A_{m-1}(t)$ the t-ordered cumulant is defined as ($A_k(t_k)$ being abbreviated as k; $t_k \geq t_{k+1}$):

$$\langle 012\ldots(m - 1)\rangle_t = \langle 0\mathcal{Q}1\mathcal{Q}2\ldots\mathcal{Q}(m - 1)\rangle; \quad \langle 0\rangle_t \equiv \langle 0\rangle. \tag{3.8}$$

In all terms contributing to $\langle 01 \ldots (m-1) \rangle_t$ the numerals are in decreasing time order (it can be obtained from $\langle 01 \ldots (m-1) \rangle_p$ by allowing only the identical permutation of $1, 2, \ldots (m-1)$; see also appendix A), which clarifies the adjective "totally time-ordered". As shown by Terwiel[4]) also the t-ordered cumulant $\langle \ldots \rangle_t$ has the cluster property.

3.2. Derivation of the differential equation for $\langle v(t) \rangle$

Now we want to convert the integro–differential equation for $\langle v(t) \rangle$ into a pure differential equation, making use of the fact that the memory effects in (3.4) are small if $\alpha \tau_c$ is small. Here τ_c is the largest of the autocorrelation time of V and the crosscorrelation times of V with g and u_0, which are all assumed to be finite.

Following Terwiel's iteration method[4]) we put

$$\langle v(s) \rangle = \langle v(t) \rangle - \int_s^t dt_1 \frac{d}{dt_1} \langle v(t_1) \rangle \tag{3.9}$$

and substitute this in (3.4). This leads to

$$\frac{d}{dt} \langle v(t) \rangle = \int_{t_0}^t ds \Gamma^{(1)}(t/s) \langle v(t) \rangle$$
$$+ \int_{t_0}^t ds \{\Gamma^{(2)}(t/s) + \Gamma^{(3)}(t/s)\} - \int_{t_0}^t ds \int_s^t dt_1 \Gamma^{(1)}(t/s) \frac{d}{dt_1} \langle v(t_1) \rangle. \tag{3.10}$$

Then we substitute again (3.4) with $t = t_1$ in the r.h.s. of (3.10), use again (3.9) and so on. Iterating this procedure one obtains

$$\frac{d}{dt} \langle v(t) \rangle = K^{(1)}(t/t_0) \langle v(t) \rangle + K^{(2)}(t/t_0) + K^{(3)}(t/t_0), \tag{3.11}$$

where $K^{(i)}(t/t_0)$ $(i = 1, 2, 3)$ is defined as

$$K^{(i)}(t/t_0) = \int_{t_0}^t ds \Gamma^{(i)}(t/s) + \sum_{n=1}^\infty (-)^n \left\{ \int_{t_0}^t ds \Gamma^{(1)}(t/s) \int_s^t dt_1 \int_{t_0}^{t_1} ds_1 \Gamma^{(1)}(t_1/s_1) \right.$$
$$\left. \ldots \int_{s_{n-2}}^t dt_{n-1} \int_{t_0}^{t_{n-1}} ds_{n-1} \Gamma^{(1)}(t_{n-1}/s_{n-1}) \int_{s_{n-1}}^t dt_n \int_{t_0}^{t_n} ds_n \Gamma^{(i)}(t_n/s_n) \right\}. \tag{3.12}$$

Note that each term of $K^{(i)}$ contains a product of $\Gamma^{(1)}$'s ending with a $\Gamma^{(i)}$. Inserting the expansion (3.6) in (3.12) and rearranging the terms according to

increasing powers of α, we find

$$K^{(i)}(t/t_0) = \sum_{m=1}^{\infty} \alpha^m K_m^{(i)}(t/t_0) \quad (i = 1, 2, 3), \tag{3.13}$$

where

$$K_m^{(i)}(t/t_0) = \int_{t_0}^{t} ds \Gamma_m^{(i)}(t/s) + \sum_{n=1}^{\infty} (-)^n \left\{ \sum_{k=1}^{\infty} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \delta_{m,k+k_1+\cdots k_n} \right.$$

$$\int_{t_0}^{t} ds \Gamma_k^{(1)}(t/s) \int_{s}^{t} dt_1 \int_{t_0}^{t_1} ds_1 \Gamma_{k_1}^{(1)}(t_1/s_1)$$

$$\cdots \int_{s_{n-2}}^{t} dt_{n-1} \int_{t_0}^{t_{n-1}} ds_{n-1} \Gamma_{k_{n-1}}^{(1)}(t_{n-1}/s_{n-1}) \int_{s_{n-1}}^{t} dt_n \int_{t_0}^{t_n} ds_n \Gamma_{k_n}^{(i)}(t_n/s_n) \left. \right\}.$$

$$\tag{3.14}$$

By careful examination of (3.7) and (3.14) one notes the following. The only difference between $K_m^{(1)}$, $K_m^{(2)}$ and $K_m^{(3)}$ lies in the fact that in each term of (3.14) the last quantity in the last t-ordered cumulant is X_1, X_2 and X_3 respectively, with the same time variable. This conclusion will be needed in the next section.

3.3. Identification of the ordered cumulants

Now we make a connection between the results (3.11)–(3.14) and those of section 2, where the homogeneous case was discussed. Let us write down explicitly the lowest order $K_m^{(1)}$'s as calculated from (3.14) and (3.3):

$$K_1^{(1)}(t/t_0) = \int_{t_0}^{t} ds \Gamma_1^{(1)}(t/s) = \langle V(t) \rangle, \tag{3.15a}$$

$$K_2^{(1)}(t/t_0) = \int_{t_0}^{t} ds \Gamma_2^{(1)}(t/s) = \int_{t_0}^{t} dt_1 \{ \langle V(t)V(t_1) \rangle - \langle V(t) \rangle \langle V(t_1) \rangle \}, \tag{3.15b}$$

$$K_3^{(1)}(t/t_0) = \int_{t_0}^{t} ds \Gamma_3^{(1)}(t/s) - \int_{t_0}^{t} ds \int_{s}^{t} dt_1 \int_{t_0}^{t_1} ds_1 \Gamma_2^{(1)}(t/s) \Gamma_1^{(1)}(t_1/s_1)$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \{ \langle V(t)V(t_1)V(t_2) \rangle - \langle V(t)V(t_1) \rangle \langle V(t_2) \rangle$$

$$- \langle V(t) \rangle \langle V(t_1)V(t_2) \rangle - \langle V(t)V(t_2) \rangle \langle V(t_1) \rangle$$

$$+ \langle V(t) \rangle \langle V(t_1) \rangle \langle V(t_2) \rangle + \langle V(t) \rangle \langle V(t_2) \rangle \langle V(t_1) \rangle \}. \tag{3.15c}$$

The integrands of the expressions (3.15) are now easily recognized as the p-ordered cumulants of V, as defined in section 2. In fact we know that (3.11) must reduce to (2.5) if we take $f(t)$ identically zero and a sure initial condition u_0. Therefore the $K_m^{(1)}$ as defined by (3.14) can be identified with (2.7)

$$K_m^{(1)}(t/t_0) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{m-2}} dt_{m-1} \langle V(t)V(t_1) \ldots V(t_{m-1}) \rangle_p. \quad (3.16)$$

This can be explicitly verified by inserting the expressions (3.7) for $i = 1$ into (3.14) and rewriting the result as an integration over the domain $t \geqslant t_1 \geqslant t_2 \cdots \geqslant t_{m-1}$. This is achieved by splitting up the original domains of integration and/or a relabeling of time variables and changing the order of integration. For example

$$\int_{t_0}^{t} ds \Gamma_k^{(1)}(t/s) \int_{s}^{t} dt_1 \int_{t_0}^{t_1} ds_1 \Gamma_{k_1}^{(i)}(t_1/s_1)$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} ds \int_{t_0}^{s} ds_1 \{ \Gamma_k^{(1)}(t/s) \Gamma_{k_1}^{(i)}(t_1/s_1) + \Gamma_k^{(1)}(t/s_1) \Gamma_{k_1}^{(i)}(t_1/s) \}.$$

The new integrand contains one or more new terms which can be obtained from the original integrand by a permutation of time variables. *The crucial point is that no interchange of operators (or matrices, vectors) is involved.*

The same manipulations which lead from (3.14) to (3.16) can also be carried out for $K_m^{(2)}/K_m^{(3)}$. Since we showed that

i) the only difference between $K_m^{(2)}/K_m^{(3)}$ and $K_m^{(1)}$ is that the last quantity in each term which contributes to $K_m^{(2)}/K_m^{(3)}$ is a $g(.)/V(.)\mathcal{Q}u_0$ instead of a $V(.)$ (section 3.2),

ii) no operators (matrices, vectors) are interchanged going from (3.14) (with $i = 1$) to (3.16),

we conclude that

$$K_m^{(2)}(t/t_0) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{m-2}} dt_{m-1} \langle V(t)V(t_1) \ldots V(t_{m-2})g(t_{m-1}) \rangle_p, \quad (3.17a)$$

where

$$K_1^{(2)}(t/t_0) \equiv \langle g(t) \rangle \quad (3.17b)$$

and

$$K_m^{(3)}(t/t_0) = \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{m-2}} dt_{m-1} \langle V(t)V(t_1) \ldots V(t_{m-2})[V(t_{m-1})\mathcal{Q}u_0] \rangle_p,$$

$$(3.18a)$$

with

$$K_1^{(3)}(t/t_0) \equiv \langle V(t) \mathcal{Q} u_0 \rangle. \tag{3.18b}$$

The integrands of (3.17) and (3.18) are multivariate p-ordered cumulants as introduced in section 2. They can easily be obtained from the $K_m^{(1)}$ in (3.15) and (3.16) by replacing the last operator $V(.)$ in each term which contributes to $K_m^{(1)}$ by the corresponding vectors $g(.)$ and $V(.) \mathcal{Q} u_0$ respectively. Note that in (3.18a) the vector $V(t_{m-1}) \mathcal{Q} u_0$ has to be considered as a single quantity with time variable t_{m-1}. The results (3.17) and (3.18) can be verified to a given order by explicit evaluation of (3.14) for $i = 2, 3$ using (3.3) and (3.7).

If g is independent of V, $K_m^{(2)} = 0$ for $m \geqslant 2$, while $K_m^{(3)}$ vanishes for all m if u_0 is non-random or statistically independent of V. This last statement follows from the fact that all *moments*, which satisfy

$$\langle V(t) V(t_1) \ldots V(t_{m-2}) V(t_{m-1}) \mathcal{Q} u_0 \rangle = \langle V(t) V(t_1) \ldots V(t_{m-1}) \rangle \langle \mathcal{Q} u_0 \rangle,$$

if the factor $\mathcal{Q} u_0$ is independent of the others, vanish since $\langle \mathcal{Q} u_0 \rangle = 0$. Therefore also the ordered cumulants

$$\langle V(t) V(t_1) \ldots V(t_{m-2}) [V(t_{m-1}) \mathcal{Q} u_0] \rangle_p \quad (m \geqslant 1)$$

vanish, although the last factor $V(t_{m-1}) \mathcal{Q} u_0$ *as a whole* need not be independent of the others, because of $V(t_{m-1})$.

Notice that the correlation of u_0 with $g(.)$ does not enter (3.17) or (3.18). However, it does come in if one studies the second moments of $v(t)$.

3.4. *Large time estimates*

The expressions (3.16), (3.17) and (3.18) still contain the initial time t_0, while (3.18) in addition depends on the initial value (distribution) u_0. At this point we use the assumption that the autocorrelation time of V, the crosscorrelation time of V with g and that of V with u_0 are all finite, the largest of them being denoted by τ_c. We will now show that after a transient time of order τ_c, the expression (3.16) for $K_m^{(1)}(t/t_0)$ approaches (2.11), while (3.17a) tends to

$$\overline{K_m^{(2)}}(t) = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{m-2}} dt_{m-1} \langle V(t) V(t_1) \ldots V(t_{m-2}) g(t_{m-1}) \rangle_p \tag{3.19}$$

and

$$K_m^{(3)}(t/t_0) \to 0 \quad (m = 1, 2, \ldots). \tag{3.20}$$

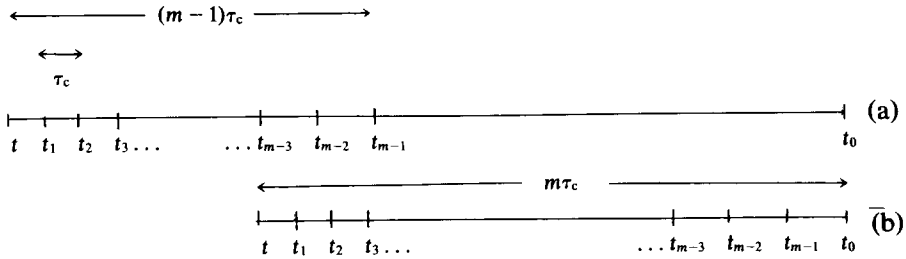The expression (3.16) is the same as (2.7a) of which it was shown in section

Fig. 1. Maximal extension of the integration domains in (a) $K_m^{(2)}$; and (b) $K_m^{(3)}$.

2.2 that it tends to (2.11) if $|t - t_0| \geq (m - 1)\tau_c$. An analogous argument can be given for $K_m^{(2)}$. In (3.17a) t and t_1 are at most a distance of order τ_c apart (otherwise $V(t)$ is independent of $V(t_1)$, but then also of $V(t_2)$, $V(t_3)$ etc. because of the ordering of times, *and* also of $g(t_{m-1})$ because τ_c is the largest correlation time: so the ordered cumulant vanishes). Repeating this argument one finds that the distance between successive time variables is at most τ_c, so that t and t_{m-1} differ at most a time of order $(m - 1)\tau_c$ (see fig. 1a). Therefore if $|t - t_0| \geq (m - 1)\tau_c$, $K_m^{(2)}(t/t_0)$ approaches the expression (3.19).

Now consider (3.18). By the same argument as above we see that t_{m-2} is at most a distance of order $(m - 2)\tau_c$ away from t. Also t_{m-1} cannot be further apart from t_{m-2} than τ_c because otherwise the whole factor $V(t_{m-1})\,2u_0$ is independent of $V(t_{m-2})$ and the cumulant vanishes. On the other hand, if t_{m-1} differs more than τ_c from t_0 the factor $2u_0$ is independent of all the operators $V(t), V(t_1), \ldots, V(t_{m-1})$, and the ordered cumulant vanishes (see the argument at the end of section 3.3). We conclude that t and t_0 can be at most a distance of order $m\tau_c$ apart, otherwise $K_m^{(3)}$ vanishes for all m (see fig. 1b).

So after a transient time* of order τ_c, $K^{(2)}(t/t_0)$ is independent of t_0, while $K^{(3)}(t/t_0)$ which was due to initial correlations between V and u_0, vanishes. This implies that the quantities (3.13) are expansions in powers of $\alpha\tau_c$ and we obtain the large time estimates

$$\alpha^m K_m^{(1)} \sim \alpha(\alpha\tau_c)^{m-1}, \quad \alpha^m K_m^{(2)} \sim \alpha(\alpha\tau_c)^{m-1}, \quad K_m^{(3)} \sim 0. \tag{3.21}$$

4. Summary of the results

Going back to the original representation u and taking $A(t) = A_0 + \alpha A_1(t)$, we have shown that, if $\alpha\tau_c \ll 1$, the average of (1.1) obeys the linear differen-

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* This time $\bar{t}$ should be such that $\tau_c \lesssim \bar{t} \ll \tau_d$, where τ_d is a measure for the duration of the process $v(t)$.

tial equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = K(t/t_0)\langle u(t)\rangle + F(t/t_0) + I(t/t_0), \tag{4.1a}$$

where

$$K(t/t_0) = A_0 + \alpha\langle A_1(t)\rangle + \sum_{m=1}^{\infty} \alpha^{m+1} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2$$

$$\dots \int_{t_0}^{t_{m-1}} \mathrm{d}t_m \, e^{(t-t_0)A_0}\langle A_1^{(1)}(t)A_1^{(1)}(t_1) \dots A_1^{(1)}(t_m)\rangle_p \, e^{-(t-t_0)A_0}, \tag{4.1b}$$

$$F(t/t_0) = \langle f(t)\rangle + \sum_{m=1}^{\infty} \alpha^{m} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2$$

$$\dots \int_{t_0}^{t_{m-1}} \mathrm{d}t_m \, e^{(t-t_0)A_0}\langle A_1^{(1)}(t)A_1^{(1)}(t_1) \dots A_1^{(1)}(t_{m-1})f^{(1)}(t_m)\rangle_p, \tag{4.1c}$$

$$I(t/t_0) = \alpha\langle A_1(t) \, e^{(t-t_0)A_0}(u_0 - \bar{u}_0)\rangle + \sum_{m=1}^{\infty} \alpha^{m+1} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2$$

$$\dots \int_{t_0}^{t_{m-1}} \mathrm{d}t_m \, e^{(t-t_0)A_0}\langle A_1^{(1)}(t)A_1^{(1)}(t_1) \dots A_1^{(1)}(t_{m-1})[A_1^{(1)}(t_m)(u - \bar{u}_0)]\rangle_p. \tag{4.1d}$$

Here the superscripts (1) denote the interaction representation

$$A_1^{(1)}(t) = e^{-(t-t_0)A_0}A_1(t) \, e^{(t-t_0)A_0}; \quad f^{(1)}(t) = e^{-(t-t_0)A_0}f(t) \tag{4.2}$$

and $\bar{u}_0$ denotes the average $\langle u(t_0)\rangle$. In calculating the ordered cumulants in (4.1d) the vector $A_1^{(1)}(t_m)(u_0 - \bar{u}_0)$ has to be considered as a single quantity with time variable t_m.

The presence of the deterministic evolution operators $e^{t_iA_0}$ in (4.1b)–(4.1d) poses a difficulty. Namely in deriving the large time estimates (3.21) we assumed that the lower limits t_0 in (3.16)–(3.18) can be replaced by $-\infty$ if the time t is large. Because of the operators $e^{t_iA_0}$ this need no longer be true in (4.1b)–(4.1d). Roughly speaking the possible growth of the deterministic evolution operators with time must not compensate the decay of the ordered cumulants. More precise criteria are derived in appendix B. If A_0 has purely imaginary eigenvalues the estimates (3.21) remain always valid.

The condition $\alpha\tau_c \ll 1$ is always necessary for the validity of the expansion (4.1) (excluding secular terms) except for the case in which the expansion is actually finite (as (2.6) is for a scalar Gaussian process). The fact that the

transient time of the mth order cumulants increases linearly with m (in general at least) indicates that the expansion may be asymptotic, unless the cumulants are negligible after a certain $m^{2,14}$).

Finally we give the result (4.1) to first order in $\alpha\tau_c$ for times exceeding the transient time (assuming that the deterministic evolution does not spoil the decay of the cumulants):

$$\frac{d}{dt}\langle u(t)\rangle = \left\{ A_0 + \alpha\langle A_1(t)\rangle + \alpha^2 \int_0^\infty d\tau \langle A_1(t) e^{\tau A_0} A_1(t-\tau)\rangle_p e^{-\tau A_0} \right\}\langle u(t)\rangle$$

$$+ \langle f(t)\rangle + \alpha \int_0^\infty d\tau \langle A_1(t) e^{\tau A_0} f(t-\tau)\rangle_p. \tag{4.3}$$

If in addition we assume that

$$\tau_c\|A_0\| \ll 1, \tag{4.4}$$

i.e. the deterministic motion of $u(t)$ is slow compared to the fluctuations in A_1, then (4.3) reduces to

$$\frac{d}{dt}\langle u(t)\rangle = \left\{ A_0 + \alpha\langle A_1(t)\rangle + \alpha^2 \int_0^\infty d\tau \langle A_1(t) A_1(t-\tau)\rangle_p \right\}\langle u(t)\rangle$$

$$+ \langle f(t)\rangle + \alpha \int_0^\infty d\tau \langle A_1(t) f(t-\tau)\rangle_p. \tag{4.5}$$

This result is *exact in the white noise limit*:

$$\alpha\tau_c \to 0, \quad \alpha^2\tau_c \text{ fixed}, \tag{4.6}$$

since the higher order terms in (4.1b) and (4.1c) are of relative order $(\alpha\tau_c)^m$, $m \geq 1$, compared to the r.h.s. of (4.5).

5. First example: decay of initial correlations

As a first illustration of the expansion (4.1) we now discuss an example of an exactly solvable stochastic differential equation in which the random coefficient matrix is correlated with the initial value. As a check of the expansion (4.1d) we show that its lower order terms for this case reproduce the exact result if this is expanded up to the same order.

Consider the equation

$$\frac{d}{dt} u(t) = \sigma_z u(t) + \alpha\xi(t)\sigma_x u(t). \tag{5.1}$$

Here $u(t)$ is a two-component vector and σ_x and σ_z are Pauli-matrices. For the sake of clarity vectors will be printed in bold-face type in this and the next section. For $\xi(t)$ we take a dichotomic Markov process[3]) with values $+1$ or -1. The composite process (u, ξ) is again a Markov process. Its joint probability density $P(u, \xi, t)$ satisfies[3])

$$\frac{\partial}{\partial t} P(u, \xi, t) = -\frac{\partial}{\partial u} \cdot [(\sigma_z u + \alpha \xi(t)\sigma_x u)P(u, \xi, t)] + \sum_{\xi'} W(\xi/\xi')P(u, \xi', t), \tag{5.2a}$$

with

$$W(\xi/\xi') = \gamma - 2\gamma\delta_{\xi,\xi'}. \tag{5.2b}$$

The evolution of the "marginal averages"

$$\langle u_i \rangle_{\pm} = \int du P(u, \pm 1, t)u_i \quad (i = 1, 2) \tag{5.3}$$

arranged as a vector with components $\langle u_1 \rangle_+$, $\langle u_1 \rangle_-$, $\langle u_2 \rangle_+$ and $\langle u_2 \rangle_-$ respectively, is determined by the matrix

$$M = \begin{pmatrix} \hat{1} + \gamma\sigma_z & \alpha\sigma_z \\ \alpha\sigma_z & -\hat{1} + \gamma\sigma_z \end{pmatrix} - \gamma \begin{pmatrix} \hat{1} & \cdot \\ \cdot & \hat{1} \end{pmatrix}. \tag{5.4}$$

Here $\hat{1}$ is the 2×2 unit matrix. The eigenvalues of (5.4) are

$$\lambda_{1,2} = -\gamma \pm \mu_+, \quad \lambda_{3,4} = -\gamma \pm \mu_-; \quad \mu_{\pm} = \sqrt{\alpha^2 + (\gamma \pm 1)^2}. \tag{5.5}$$

In terms of the sum and difference of the marginal averages

$$\langle u_i \rangle = \langle u_i \rangle_+ + \langle u_i \rangle_-, \quad \langle v_i \rangle = \langle u_i \rangle_+ - \langle u_i \rangle_-, \tag{5.6}$$

we find

$$\langle u_1(t) \rangle = c_+(t)\langle u_1(0) \rangle + d_+(t)\langle v_2(0) \rangle, \tag{5.7a}$$

$$\langle u_2(t) \rangle = c_-(t)\langle u_2(0) \rangle + d_-(t)\langle v_1(0) \rangle, \tag{5.7b}$$

where

$$c_{\pm}(t) = \left\{ \frac{\mu_{\pm} + (\gamma \pm 1)}{2\mu_{\pm}} \right\} e^{(-\gamma+\mu_{\pm})t} + \left\{ \frac{\mu_{\pm} - (\gamma \pm 1)}{2\mu_{\pm}} \right\} e^{(-\gamma-\mu_{\pm})t}, \tag{5.8a}$$

$$d_{\pm}(t) = \frac{\alpha}{2\mu_{\pm}} e^{-\gamma t} \{ e^{\mu_{\pm}t} - e^{-\mu_{\pm}t} \}. \tag{5.8b}$$

To compare the result (5.7) with that of the cumulant expansion, we differentiate (5.7) and express the result in terms of $\langle u(t) \rangle$ and $\langle v(0) \rangle$ by eliminating $\langle u(0) \rangle$ from (5.7) and its derivative:

$$\langle \dot{u}_1(t) \rangle = k_+(t)\langle u_1(t) \rangle + i_+(t)\langle v_2(0) \rangle, \tag{5.9a}$$

$$\langle \dot{u}_2(t) \rangle = k_-(t)\langle u_2(t) \rangle + i_-(t)\langle v_1(0) \rangle, \tag{5.9b}$$

where

$$k_\pm(t) = \frac{d}{dt}\{\ln c_\pm(t)\}; \quad i_\pm(t) = \dot{d}_\pm(t) - d_\pm(t)k_\pm(t). \tag{5.10}$$

For large t,

$$k_\pm(t) \to -\gamma + \mu_\pm, \quad i_\pm(t) \to \left\{\frac{2\alpha}{1 + (\gamma \pm 1)/\mu_\pm}\right\} e^{(-\gamma - \mu_\pm)t}. \tag{5.11}$$

If t really goes to infinity, $i_\pm(t) \to 0$, but for comparison with the cumulant expansion we retained its asymptotic time dependence. Expanding the result (5.11) to third order in α, one finds in the case that $\gamma > 1$:

$$k_\pm = \pm 1 + \frac{\alpha^2}{2(\gamma \pm 1)}, \tag{5.12a}$$

$$i_\pm(t) = \alpha \, e^{(-2\gamma \mp 1)t}\left\{1 + \frac{\alpha^2}{4(\gamma \pm 1)^2}(1 - 2(\gamma \pm 1)t)\right\}. \tag{5.12b}$$

Now we apply the result of the cumulant expansion to (5.1). The term (4.1d) only yields a nonzero result if the initial condition u_0 is correlated with $\xi(t)$. As an example we take the following initial distribution for u and ξ:

$$P(u, \xi, 0) = \tfrac{1}{2}\delta_{\xi,+}\delta(u - p) + \tfrac{1}{2}\delta_{\xi,-}\delta(u - q) \quad (p \neq q). \tag{5.13}$$

From (5.13) follows the initial distribution of ξ,

$$P(\xi, 0) = \tfrac{1}{2}\delta_{\xi,+} + \tfrac{1}{2}\delta_{\xi,-}, \tag{5.14}$$

and one finds that

$$\langle u(0)\rangle = \tfrac{1}{2}(p + q), \quad \langle v(0)\rangle = \tfrac{1}{2}(p - q) = \langle \xi(0)u(0)\rangle. \tag{5.15}$$

Since the process $\xi(t)$ evolves independently of $u(t)$, the many-time distributions are easily obtained as

$$P(\xi_1, t_1; \xi_2, t_2; \ldots; \xi_n, t_n; u, 0) = \sum_{\xi'} T(\xi_1 t_1/\xi_2 t_2)T(\xi_2 t_2/\xi_3 t_3)$$

$$\ldots T(\xi_n t_n/\xi' 0)P(\xi', u, 0), \tag{5.16}$$

where $t_1 \geq t_2 \ldots \geq t_n \geq 0$, and T is the transition-matrix of the dichotomic Markov process ($\xi_i = \pm 1$)

$$T(\xi_1 t_1/\xi_2 t_2) = \tfrac{1}{2}\{1 + \xi_1\xi_2 \, e^{-2\gamma(t_1 - t_2)}\}, \quad (t_1 \geq t_2). \tag{5.17}$$

Using (5.13)–(5.17) we find* ($n = 1, 2, 3 \ldots$)

$$\langle \xi(t_1)\xi(t_2) \ldots \xi(t_{2n-1}) \rangle = 0, \tag{5.18a}$$

$$\langle \xi(t_1)\xi(t_2) \ldots \xi(t_{2n}) \rangle = \langle \xi(t_1)\xi(t_2) \rangle \langle \xi(t_3)\xi(t_4) \rangle \ldots \langle \xi(t_{2n-1})\xi(t_{2n}) \rangle, \tag{5.18b}$$

$$\langle \xi(t_1)\xi(t_2) \ldots \xi(t_{2n})u(0) \rangle = \langle \xi(t_1) \ldots \xi(t_{2n}) \rangle \langle u(0) \rangle, \tag{5.18c}$$

$$\langle \xi(t_1)\xi(t_2) \ldots \xi(t_{2n-1})u(0) \rangle = \langle \xi(t_1) \ldots \xi(t_{2n-2}) \rangle \langle \xi(t_{2n-1})u(0) \rangle, \tag{5.18d}$$

where in particular

$$\langle \xi(t_1)\xi(t_2) \rangle = e^{-2\gamma(t_1-t_2)} \quad (t_1 \geqslant t_2), \tag{5.19a}$$

$$\langle \xi(t_1)u(0) \rangle = e^{-2\gamma t_1}\langle v(0) \rangle \quad (t_1 \geqslant 0). \tag{5.19b}$$

The cumulant expansion (4.1) to third order in α yields in this case ($t_0 = 0$):

$$\langle \dot{u}(t) \rangle = \left[\sigma_z + \alpha^2 \int_0^t dt_1\, e^{t\sigma_z}\langle \eta(t)\eta(t_1) \rangle_p\, e^{-t\sigma_z} \right] \langle u(t) \rangle$$

$$+ \alpha\, e^{t\sigma_z}\langle \eta(t)(u_0 - \bar{u}_0) \rangle + \alpha^2 \int_0^t dt_1\, e^{t\sigma_z}\langle \eta(t)[\eta(t_1)(u_0 - \bar{u}_0)] \rangle_p$$

$$+ \alpha^3 \int_0^t dt_1 \int_0^{t_1} dt_2\, e^{t\sigma_z}\langle \eta(t)\eta(t_1)[\eta(t_2)(u_0 - \bar{u}_0)] \rangle_p, \tag{5.20}$$

where

$$\eta(t) = \xi(t)\, e^{-t\sigma_z}\sigma_x\, e^{t\sigma_z} \equiv \xi(t)\tilde{\sigma}_x(t)$$

and we already used that $\langle \xi(t) \rangle = 0$. From (5.18) and (5.19) we find

$$\langle \eta(t)\eta(t_1) \rangle_p = e^{-2\gamma(t-t_1)}\, e^{t\sigma_z}\tilde{\sigma}_x(t)\tilde{\sigma}_x(t_1)\, e^{-t\sigma_z},$$

$$\langle \eta(t)(u_0 - \bar{u}_0) \rangle = \tilde{\sigma}_x(t)\, e^{-2\gamma t}\langle v(0) \rangle$$

$$\langle \eta(t)[\eta(t_1)(u_0 - \bar{u}_0)] \rangle_p = 0,$$

$$\langle \eta(t)\eta(t_1)[\eta(t_2)(u_0 - \bar{u}_0)] \rangle_p = -\tilde{\sigma}_x(t)\tilde{\sigma}_x(t_2)\tilde{\sigma}_x(t_1)\, e^{-2\gamma(t-t_2)}\, e^{-2\gamma t_1}\langle v(0) \rangle.$$

Inserting this in (5.20) and taking $t \to \infty$ in the first line of (5.20)† and t large

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* The eqs. (5.18) imply that[17] $\langle \xi(t_1) \ldots \xi(t_n) \rangle_t = \langle \xi(t_1) \ldots \xi(t_{n-1})[\xi(t_n)u(0)] \rangle_t = 0$, $n > 2$.
† This is possible if $\gamma > 1$, i.e. $2\gamma = \tau_c^{-1} > 2$ in agreement with (B.3).

but finite in the remaining lines, one finds for $\gamma > 1$:

$$\langle \dot{u}(t) \rangle = \left\{ \sigma_z + \frac{\alpha^2}{2(\gamma + \sigma_z)} \right\} \langle u(t) \rangle$$

$$+ \alpha \, e^{-(2\gamma + \sigma_z)t} \sigma_x \left\{ 1 + \frac{\alpha^2}{2(\sigma_z - \gamma)} \left(t + \frac{1}{2(\sigma_z - \gamma)} \right) \right\} \langle v(0) \rangle, \tag{5.21}$$

which is easily shown to agree with the exact results (5.9) and (5.12) for the individual components u_1 and u_2.

6. Application to the calculation of second moments

6.1. The general case

Consider again the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = \{A_0 + \alpha A_1(t, \omega)\} u(t) + f(t, \omega), \tag{6.1}$$

with a fixed initial condition. In studying the second moments of u one again is lead to an equation of type (6.1) (see below), which can be treated by the method of section 3 (the same is true for all higher moments of u). We will restrict ourselves here to the case in which A_1 and f are *statistically independent*, and show that even then the equation for the second moments is of the general type (1.1) with correlated multiplicative and additive noise.

To study the second moments of $u(t)$ we construct the vector

$$U(t) = \begin{pmatrix} u(t) \\ u(t) \otimes u(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u_1(t)u(t) \\ \vdots \\ u_n(t)u(t) \end{pmatrix}, \tag{6.2}$$

where the symbol $\otimes$ indicates a Kronecker product. The vector $U(t)$ has $(n+1)n$ components (if the symmetry condition $u_i u_j = u_j u_i$ is satisfied it is sufficient to consider a vector U with $n + \frac{1}{2}n(n+1) = \frac{1}{2}n(n+3)$ components). As a consequence of (6.1) $U(t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t) = \{\mathbf{A}_0 + \mathbf{A}_1(t)\} U(t) + \mathbf{F}(t), \tag{6.3}$$

where

$$\mathbf{A}_0 = \begin{pmatrix} A_0 & \emptyset \\ \hline \emptyset & \bar{A}_0 \end{pmatrix}; \quad \mathbf{A}_1(t) = \begin{pmatrix} \alpha A_1(t) & \emptyset \\ \hline \bar{f}(t) & \alpha \bar{A}_1(t) \end{pmatrix}; \quad \mathbf{F}(t) = \begin{pmatrix} f(t) \\ \emptyset \end{pmatrix} \tag{6.4}$$

and the $\sim$ symbol is defined for a $n \times n$-matrix or n dimensional vector C by

$$\bar{C} = \hat{1} \otimes C + C \otimes \hat{1}. \tag{6.5}$$

Here $\hat{1}$ denotes the $n \times n$-unit matrix.

Eq. (6.3) is again of type (6.1) but now with $\mathbf{A}_1(t)$ and $\mathbf{F}(t)$ correlated as they both contain $f(t)$. Hence the method of section 3 and 4 must be employed. So consider the series (4.1b) and (4.1c) for the case (6.3) (in this case $I(t/t_0) \equiv 0$ since we took a fixed initial condition). The lowest order terms of (4.1b) involve

$$\langle \mathbf{A}_1(t) \rangle = \begin{pmatrix} \alpha \langle A_1(t) \rangle & \emptyset \\ \hline \langle \bar{f}(t) \rangle & \alpha \langle \bar{A}_1(t) \rangle \end{pmatrix}, \tag{6.6a}$$

$$\langle \mathbf{A}_1^{(1)}(t) \mathbf{A}_1^{(1)}(t_1) \rangle_p$$

$$= \begin{pmatrix} \alpha^2 \langle A_1^{(1)}(t) A_1^{(1)}(t_1) \rangle_p & \emptyset \\ \hline \alpha \langle \bar{f}^{(1)}(t) A_1^{(1)}(t_1) + \bar{A}_1^{(1)}(t) \bar{f}^{(1)}(t_1) \rangle_p & \alpha^2 \langle \bar{A}_1^{(1)}(t) \bar{A}_1^{(1)}(t_1) \rangle_p \end{pmatrix}, \tag{6.6b}$$

where

$$A_1^{(1)}(t) = e^{-(t-t_0)A_0} A_1(t) \, e^{(t-t_0)A_0},$$

$$\bar{A}_1^{(1)}(t) = e^{-(t-t_0)\bar{A}_0} \bar{A}_1(t) \, e^{(t-t_0)\bar{A}_0} \tag{6.7a}$$

and

$$\bar{f}^{(1)}(t) = e^{-(t-t_0)\bar{A}_0} \bar{f}(t) \, e^{(t-t_0)A_0}. \tag{6.7b}$$

The lower left submatrix in (6.6b) vanishes because of the presupposed statistical independence of $A_1(t)$ and $f(t)$. This remains true for the higher cumulants of $\mathbf{A}_1^{(1)}(t)$: they too are of the form (6.6b) with an upper right submatrix which is identically zero and a lower left submatrix which vanishes because it contains the ordered cumulants of one f with two or more A_1's. In the upper left submatrix the mth order cumulant of $\alpha A_1^{(1)}$ appears, and from this the lower right submatrix in mth order can be obtained by replacing A_0 and A_1 by $\bar{A}_0$ and $\bar{A}_1$, respectively.

From the series (4.1c) only two terms survive:

$$\langle \mathbf{F}(t) \rangle = \begin{pmatrix} \langle f(t) \rangle \\ \emptyset \end{pmatrix} \tag{6.8}$$

and

$$\langle \mathbf{A}_1(t) \, e^{(t-t_1)\mathbf{A}_0} \mathbf{F}(t_1) \rangle_p = \begin{pmatrix} \alpha \langle A_1(t) \, e^{(t-t_1)A_0} f(t_1) \rangle_p \\ \langle \bar{f}(t) \, e^{(t-t_1)A_0} f(t_1) \rangle_p \end{pmatrix}. \tag{6.9}$$

The first n components of (6.9) are again zero because of the statistical independence of A_1 and f, and for the same reason all higher order cumulants

of $\mathbf{A}_1^{(1)}$ with $\mathbf{F}^{(1)}$ are zero since always ordered cumulants of one or more A_1's with one or two f's are involved.

Summarizing the above results we have found that $\langle U(t) \rangle$ obeys

$$\frac{d}{dt} \langle U(t) \rangle = \mathsf{K}(t/t_0) \langle U(t) \rangle + \mathsf{F}(t/t_0), \tag{6.10}$$

where

$$\mathsf{K}(t/t_0) = \left(\begin{array}{c|c} K(t/t_0) & \emptyset \\ \hline \langle \tilde{f}(t) \rangle & \tilde{K}(t/t_0) \end{array} \right),$$

$$\mathsf{F}(t/t_0) = \left(\begin{array}{c} \langle f(t) \rangle \\ \displaystyle\int_{t_0}^{t} dt_1 \langle \tilde{f}(t) \, e^{(t-t_1)A_0} f(t_1) \rangle_p \end{array} \right). \tag{6.11}$$

The matrix $K(t/t_0)$ is the same as (4.1b), while $\tilde{K}(t/t_0)$ is obtained from $K(t/t_0)$ by the replacements $A_0 \rightarrow \tilde{A}_0$, $A_1(t) \rightarrow \tilde{A}_1(t)$ (the $\sim$ symbol is defined in (6.5)).

We will discuss now an example for which the matrices K and $\tilde{K}$ of (6.11) can be exactly calculated (only a finite number of terms contributes).

6.2. Example: harmonic oscillator with stochastic frequency

Recently West et al.[7] treated the driven harmonic oscillator with stochastic frequency modelled by

$$\dot{x}(t) = p(t), \tag{6.12a}$$

$$\dot{p}(t) = -2\lambda p(t) - \Omega^2 x(t) - \gamma(t) x(t) + f_2(t). \tag{6.12b}$$

Here $\gamma(t)$ is a process with zero mean and delta-correlated cumulants*

$$\langle\langle \gamma(t_1) \gamma(t_2) \ldots \gamma(t_m) \rangle\rangle = m! D_m \delta(t_1 - t_2) \ldots \delta(t_{m-1} - t_m) \quad (m \geq 2), \tag{6.13}$$

where the brackets $\langle\langle \ldots \rangle\rangle$ denote ordinary cumulants, and $f_2(t)$ is a stationary process with zero mean, but otherwise unspecified. Its correlation function is denoted by

$$\langle f_2(t) f_2(t - \tau) \rangle = 2\tilde{D} \phi(\tau). \tag{6.14}$$

It is assumed that $\gamma(t)$ and $f_2(t)$ are statistically independent. The vector

$$U(t) = \begin{pmatrix} x(t) \\ p(t) \\ x^2(t) \\ x(t)p(t) \\ p(t)x(t) \\ p^2(t) \end{pmatrix} \tag{6.15}$$

* We use a coefficient $m! D_m$ instead of $2^m D_m$ as in ref. 7.

satisfies eq. (6.3), where the quantities in (6.4) are now specified as follows (dots denoting zeroes):

$$A_0 = \begin{pmatrix} \cdot & 1 \\ -\Omega^2 & -2\lambda \end{pmatrix}, \qquad \tilde{A}_0 = \begin{pmatrix} \cdot & 1 & 1 & \cdot \\ -\Omega^2 & -2\lambda & \cdot & 1 \\ -\Omega^2 & \cdot & -2\lambda & 1 \\ \cdot & -\Omega^2 & -\Omega^2 & -4\lambda \end{pmatrix}, \qquad (6.16)$$

and

$$A_1(t) = \gamma(t) B_1, \quad \tilde{A}_1(t) = \gamma(t) \tilde{B}_1,$$
$$f(t) = f_2(t) f_0, \quad \tilde{f}(t) = f_2(t) \tilde{f}_0, \qquad (6.17a)$$

where

$$B_1 = \begin{pmatrix} \cdot & \cdot \\ -1 & \cdot \end{pmatrix}; \quad \tilde{B}_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & -1 & -1 & \cdot \end{pmatrix}; \quad f_0 = \begin{pmatrix} \cdot \\ 1 \end{pmatrix}; \quad \tilde{f}_0 = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ 1 & \cdot \\ \cdot & 2 \end{pmatrix}$$
$$(6.17b)$$

Now we have to calculate the quantities in (6.11). First consider $K(t/t_0)$ which is given by (4.1b) with $\alpha \equiv 1$. Since $\gamma(t)$ is delta-correlated the integrations can be performed, because the integrands of (4.1b) only contribute if all t_1, $t_2, \ldots, t_m$ are equal to t. The mth order term then contains the matrix $D_m B_1^{m+1}$. Explicit calculation shows that $B_1^2 = 0$, so $B_1^{m+1} = 0$ if $m \geq 1^*$. Since $\langle \gamma(t) \rangle = 0$, we find that

$$K(t/t_0) = A_0. \qquad (6.18)$$

In the same way we find that from the series for $\tilde{K}(t/t_0)$ only the $m = 1$ term survives since $\tilde{B}_1^{m+1} = 0$, $m \geq 2$.

$$\tilde{K}(t/t_0) = \tilde{A}_0 + D_2 \tilde{B}_1^2. \qquad (6.19)$$

Furthermore we have that

$$\langle f(t) \rangle = 0, \quad \langle \tilde{f}(t) \rangle = 0, \qquad (6.20)$$

because $f_2(t)$ has zero average.

So we have found that the average of (6.15) obeys

$$\frac{d}{dt} \langle U(t) \rangle = \left(\begin{array}{c|c} A_0 & \emptyset \\ \hline \emptyset & \tilde{A}_0 + D_2 \tilde{B}_1^2 \end{array} \right) \langle U(t) \rangle + \int_0^{t-t_0} d\tau \begin{pmatrix} \emptyset \\ 2\tilde{D}\phi(\tau) \tilde{f}_0 \, e^{\tau A_0} f_0 \end{pmatrix}. \qquad (6.21)$$

* The nilpotency of B_1 also implies that the Itô- and Stratonovich interpretations of (6.12) yield the same result[8]).

The eigenvalues of A_0 are (assuming subcritical damping)

$$\lambda_\pm = -\lambda \pm i\omega_1, \quad \omega_1 = (\Omega^2 - \lambda^2)^{1/2} \tag{6.22}$$

and the deterministic evolution matrix is easily found as

$$e^{\tau A_0} = \begin{pmatrix} \cos \tau\omega_1 + \dfrac{\lambda}{\omega_1} \sin \tau\omega_1 & \dfrac{1}{\omega_1} \sin \tau\omega_1 \\ -\dfrac{\Omega^2}{\omega_1} \sin \tau\omega_1 & \cos \tau\omega_1 - \dfrac{\lambda}{\omega_1} \sin \tau\omega_1 \end{pmatrix} e^{-\tau\lambda}. \tag{6.23}$$

Combining (6.16), (6.17b), (6.21) and (6.23) we write down the result for the individual components of (6.15), where we make use of the symmetry $xp = px$:

$$\frac{d}{dt} \langle x \rangle_t = \langle p \rangle_t,$$

$$\frac{d}{dt} \langle p \rangle_t = -\Omega^2 \langle x \rangle_t - 2\lambda \langle p \rangle_t,$$

$$\frac{d}{dt} \langle x^2 \rangle_t = 2\langle xp \rangle_t,$$

$$\frac{d}{dt} \langle xp \rangle_t = -\Omega^2 \langle x^2 \rangle_t - 2\lambda \langle xp \rangle_t + \langle p^2 \rangle_t + \int_0^{t-t_0} d\tau\, 2\tilde{D}\phi(\tau) \frac{1}{\omega_1} \sin \omega_1\tau\, e^{-\tau\lambda},$$

$$\frac{d}{dt} \langle p^2 \rangle_t = -2\Omega^2 \langle xp \rangle_t - 4\lambda \langle p^2 \rangle_t + 2D_2 \langle x^2 \rangle_t$$

$$+ 2 \int_0^{t-t_0} d\tau\, 2\tilde{D}\phi(\tau) \left(\cos \tau\omega_1 - \frac{\lambda}{\omega_1} \sin \tau\omega_1 \right) e^{-\tau\lambda}. \tag{6.24}$$

Here $\langle x \rangle_t = \langle x(t) \rangle$ etc. From (6.24) we derive the equilibrium values (assuming they exist, i.e. $D_2 < 2\lambda\Omega^2$)

$$\langle x \rangle_{eq} = \langle p \rangle_{eq} = \langle xp \rangle_{eq} = 0 \tag{6.25a}$$

$$\langle x^2 \rangle_{eq} = \frac{2\lambda}{2\lambda\Omega^2 - D_2} \left\{ \tilde{D}_c + \frac{\lambda}{\omega_1} \tilde{D}_s \right\}, \tag{6.25b}$$

$$\langle p^2 \rangle_{eq} = \frac{2\lambda}{2\lambda\Omega^2 - D_2} \left\{ \tilde{D}_c \Omega^2 + \frac{\tilde{D}_s}{\omega_1} (D_2 - \lambda\Omega^2) \right\}, \tag{6.25c}$$

where

$$\tilde{D}_c = \frac{\tilde{D}}{\lambda} \phi_c(\infty), \quad \phi_c(\infty) = \int_0^\infty d\tau\, \phi(\tau) \cos \tau\omega_1\, e^{-\tau\lambda},$$

$$\tilde{D}_s = \frac{\tilde{D}}{\lambda} \phi_s(\infty), \quad \phi_s(\infty) = \int_0^\infty d\tau\, \phi(\tau) \sin \tau\omega_1\, e^{-\tau\lambda}.$$

These equilibrium results agree with the result (A.38) of West et al.[7]), which was obtained by a different method.

In the case that $\gamma(t)$ and $f_2(t)$ are correlated, we cannot use (6.11), but we have to apply the cumulant expansion (4.1) to (6.3). However, if $\gamma(t)$ and $f_2(t)$ are also delta-correlated to all orders, i.e.

$$\langle\langle\gamma(t_1)\gamma(t_2)\ldots\gamma(t_{m-1})f_2(t_m)\rangle\rangle = m\,!C_m\delta(t_1 - t_2)\ldots\delta(t_{m-1} - t_m) \qquad (6.26)$$

one easily finds that the only modification in (6.24) is an extra contribution $-4C_2\langle x\rangle_t$ to the last line of (6.24). But since $\langle x\rangle_{eq} = 0$, the equilibrium results (6.25) are unaffected.

7. Alternative derivation via the stochastic Liouville equation

The results (4.1a)–(4.1c) (fixed initial condition) can also be derived via the so-called "stochastic Liouville equation"[9]) which is a continuity-equation for the probability density $\rho(u, t)$ in the *state space* of u (also called "phase-space" or "u-space"). If the vector $u(t)$ obeys

$$\frac{d}{dt}u(t) = \{A_0 + \alpha A_1(t, \omega)\}u(t) + f(t, \omega), \qquad (7.1)$$

with a *fixed* initial condition u_0, the corresponding equation for $\rho(u, t)$ is[3,9])

$$\frac{\partial\rho}{\partial t}(u, t) = -\frac{\partial}{\partial u}\cdot\{(A_0 + \alpha A_1(t, \omega))u(t) + f(t, \omega)\}\rho(u, t), \qquad (7.2a)$$

with

$$\rho(u, t_0) = \delta(u - u_0). \qquad (7.2b)$$

Eq. (7.2a) can be written as

$$\frac{\partial}{\partial t}\rho(u, t) = \{L_0 + L_1(t, \omega)\}\rho(u, t), \qquad (7.3a)$$

where

$$L_0\ldots = -\frac{\partial}{\partial u}\cdot[\{A_0 u\}\ldots], \qquad (7.3b)$$

$$L_1\ldots = -\frac{\partial}{\partial u}\cdot[\{A_1(t, \omega)u + f(t, \omega)\}\ldots]. \qquad (7.3c)$$

According to Van Kampen's lemma[3]) the average $\langle\rho(u, t)\rangle = \int_\Omega d\omega P(\omega)\rho(u, t, \omega)$ equals the probability density $P(u, t)$ (with $P(u, t_0) = \delta(u - u_0)$) as determined by (7.1).

Since (7.3a) is a homogeneous stochastic differential equation for $\rho(u, t)$, the ordinary cumulant expansion (2.5)–(2.7) can be employed. So we have

$$\frac{\partial}{\partial t} P(u, t) = \left[L_0 + \langle L_1(t) \rangle + \sum_{m=1}^{\infty} \left\{ \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \right. \right.$$

$$\left. \left. \dots \int_{t_0}^{t_{m-1}} dt_m \, e^{(t-t_0)L_0} \langle L_1^{(1)}(t) L_1^{(1)}(t_1) \dots L_1^{(1)}(t_m) \rangle_p \, e^{-(t-t_0)L_0} \right\} \right] P(u, t), \qquad (7.4)$$

where

$$L_1^{(1)}(t) = e^{-(t-t_0)L_0} L_1(t) \, e^{(t-t_0)L_0}.$$

The mean $\langle u(t) \rangle$ is obtained as

$$\langle u(t) \rangle = \int du \, u P(u, t). \qquad (7.5)$$

We assume that $P(u, t)$ and its derivatives* $P^{(\kappa)}(u, t) = (\partial^{\kappa}/\partial u^{\kappa}) P(u, t)$ decrease sufficiently fast near the boundaries of the integration domain, so that for any polynomial $\psi(u) = \Sigma_{\{\lambda_i\}} c\{\lambda_i\} u_1^{\lambda_1} \dots u_n^{\lambda_n} (\lambda_i = 0, 1, 2 \dots)$

$$\int du \, \frac{\partial}{\partial u_j} \{ \psi(u) P^{(\kappa)}(u, t) \} = 0 \quad (j = 1, 2, \dots n).$$

This implies that for all $\kappa_i \geqslant 0$,

$$\int du L_0 \{ \psi(u) P^{(\kappa)}(u, t) \} = 0, \quad \int du L_1(t) \{ \psi(u) P^{(\kappa)}(u, t) \} = 0 \qquad (7.6)$$

and

$$\int du \, u L_0 \{ \psi(u) P^{(\kappa)}(u, t) \} = \int du \{ A_0 u \} \psi(u) P^{(\kappa)}(u, t), \qquad (7.7)$$

$$\int du \, u L_1(t) \{ \psi(u) P^{(\kappa)}(u, t) \} = \int du \{ \alpha A_1(t) u + f(t) \} \psi(u) P^{(\kappa)}(u, t). \qquad (7.8)$$

By repeated application of (7.6)–(7.8) we finally have

$$\int du \, e^{tL_0} \psi(u) P^{(\kappa)}(u, t) = \int du \psi(u) P^{(\kappa)}(u, t), \qquad (7.9)$$

$$\int du \, u \, e^{tL_0} \psi(u) P^{(\kappa)}(u, t) = e^{tA_0} \int du \, u \psi(u) P^{(\kappa)}(u, t) \qquad (7.10)$$

* $\dfrac{\partial^{\kappa}}{\partial u^{\kappa}} \equiv \dfrac{\partial^{\kappa_1}}{\partial u_1^{\kappa_1}} \dots \dfrac{\partial^{\kappa_n}}{\partial u_n^{\kappa_n}}; \kappa_i = 0, 1, 2 \dots.$

and

$$\int du u L_1^{(1)}(t) \psi(u) P^{(\kappa)}(u, t) = \int du \{\alpha A_1^{(1)}(t) u + f^{(1)}(t)\} \psi(u) P^{(\kappa)}(u, t), \qquad (7.11)$$

where $A_1^{(1)}(t)$ and $f^{(1)}(t)$ are defined in (4.2). Now from (7.4) and (7.5) the differential equation for $\langle u(t) \rangle$ can be derived by calculating

$$\langle \dot{u}(t) \rangle = \int du u \frac{\partial}{\partial t} P(u, t), \qquad (7.12)$$

where the r.h.s. of (7.4) is substituted in (7.12). As a consequence of (7.7) and (7.8), the first two terms of (7.12) are

$$\int du u \{L_0 + L_1(t)\} P(u, t) = \{A_0 + \alpha \langle A_1(t) \rangle\} \langle u(t) \rangle + \langle f(t) \rangle, \qquad (7.13)$$

where we used (7.5) and the normalization of $P(u, t)$. To calculate the higher order terms in (7.12) consider the quantity

$$B_m = \int du u \{e^{(t-t_0)L_0} \langle L_1^{(1)}(t) L_1^{(1)}(t_1) \ldots L_1^{(1)}(t_m) \rangle_p e^{-(t-t_0)L_0} P(u, t)\} \quad (m \geq 1). \qquad (7.14)$$

Making use of (7.10) and (7.11) we can move the factor u through the first operators

$$B_m = \int du \{e^{(t-t_0)A_0} \langle (\alpha A_1^{(1)}(t) u + f^{(1)}(t)) L_1^{(1)}(t_1) \ldots L_1^{(1)}(t_m) \rangle_p e^{-(t-t_0)L_0} P(u, t)\}.$$

From the first factor between brackets $\langle \ldots \rangle_p$ only the first term survives, since $f^{(1)}(t)$ is "annihilated" by $L_1^{(1)}(t_1)$ (see (7.6) and (7.9)). By repeating this the factor u can be moved through all the operators $L_1^{(1)}$:

$$B_m = \int du \alpha^m \{e^{(t-t_0)A_0} \langle A_1^{(1)}(t) A_1^{(1)}(t_1) \ldots \{\alpha A_1^{(1)}(t_m) u$$
$$+ f^{(1)}(t_m)\} \rangle_p e^{-(t-t_0)L_0} P(u, t)\}$$
$$= \alpha^{m+1} \{e^{(t-t_0)A_0} \langle A_1^{(1)}(t) A_1^{(1)}(t_1) \ldots A_1^{(1)}(t_m) \rangle_p e^{-(t-t_0)A_0}\} \langle u(t) \rangle$$
$$+ \alpha^m e^{(t-t_0)A_0} \langle A_1^{(1)}(t) A_1^{(1)}(t_1) \ldots A_1^{(1)}(t_{m-1}) f^{(1)}(t_m) \rangle_p, \qquad (7.15)$$

since (7.9) and (7.10) imply that

$$\int du u \, e^{-(t-t_0)L_0} P(u, t) = e^{-(t-t_0)A_0} \langle u(t) \rangle,$$

$$\int du \, e^{-(t-t_0)L_0} P(u, t) = 1.$$

Substituting the results (7.13) and (7.15) into the r.h.s. of (7.12) one again finds the previous equations (4.1a)–(4.1c).

Note however that (7.4) now also provides an equation for the *distribution function* of u, not only for its average. Let us apply this to the harmonic oscillator (6.12) for which

$$L_0 = -\frac{\partial}{\partial x} p + \frac{\partial}{\partial p} \{\Omega^2 x + 2\lambda p\} \tag{7.16a}$$

and

$$L_1 = \gamma(t) \frac{\partial}{\partial p} x - f_2(t) \frac{\partial}{\partial p}. \tag{7.16b}$$

Using the statistical properties of $\gamma(t)$ and specializing to the case that $f_2(t)$ is Gaussian white noise ($\phi(\tau) = \delta(\tau)$), one finds from (7.4)

$$\frac{\partial}{\partial t} P(x, p, t) = \left\{ -\frac{\partial}{\partial x} p + \frac{\partial}{\partial p} (\Omega^2 x + 2\lambda p) \right.$$

$$\left. + \sum_{m=2}^{\infty} D_m \left(\frac{\partial}{\partial p} x \right)^m + \bar{D} \frac{\partial^2}{\partial p^2} \right\} P(x, p, t). \tag{7.17}$$

Now we can write

$$\sum_{m=2}^{\infty} D_m \left(\frac{\partial}{\partial p} x \right)^m P(x, p, t) = \rho \int d\xi \phi(\xi) P(x, p + \xi x, t) - \rho \int d\xi \phi(\xi) P(x, p, t),$$

$$\tag{7.18}$$

where

$$m! D_m = \rho \int d\xi \phi(\xi) \xi^m, \tag{7.19}$$

since the l.h.s. of (7.18) is just the Taylor expansion of its r.h.s. Here $\phi(\xi)$ is the probability density of the jump size ξ (note that $\int d\xi \phi(\xi) \xi = 0$ since $\langle \gamma(t) \rangle = 0$). Substitution of (7.18) in (7.17) leads to a master equation for the process $(x(t), p(t))$, which is a consequence of the fact that both $\gamma(t)$ and $f_2(t)$ were assumed to be delta-correlated. It was derived by Van Kampen in a different way[8]).

A final remark is that the method of this section can also be applied to nonlinear equations[3]). The approximation of (7.4) to second order in L_1 produces a partial differential equation of second order in u, with coefficients which in general still depend on t_0 and t. On a time scale large compared to τ_c the dependence on t_0 vanishes and one obtains a Fokker-Planck like equation with (in general) time-dependent coefficients. In the white noise limit (4.6) this becomes a genuine (nonlinear) Fokker–Planck equation for a Markov process $u(t)$.

Acknowledgements

Appendix A

Connection between partial and total time ordered cumulants

To find the connection between partial and total time ordered cumulants we first note that the rules (ii)–(iv) following eq. (2.8) for constructing the p-ordered cumulants can be replaced by three equivalent rules:

(ii') Write a zero on the first dot and any permutation of the numerals 1, $2, \ldots, (m-1)$ on the remaining dots.

(iii') In every permutation insert an operator $\mathscr{P}$ (defined by $\mathscr{P} \ldots = \langle \ldots \rangle$) between two successive numerals i and j if $i > j$ and an operator $\mathscr{Q} = 1 - \mathscr{P}$ if $i < j$ and place this whole expression between brackets $\langle \ldots \rangle$.

(iv') For each permutation with $p - 1$ operators $\mathscr{P}$ (p subsequences) supply a factor $(-)^{p-1}$.

So each permutation of $1, 2, \ldots (m-1)$ yields a contribution to $\langle 01 \ldots (m-1) \rangle_p$ which is of the form

$$A_\kappa = (-1)^{p-1} \langle 0\mathscr{Q} \ldots j_0 \mathscr{P} i_1 \mathscr{Q} \ldots j_1 \mathscr{P} i_2 \ldots \ldots \mathscr{P} i_{p-1} \ldots j_p \rangle$$
$$= (-)^{p-1} \langle 0\mathscr{Q} \ldots j_0 \rangle \langle i_1 \mathscr{Q} \ldots j_1 \rangle \ldots \langle i_{p-1} \ldots j_p \rangle \qquad (A.1)$$

if there are precisely $(p-1)$ pairs $(j_\kappa, i_{\kappa+1})$ with $j_\kappa > i_{\kappa+1}$. Now each of the p subsequences in (A.1) is precisely a t-ordered cumulant (as defined in section 3.1) containing one numeral or a number of increasing numerals. Therefore the rules for obtaining the p-ordered cumulant $\langle A_0(t) A_1(t_1) \ldots A_{m-1}(t_{m-1}) \rangle_p$, $t \geqslant t_1 \ldots \geqslant t_{m-1}$, from the t-ordered ones are as follows:

(A.i) Write a sequence of m dots.

(A.ii) Write a zero on the first dot and any permutation of the numerals 1, $2, \ldots (m-1)$ on the remaining dots.

(A.iii) Partition each of the $(m-1)!$ permutations of numerals into subsequences by inserting angular brackets $\langle\ldots\rangle_t$ in such a way that two successive numerals belong to the same subsequence if and only if the first one is smaller than the second.

(A.iv) For each partition consisting of p subsequences supply a factor $(-)^{p-1}$.

(A.v) Replace each numeral n on the κth dot by $A_{\kappa-1}(t_n)$.

For the lowest order cumulants this yields:

$$\langle 0\rangle_p = \langle 0\rangle_t, \quad \langle 01\rangle_p = \langle 01\rangle_t, \quad \langle 012\rangle_p = \langle 012\rangle_t - \langle 02\rangle_t\langle 1\rangle_t, \tag{A.2}$$

$$\langle 0123\rangle_p = \langle 0123\rangle_t - \langle 023\rangle_t\langle 1\rangle_t - \langle 013\rangle_t\langle 2\rangle_t - \langle 03\rangle_t\langle 12\rangle_t$$
$$- \langle 02\rangle_t\langle 13\rangle_t + \langle 03\rangle_t\langle 2\rangle_t\langle 1\rangle_t. \tag{A.3}$$

Note that these connections are also given by eq. (3.14) in a somewhat hidden form.

Proof of the cluster property for the individual terms A_κ

As shown by Terwiel[4] the t-cumulants of identical quantities $A(t)$ have the cluster property. This remains true for the t-cumulant of non-identical quantities if they all have a finite crosscorrelation time (or autocorrelation time for those which are identical) with maximum τ_c. Then the t-cumulants $\langle 01\ldots(m-1)\rangle_t$ vanish as soon as $t_i - t_{i+1} \geqslant \tau_c$, $i = 0,\ldots,(m-2)$.

We will now show that this implies that each of the $(m-1)!$ terms in which the p-cumulant $\langle 01\ldots(m-1)\rangle_p$ is split up by the rules just defined, also has the cluster property.

Since the times are ordered, $t \geqslant t_1 \geqslant \cdots \geqslant t_{m-1}$, it is sufficient to prove that each term vanishes as $t_i - t_{i+1} \geqslant \tau_c$, $i = 0,\ldots,m-2$. We distinguish two cases (for each term):

(I) The numeral i is in the same subsequence as $(i+1)$, (consequently it comes next after i), so this subsequence vanishes as $t_i - t_{i+1} \geqslant \tau_c$ (each t-cumulant with more than one numeral has the cluster property).

(II) The numeral i is in a different subsequence as $(i+1)$. There occur three cases ((i) and (ii) are not mutually exclusive):

(i) There is a numeral p succeeding i within the same subsequence as i. Then it must be $(i+2)$ or higher, so if $t_i - t_{i+1} \geqslant \tau_c$, then certainly $t_i - t_p \geqslant \tau_c$ and the subsequence vanishes. (This is always the case if $i = 0$).

(ii) There is a numeral p preceding $(i+1)$ within the same subsequence as $(i+1)$. Then it must be $(i-1)$ or lower, so if $t_i - t_{i+1} \geqslant \tau_c$, also $t_p - t_{i+1} \geqslant \tau_c$ and the subsequence vanishes.

(iii) Neither (i) nor (ii) is the case (so in particular $i \neq 0$). Then there occur two subsequences of the form $\langle \ldots i \rangle_t$ and $\langle (i+1) \ldots \rangle_t$. If these were the only ones then because of rule A.iii) the combination should be $\langle (i+1) \ldots \rangle_t \langle \ldots i \rangle_t$, but this is impossible since on the first dot there must be a zero. So there are more subsequences. We will demonstrate that the following proposition R holds: at least one of these other subsequences is of the form $\langle q_1 q_2 \ldots q_k \rangle_t$, $k > 1$, with $q_1 < i$ and $q_k > (i+1)$ (so that it vanishes if $t_i - t_{i+1} \geq \tau_c$, because then $t_{q_1} - t_{q_k} \geq \tau_c$). To prove this, we show that the negation $\bar{R}$ of the proposition leads to a contradiction. There are again two possibilities:

a) The subsequence $\langle \ldots i \rangle$ precedes $\langle (i+1) \ldots \rangle$. Then between them there are m subsequences, $m \geq 1$, so we have an expression like

$$\ldots \langle \ldots i \rangle_t \langle q_{1,1} \ldots q_{1,k_1} \rangle_t \langle q_{2,1} \ldots q_{2,k_2} \rangle_t$$
$$\ldots \langle q_{m,1} \ldots q_{m,k_m} \rangle_t \langle (i+1) \ldots \rangle_t \ldots \quad (k_i \geq 1).$$

Now we apply rule (A.iii) and the proposition $\bar{R}$, and infer the following chain of deductions

$$\xrightarrow{\text{(A.iii)}} q_{1,1} < i \xrightarrow{\bar{R}} q_{1,k_1} < i \xrightarrow{\text{(A.iii)}} q_{2,1} < i \rightarrow \ldots \rightarrow q_{m,k_m} < i.$$

But the last statement cannot be true since by rule A.iii) we have that $q_{m,k_m} > (i+1)$: contradiction.

b) The subsequence $\langle \ldots i \rangle$ occurs after $\langle (i+1) \ldots \rangle$. Then we have an expression like

$$\langle 0 \ldots q_{1,k_1} \rangle_t \langle q_{2,1} \ldots q_{2,k_2} \rangle_t$$
$$\ldots \langle q_{m,1} \ldots q_{m,k_m} \rangle_t \langle (i+1) \ldots \rangle_t \ldots \langle \ldots i \rangle_t \ldots \quad (k_i \geq 1).$$

By the same reasoning as above we find that $q_{m,k_m} < i$ since $0 < i$. But this is impossible since $q_{m,k_m} > (i+1)$ (rule (A.iii)) and we have again a contradiction.

So we have proven that all the terms A_κ in (A.1) have the cluster property.

Appendix B

Here we will derive sufficient conditions for the estimates (3.21) to remain valid if there occur deterministic evolution matrices $e^{t_i A_0}$ in the ordered cumulants in (4.1b)–(4.1d).

To this end we assume that the t-ordered cumulants of $A_1(t)$ satisfy

$$\langle |A_1(t_1)| \rangle_t \leq C_1, \tag{B.1a}$$

$$\langle |A_1(t_1)| \ldots |A_1(t_m)| \rangle_t \leq C_m \, e^{-(t_1 - t_m)1/\tau_c}, \tag{B.1b}$$

where $t \geq t_1 \geq t_2 \ldots \geq t_m \geq t_0$. The C_i's are positive constants and $|\ldots|$

denotes a matrix norm. For the cumulants of A with f we assume a similar condition with $|A_1(t_m)|$ replaced by $\|f(t_m)\|$, $\|\ldots\|$ denoting the norm of the vector $f(t_m)$. Finally for the t-ordered cumulants of A with u_0 we assume

$$\langle |A_1(t_1)| \ldots |A_1(t_{m-1})| |A_1(t_m)| \|u_0 - \bar{u}_0\| \rangle_t \leq C'_m \, e^{-((t_1-t_0)/\tau_c)}, \tag{B.1c}$$

the C''_i's again being positive constants.

The matrix e^{tA_0} can be expressed in the following way:

$$e^{tA_0} = \sum_{k=1}^{\sigma} \sum_{l=0}^{m_k-1} t^l \, e^{\lambda_k t} Z_{kl}, \tag{B.2}$$

where $\lambda_1, \ldots \lambda_\sigma$ are distinct eigenvalues of A_0 with λ_k having multiplicity m_k. The matrices Z_{kl} have constant elements and depend only on A_0.

Using (B.1)–(B.2) we will now show that the estimates (3.21) remain valid if

$$\text{(I)} \ (\text{Re } \lambda)_{max} - (\text{Re } \lambda)_{min} < \frac{1}{\tau_c}, \tag{B.3}$$

$$\text{(II)} \ (\text{Re } \lambda)_{max} < \frac{1}{\tau_c}, \tag{B.4}$$

where "max" and "min" denote respectively the largest and smallest value of the real part of λ_k, $k = 1, \ldots, \sigma$.

If condition (I) is satisfied all limits t_0 in (4.1b) can be replaced by $-\infty$ after a transient time and the estimate (3.21) for $K_m^{(1)}$ becomes*

$$\alpha^m K_m \sim \alpha^m \tau_c'^{m-1}, \tag{B.5}$$

where

$$\tau_c' = \tau_c [1 - \overline{\Delta\lambda_R} \, \tau_c]^{-1}, \tag{B.6}$$

with

$$\overline{\Delta\lambda_R} = (\text{Re } \lambda)_{max} - (\text{Re } \lambda)_{min}. \tag{B.7}$$

Similarly, conditions (I) and (II) *together* imply that for $t \gg \tau_c$ all limits t_0 in (4.1c) can be replaced by $-\infty$ and that $I(t/t_0)$ vanishes*:

$$\alpha^m F_m \sim (\alpha \tau_c'')^m; \quad I_m \sim 0, \tag{B.8}$$

where

$$\tau_c'' = \max \{ \tau_c [1 - \overline{\lambda_R} \, \tau_c]^{-1}, \tau_c' \}, \tag{B.9}$$

with

$$\overline{\lambda_R} = (\text{Re } \lambda)_{max}. \tag{B.10}$$

*K_m, F_m and I_m are the coefficients of α^m in the expansions (4.1b)–(4.1d).

The conditions (B.3) and (B.4) may be weakened in specific cases, e.g. if A_0 and $A_1(t)$ commute (B.3) can be dropped. If A_0 has purely imaginary eigenvalues both (B.3) and (B.4) are satisfied.

To derive (B.3) consider the quantity

$$B = e^{(t-t_0)A_0}\langle A_1^{(1)}(t)A_1^{(1)}(t_1)\dots A_1^{(1)}(t_m)\rangle_p \, e^{-(t-t_0)A_0}, \tag{B.11}$$

which occurs in (4.1b). The p-cumulant $\langle 01\dots m\rangle_p$ is a sum of $m!$ products of t-cumulants (see appendix A). In each of these terms the time variables of the matrices $A_1^{(1)}(.)$ have a definite order which is a permutation P of $0, 1,\dots, m$ with $P(0) = 0$. Defining the kth permutation $(k = 1, 2,\dots, m!)$ by $P_k(l) = k_l(l = 1, 2,\dots, m)$ we find

$$B = \sum_{k=1}^{m!} B_k,$$
$$B_k = \langle A_1(t) \, e^{(t-t_{k_1})A_0}A_1(t_{k_1}) \, e^{(t_{k_1}-t_{k_2})A_0}\dots\rangle_t$$

$$\dots\langle\dots\rangle_t\dots\langle\dots e^{(t_{k_{m-1}}-t_{k_m})A_0}A_1(t_{k_m})\rangle_t \, e^{(t_{k_m}-t)A_0}. \tag{B.12}$$

Taking the norm of B_k this gives

$$|B_k| \leqslant \prod_{i=1}^{m} |e^{(t_{k_{i-1}}-t_{k_i})A_0}\|e^{(t_{k_m}-t)A_0}|\langle|A_1(t)|\dots\rangle_t\dots\langle\dots|A_1(t_{k_m})|\rangle_t, \tag{B.13}$$

where $t_{k_0} \equiv t$.

Now the r.h.s. of (B.12) contains a product of a number, say p, of t-cumulants. The first and last numeral (each numeral i stands for $A_1(t_i)$) of the lth t-cumulant is denoted by i_l, resp. j_l (where $t_{i_1} \equiv t$, $t_{j_p} \equiv t_{k_m}$). If there is only one numeral in a t-cumulant we put $i_l = j_l$. Within each t-cumulant the numerals increase, so from (B.2) we can deduce the estimate (let A_0 be a $N \times N$-matrix)

$$|e^{(t_{k_{i-1}}-t_{k_i})A_0}| \leqslant \beta \, e^{(t_{k_{i-1}}-t_{k_i})(\mathrm{Re}\,\lambda)_{\max}}|t_{k_{i-1}} - t_{k_i}|^{N-1}$$

$$\leqslant \beta(t - t_m)^{N-1} \, e^{(t_{k_{i-1}}-t_{k_i})(\mathrm{Re}\,\lambda)_{\max}} \quad (t_{k_{i-1}} - t_{k_i} \geqslant 0) \tag{B.14}$$

for the evolution operators occurring within a t-cumulant. Here β is a positive constant and we used that $t_m \leqslant t_{k_i} \leqslant t$ for all $i \in \{0, 1,\dots, m\}$. Going from one t-cumulant to the next the numerals decrease, so for the remaining evolution operators "between" successive t-cumulants we have:

$$|e^{(t_{j_{l-1}}-t_{i_l})A_0}| \leqslant \beta \, e^{(t_{j_{l-1}}-t_{i_l})(\mathrm{Re}\,\lambda)_{\min}} |t_{j_{l-1}} - t_{i_l}|^{N-1}$$

$$\leqslant \beta(t - t_m)^{N-1} \, e^{(t_{j_{l-1}}-t_{i_l})(\mathrm{Re}\,\lambda)_{\min}} \quad (t_{j_{l-1}} - t_{i_l} \leqslant 0). \tag{B.15}$$

Using (B.14), (B.15) and (B.1) we find

$$|B_k| \leqslant \beta^{m+1}(t-t_m)^{(m+1)(N-1)} \prod_{l=1}^{p} \{e^{(t_{i_l}-t_{j_l})(\operatorname{Re}\lambda)_{\max}} \cdot e^{-(t_{i_{l+1}}-t_{j_l})(\operatorname{Re}\lambda)_{\min}}\}$$

$$\times D \prod_{l=1}^{p} e^{-(t_{i_l}-t_{j_l})1/\tau_c}, \tag{B.16}$$

where $t_{i_{p+1}} \equiv t$ and D is a positive constant.

Since $t_{i_1} = t_{i_{p+1}} = t$ it follows that

$$\sum_{l=1}^{p} \{t_{i_{l+1}} - t_{j_l}\} = \sum_{l=1}^{p} t_{i_l} - t_{j_l}, \tag{B.17}$$

so (B.16) leads to

$$|B_k| \leqslant D\beta^{m+1}(t-t_m)^{(m+1)(N-1)} \exp\left[\sum_{l=1}^{p}(t_{i_l}-t_{j_l})\left\{(\operatorname{Re}\lambda)_{\max}-(\operatorname{Re}\lambda)_{\min}-\frac{1}{\tau_c}\right\}\right]. \tag{B.18}$$

Now $t_{i_l} - t_{j_l} \geqslant 0$, and there is a numeral l' such that $t_{j_{l'}} = t_m$, hence

$$\sum_{l=1}^{p} t_{i_l} - t_{j_l} \geqslant \sum_{l=1}^{l'} t_{i_l} - t_{j_l}. \tag{B.19}$$

Furthermore $t_{i_{l+1}} - t_{j_l} \geqslant 0$, so

$$t - t_{j_1} + t_{i_2} - t_{j_2} \geqslant t - t_{j_2}.$$

Repeating this argument we find from (B.19)

$$\sum_{l=1}^{p} t_{i_l} - t_{j_l} \geqslant t - t_m. \tag{B.20}$$

Therefore if $\overline{\Delta\lambda_R} - 1/\tau_c \leqslant 0$,

$$|B_k| \leqslant D\beta^{m+1}(t-t_m)^{(m+1)(N-1)} \exp\left[(t-t_m)\left(\overline{\Delta\lambda_R}-\frac{1}{\tau_c}\right)\right]. \tag{B.21}$$

The estimate (B.21) leads to the conclusion that in (4.1b) $\lim_{t_0 \to -\infty} K(t/t_0)$ exists if $\int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{m-1}} dt_m |B_k|$ exists for all k. This is the case if $\overline{\Delta\lambda_R} < 1/\tau_c$, which is condition (B.3).

In the same way we find for the kth term of the quantity

$$B' = e^{(t-t_0)A_0}\langle A_1^{(1)}(t) \ldots A_1^{(1)}(t_{m-1})f^{(1)}(t_m)\rangle_p, \tag{B.22}$$

$$|B_k'| \leqslant \prod_{i=1}^{m} |e^{(t_{k_{i-1}}-t_{k_i})A_0}|\langle(|A_1(t)| \ldots)_t\langle \ldots)_t \ldots \langle \ldots \|f(t_{k_m})\|\rangle_t. \tag{B.23}$$

Using (B.14), (B.15) and (B.1) one has

$$|B_k^i| \leqslant \beta^m (t - t_m)^{m(N-1)} \left\{ \prod_{l=1}^{p} e^{(t_{i_l} - t_{j_l})(\text{Re}\,\lambda)_{\max}} \right\}$$

$$\times \left\{ \prod_{l=1}^{p-1} e^{-(t_{i_{l+1}} - t_{j_l})(\text{Re}\,\lambda)_{\min}} \right\} D' \prod_{l=1}^{p} e^{-(t_{i_l} - t_{j_l})1/\tau_c}, \tag{B.24}$$

where D' is again a positive constant. By virtue of (B.17) we can write

$$|B_k^i| \leqslant D' \beta^m (t - t_m)^{m(N-1)} \left[\prod_{l=1}^{p-1} e^{(t_{i_{l+1}} - t_{j_l})\{(\text{Re}\,\lambda)_{\max} - (\text{Re}\,\lambda)_{\min} - 1/\tau_c\}} \right] e^{(t - t_{k_m})\{(\text{Re}\,\lambda)_{\max} - 1/\tau_c\}}. \tag{B.25}$$

Now we distinguish between two cases:

(i) $\underline{(\text{Re}\,\lambda)_{\min} \geqslant 0}$: Then

$$(\text{Re}\,\lambda)_{\max} - (\text{Re}\,\lambda)_{\min} \leqslant (\text{Re}\,\lambda)_{\max}, \tag{B.26}$$

hence

$$|B_k^i| \leqslant D' \beta^m (t - t_m)^{m(N-1)} \prod_{l=1}^{p} e^{(t_{i_{l+1}} - t_{j_l})\{(\text{Re}\,\lambda)_{\max} - 1/\tau_c\}}$$

$$\leqslant D' \beta^m (t - t_m)^{m(N-1)} e^{(t - t_m)(\overline{\lambda_R} - 1/\tau_c)} \quad \left(\overline{\lambda_R} \leqslant \frac{1}{\tau_c} \right), \tag{B.27}$$

where we used (B.17) and (B.20) and the assumption $\overline{\lambda_R} \leqslant 1/\tau_c$ to get the last inequality.

(ii) $\underline{(\text{Re}\,\lambda)_{\min} < 0}$: Then

$$(\text{Re}\,\lambda)_{\max} - (\text{Re}\,\lambda)_{\min} > (\text{Re}\,\lambda)_{\max},$$

so by a similar argument as in (i):

$$|B_k^i| \leqslant D' \beta^m (t - t_m)^{m(N-1)} e^{(t - t_m)(\overline{\Delta\lambda_R} - 1/\tau_c)} \quad \left(\overline{\Delta\lambda_R} \leqslant \frac{1}{\tau_c} \right). \tag{B.28}$$

Finally consider the kth term of the quantity

$$B'' = e^{(t - t_0)A_0} \langle A_1^{(1)}(t) \dots A_1^{(1)}(t_{m-1})[A_1^{(1)}(t_m)(u_0 - \bar{u}_0)] \rangle_p, \tag{B.29}$$

for which

$$|B_k''| \leqslant \prod_{i=1}^{m} |e^{(t_{k_{i-1}} - t_{k_i})A_0}| |e^{(t_{k_m} - t_0)A_0}|$$

$$\times \langle |A_1(t)| \dots \rangle_t \langle \dots \rangle_t \dots \langle \dots |A_1(t_{k_m})| \, \|u_0 - \bar{u}_0\| \rangle_t$$

$$\leqslant \beta^m (t - t_m)^{m(N-1)} \left\{ \prod_{l=1}^{p} e^{(t_{i_l} - t_{j_l})(\text{Re}\,\lambda)_{\max}} \right\} \left\{ \prod_{l=1}^{p-1} e^{-(t_{i_{l+1}} - t_{j_l})(\text{Re}\,\lambda)_{\min}} \right\}$$

$$\times \beta(t_{k_m} - t_0)^{N-1} e^{(t_{k_m} - t_0)(\text{Re}\,\lambda)_{\max}} \times D'' \left\{ \prod_{l=1}^{p-1} e^{-(t_{i_l} - t_{j_l})/\tau_c} \right\} e^{-(t_{i_p} - t_0)/\tau_c}, \tag{B.30}$$

where again use has been made of (B.14), (B.15) and (B.1) (D'' is a positive constant). By rearranging the time variables in the r.h.s. of (B.30) using (B.17), one is led to

$$|B_k''| \leq D'' \beta^{m+1} (t - t_0)^{(m+1)(N-1)} \left[\prod_{l=1}^{p-1} e^{(t_{i_{l+1}} - t_{i_l})\{(\mathrm{Re}\,\lambda)_{max} - (\mathrm{Re}\,\lambda)_{min} - 1/\tau_c\}} \right]$$

$$\times e^{(t - t_{k_m})\{(\mathrm{Re}\,\lambda)_{max} - 1/\tau_c\}} e^{(t_{k_m} - t_0)\{(\mathrm{Re}\,\lambda)_{max} - 1/\tau_c\}}$$

$$\leq D'' \beta^{m+1} (t - t_0)^{(m+1)(N-1)} e^{(t-t_0)(\overline{\lambda_R} - 1/\tau_c)} \left(\overline{\Delta\lambda_R} \leq \frac{1}{\tau_c} \right), \qquad (B.31)$$

where the assumption $\overline{\Delta\lambda_R} \leq 1/\tau_c$ has been made to get the last inequality.

Therefore if (B.3) and (B.4) are satisfied we conclude from (B.27), (B.28) and (B.31) that the expression (4.1c) with all limits t_0 replaced by $-\infty$ exists, and that $\lim_{t \to \infty} I(t/t_0) = 0$.

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