A new class of morphological pyramids for multiresolution image analysis

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Abstract. We study nonlinear multiresolution signal decomposition based on morphological pyramids. Motivated by a problem arising in multiresolution volume visualization, we introduce a new class of morphological pyramids. In this class the pyramidal synthesis operator always has the same form, i.e. a dilation by a structuring element A, preceded by upsampling, while the pyramidal analysis operator is a certain operator $R_A^{(n)}$ indexed by an integer n, followed by downsampling. For $n = 0$, $R_A^{(n)}$ equals the erosion ε_A with structuring element A, whereas for $n > 0$, $R_A^{(n)}$ equals the erosion ε_A followed by n conditional dilations, which for $n \to \infty$ is the opening by reconstruction. The resulting pair of analysis and synthesis operators is shown to satisfy the pyramid condition for all n. The corresponding pyramids for $n = 0$ and $n = 1$ are known as the adjunction pyramid and Sun-Maragos pyramid, respectively. Experiments are performed to study the approximation quality of the pyramids as a function of the number of iterations n of the conditional dilation operator.

1 Introduction

Multiresolution signal decomposition schemes have enjoyed a long standing interest. Analyzing signals at multiple scales may be used to suppress noise and can lead to more robust detection of signal features, such as transitions in sound data, or edges in images. Multiresolution algorithms also may offer computational advantages, when the analysis of the signal is performed in a coarse-to-fine fashion. Examples of linear multiresolution schemes are the Laplacian pyramid [**?**] and decomposition methods based on wavelets [**?**].

This paper is concerned with nonlinear multiresolution signal decomposition based on morphological pyramids. A detailed study of such pyramids was recently made by Goutsias and Heijmans [**?**, **?**]. Morphological pyramids systematically split the input signal into approximation and detail signals by repeatedly applying a pyramidal analysis operator which involves morphological filtering followed by downsampling. As the level of the pyramid is increased, spatial features of increasing size are extracted. The original signal can be recovered from the pyramid decomposition by repeated application of a pyramid synthesis operator. If the analysis and synthesis operators satisfy the so-called pyramid condition, then perfect reconstruction holds, i.e. the original signal can be exactly recovered from the pyramidal decomposition data.

The goal of this paper is to derive a class of morphological pyramids in which the pyramidal synthesis operator $\psi_A^{\downarrow} = \delta_A\, \sigma^{\downarrow}$ is fixed to be a dilation δ_A by a structuring element A (preceded by an upsampling operator $\sigma^{\downarrow}$), but where the pyramid analysis operator has the form $\psi_A^{\uparrow} = \sigma^{\uparrow}\, \eta_A$ where $\sigma^{\uparrow}$ denotes downsampling and η_A may be chosen in different ways. Two particular cases of this type of pyramid were mentioned in [?, ?]: (i) the adjunction pyramid, where η_A equals an erosion ε_A by a structuring element A; (ii) the Sun-Maragos pyramid, where η_A is an opening $\alpha_A = \delta_A\, \varepsilon_A$. As we will show below, choosing the operator η_A to be an erosion ε_A, followed by an arbitrary number of conditional dilations with structuring element A also leads to a valid synthesis operator, that is, the pair $(\psi_A^{\uparrow}, \psi_A^{\downarrow})$ satisfies the pyramid condition. Note that this class also contains the opening by reconstruction, which is the connected filter obtained by iterating the conditional dilations until idempotence [**?**].

The motivation to study this class of pyramids stems from our work on multiresolution volume visualization. Volume visualization or volume rendering is a technique to produce two-dimensional images of three-dimensional data from different viewpoints, using advanced computer graphics techniques such as illumination, shading and colour. Interactive rendering of volume data is a demanding problem due to the large sizes of the signals. For this purpose multiresolution models are developed, which can be used to visualize data incrementally (‘progressive refinement’). In preview mode, when a user is exploring the data from different viewpoints, a coarse representation is used whose data size is smaller than that of the original data, so that rendering is accelerated and thus user interaction is improved. For the case of *X-ray volume rendering*, which is a linear transform based upon integrating the 3-D data along the line of sight, wavelets have been studied extensively for multiresolution visualization [**?**, **?**].

Another volume rendering method widely used in medical imaging is *maximum intensity projection* (MIP) where one computes the maximum, instead of the integral, along the line of sight. Since this transform is nonlinear, morphological pyramids are a suitable tool for multiresolution analysis. More in particular, pyramids where the synthesis operator is a dilation are particularly appropriate, because in this case the maxima along the line of sight can be computed on a coarse level (where the size of the data is reduced), before applying a two-dimensional synthesis operator to perform reconstruction of the projection image to full grid resolution. Two cases we have recently investigated for MIP volume rendering are the adjunction pyramid [**?**, **?**] and the Sun-Maragos pyramid [**?**]. One of the problems with the adjunction pyramid is that too few small features are retained in higher levels of the pyramid. The basic reason is that the initial erosion of the analysis operator removes fine details. The subsequent downsampling step only aggravates the situation. In the Sun-Maragos pyramid this situation improved, essentially because erosions are replaced by openings, which keep image features to a larger extent, so that the chance that (parts of) these features survive the downsampling step is larger. From here, it is only a small step to conjecture that perhaps a number of conditional dilations after the erosion might do even better, because such operators reconstruct more of a certain feature provided some part of it survives the initial erosion. In future work, we intend to apply the new class of pyramids derived here to the MIP volume rendering problem to see whether further improvements can be obtained.

The remainder of this paper is organized as follows. Section **??** recalls a few preliminaries on morphological pyramids. In section **??** we derive the new class of morphological pyramids. Some examples are discussed in section **??**. Section **??** contains a summary and discussion of future work.

2 Preliminaries

Consider signals in a d-dimensional signal space V_0, which is assumed to be the set of functions on (a subset of) the discrete grid $\mathbb{Z}^d$, where $d = 2$ or $d = 3$ (image and volume data), that take values in a finite set of nonnegative integers.

The general structure of linear as well as nonlinear pyramids is as follows. From an initial signal f_0, approximations $\{f_j\}$ of increasingly reduced size are computed by a decomposition or analysis operator $\psi^{\uparrow}$:

$$f_j = \psi^{\uparrow}(f_{j-1}), \qquad j = 1, 2, \ldots L.$$

Here j is called the level of the decomposition. In the case of a Gaussian pyramid, the analysis operator consists of Gaussian low-pass filtering, followed by downsampling [?]. An approximation error associated to f_{j+1} may be defined by taking the difference between f_j and an expanded version of f_{j+1}:

$$d_j = f_j \dot{-} \psi^{\downarrow}(f_{j+1}). \tag{1}$$

Here $\dot{-}$ is a generalized subtraction operator. The set $d_0, d_1, \ldots, d_{L-1}, f_L$ is referred to as a *detail pyramid*. Assuming there exists an associated generalized addition operator $\dot{+}$ such that, for all j,

$$\hat{f}_j \dot{+} (f_j \dot{-} \hat{f}_j) = f_j, \quad \text{where } \hat{f}_j = \psi^{\downarrow}(\psi^{\uparrow}(f_j)),$$

we have *perfect reconstruction*, that is, f_0 can be exactly reconstructed by the recursion

$$f_j = \psi^{\downarrow}(f_{j+1}) \dot{+} d_j, \quad j = L-1, \ldots, 0. \tag{2}$$

For the linear case, the detail pyramid is called a Laplacian pyramid, and the synthesis operation consists of upsampling, followed by Gaussian low-pass filtering [?]. In the case of morphological pyramids, the analysis and synthesis operators involve morphological filtering instead of Gaussian filtering [?, ?]. It should be noted that, in principle, the analysis and synthesis operators may depend on level, but we assume them to be the same for all levels j throughout this paper.

To guarantee that information lost during analysis can be recovered in the synthesis phase in a non-redundant way, one needs the so-called *pyramid condition*:

$$\psi^{\uparrow}(\psi^{\downarrow}(f)) = f \text{ for all } f. \tag{3}$$

By approximations of f we will mean signals in V_0 of the same size as the initial signal f which are reconstructed from higher levels of the pyramid by omitting some of the detail signals. More precisely, a level-j approximation $\hat{f}_j^{(0)}$ of f is defined as

$$\hat{f}_j^{(0)} = \psi^{\downarrow j}(f_j), \tag{4}$$

where $\psi^{\downarrow j}$ means repeating the $\psi^{\downarrow}$ operator j times.

The generalized addition and subtraction operators $\dot{+}$ and $\dot{-}$ appearing in the definition (**??**) of the detail signals and the reconstruction equation (**??**) may be taken as ordinary addition and subtraction, although other choices are sometimes possible as well [?, ?, ?].

2.1 Adjunction pyramid

Morphological adjunction pyramids [?] involve the morphological operators of dilation $\delta_A(f)$ and erosion $\varepsilon_A(f)$ with structuring element A. In this case the analysis and synthesis operators are denoted by $\psi_A^{\uparrow}$ and $\psi_A^{\downarrow}$, respectively, and have the form

$$\psi_A^{\uparrow}(f) = \sigma^{\uparrow}\left(\varepsilon_A(f)\right), \tag{5}$$

$$\psi_A^{\downarrow}(f) = \delta_A\left(\sigma^{\downarrow}(f)\right), \tag{6}$$

where the arrows indicate transformations to higher (coarser) or lower (finer) levels of the pyramid. Here $\sigma^{\uparrow}$ and $\sigma^{\downarrow}$ denote downsampling and upsampling by a factor of 2 in each spatial dimension:

$$\sigma^{\uparrow}(f)(n) = f(2n)$$

$$\sigma^{\downarrow}(f)(m) = \begin{cases} f(n), & \text{if } m = 2n \\ 0, & \text{otherwise} \end{cases}$$

The pyramid condition (**??**) is satisfied, if there exists an $a \in A$ such that the translates of a over an even number of grid steps are never contained in the structuring element A, cf. [?]. Introducing the notation

$$\mathbb{Z}^d[n] = \{k \in \mathbb{Z}^d | k - n \in 2\mathbb{Z}^d\}$$

$$A[n] = A \cap \mathbb{Z}^d[n]$$

the pyramid condition can be expressed as

$$A[a] = \{a\} \text{ for some } a \in A. \tag{7}$$

2.2 Sun-Maragos pyramid

This pyramid is defined by the following choice of analysis and synthesis operators:

$$\psi_A^{\uparrow}(f) = \sigma^{\uparrow}\left(\alpha_A(f)\right), \tag{8}$$

$$\psi_A^{\downarrow}(f) = \delta_A\left(\sigma^{\downarrow}(f)\right), \tag{9}$$

where $\alpha_A = \delta_A\, \varepsilon_A$ is the opening by structuring element A. Note that the synthesis operator is identical to that of the adjunction pyramid, cf. (**??**). Under the condition that

$$A[\mathbf{0}] = \{\mathbf{0}\}, \tag{10}$$

where **0** is the origin of $\mathbb{Z}^d$, the pyramid condition (**??**) is satisfied (see [?, Proposition 5.9]), that is,

$$\psi_A^{\uparrow}\,\psi_A^{\downarrow} = \sigma^{\uparrow}\,\delta_A\,\varepsilon_A\,\delta_A\sigma^{\downarrow} = \mathsf{id},$$

where id denotes the identity operator. Since $(\varepsilon_A, \delta_A)$ is an adjunction, we have that $\delta_A\,\varepsilon_A\,\delta_A = \delta_A$. Therefore, when A satisfies (**??**), the previous formula implies that

$$\sigma^{\uparrow}\,\delta_A\,\sigma^{\downarrow} = \mathsf{id}. \tag{11}$$

3 A new class of morphological pyramids

In this section we present the main contribution of this paper, which is the derivation of a new class of morphological pyramids containing the adjunction pyramid and Sun-Maragos pyramid as special cases.

We start by recalling the definition of the opening by reconstruction. Let f be a d-dimensional signal (e.g. image or volume data). We define a sequence of operators $R_A^{(n)}$ for $n = 0, 1, 2, \ldots$ by the following recursion:

$$R_A^{(0)}(f) = \varepsilon_A(f) \tag{12}$$

$$R_A^{(n)}(f) = f \wedge \delta_A(R_A^{(n-1)}(f)), \qquad n = 1, 2, \ldots. \tag{13}$$

The operator in (**??**) is a conditional dilation, that is, after each dilation step the infimum with the original signal f is taken. Then $R_A^{(\infty)}(f)$ is the opening by reconstruction of f from its erosion $\varepsilon_A(f)$. In practice, f is defined on a finite subset $D \subseteq \mathbb{Z}^d$ and the recursion terminates after a finite number of steps.

We now consider the class of pyramids whose analysis/synthesis operator pairs have the form

$$\psi_A^{\uparrow}(f) = \sigma^{\uparrow}\,(R_A^{(n)}(f)), \tag{14}$$

$$\psi_A^{\downarrow}(f) = \delta_A\,(\sigma^{\downarrow}\,(f)), \tag{15}$$

where $\sigma^{\uparrow}$ and $\sigma^{\downarrow}$ denote dyadic downsampling and upsampling as introduced in section **??**, and the structuring element A satisfies condition (**??**). By observing that $R_A^{(0)}(f)$ is the erosion ε_A of f and $R_A^{(1)}(f)$ is the opening $\alpha_A = \delta_A\,\varepsilon_A$ of f, we see that the cases $n = 0$ and $n = 1$ correspond to the adjunction pyramid and Sun-Maragos pyramid, respectively. Our task is to prove that the pair of operators (**??**), (**??**) satisfies the pyramid condition.

First, the following lemma is proved.

Lemma 1. *Consider a morphological pyramid with analysis operator $\psi^{\uparrow} = \sigma^{\uparrow}\,\eta$ and synthesis operator $\psi^{\downarrow} = \delta_A\,\sigma^{\downarrow}$, satisfying the following assumptions:*

1. *η is an anti-extensive operator*
2. *$\eta\,\delta_A \geq \mathsf{id}$*
3. *The structuring element A satisfies condition (**??**).*

Then the pyramid condition holds.

Proof. By assumption 2 we have that $\psi^{\uparrow}\,\psi^{\downarrow} = \sigma^{\uparrow}\,\eta\,\delta_A\,\sigma^{\downarrow} \geq \sigma^{\uparrow}\,\sigma^{\downarrow} = \mathsf{id}$. On the other hand, from assumption 1, $\psi^{\uparrow}\,\psi^{\downarrow} \leq \sigma^{\uparrow}\,\delta_A\,\sigma^{\downarrow}$. By assumption 3, formula (**??**) holds, that is, $\sigma^{\uparrow}\,\delta_A\,\sigma^{\downarrow} = \mathsf{id}$. Hence we found that $\psi^{\uparrow}\,\psi^{\downarrow} \leq \mathsf{id}$ and that $\psi^{\uparrow}\,\psi^{\downarrow} \geq \mathsf{id}$, so $\psi^{\uparrow}\,\psi^{\downarrow} = \mathsf{id}$.

Now we return to the problem of showing that the pair (**??**), (**??**) satisfies the pyramid condition for each n. It is sufficient to show that the operator $\eta = R_A^{(n)}$ satisfies assumptions 1 and 2 of the lemma, since assumption 3 was assumed to hold anyhow.

1. The operator $R_A^{(0)} = \varepsilon_A$ is anti-extensive, because (**??**) implies that $\mathbf{0} \in A$, and hence the erosion ε_A is anti-extensive. For $n > 0$, equation (**??**) trivially implies that $R_A^{(n)}(f) \leq f$. Hence $R_A^{(n)}$ is anti-extensive for all $n \geq 0$.
2. We prove by induction that assumption 2 holds. First, $R_A^{(0)}(\delta_A(f)) = \varepsilon_A(\delta_A(f)) \geq f$ since $\varepsilon_A\,\delta_A$ is a closing. Second, for $n > 0$

$$R_A^{(n)}(\delta_A(f)) = \delta_A(f) \wedge \delta_A\left(R_A^{(n-1)}(\delta_A(f))\right).$$

Applying the induction hypothesis, i.e. $R_A^{(n-1)}\delta_A \geq \mathsf{id}$, we find

$$R_A^{(n)}(\delta_A(f)) \geq \delta_A(f) \wedge \delta_A(f) = \delta_A(f).$$

Finally, (**??**) implies that $\mathbf{0} \in A$, and hence the dilation δ_A is extensive. Therefore $R_A^{(n)}(\delta_A(f)) \geq f$, and we are done.

4 Example

In this section, we apply the pyramids discussed in the previous section for image analysis. We computed image decompositions according to a number of analysis/synthesis operator pairs (**??**), (**??**) corresponding to various values of n. Two aspects were considered in the experiments. First, the error $\mathcal{E}_k^{(j)}$ of a level-j approximation $\hat{f}_j^{(0)}$ as defined in (**??**):

$$\mathcal{E}_k^{(j)} = \|f - \hat{f}_j^{(0)}\|_k / \|f\|_k, \tag{16}$$

for $k = 1, 2, \infty$ corresponding to the L_1, L_2 and L_∞ norms, respectively. Here $\hat{f}_j^{(0)}$ is computed by $\hat{f}_j^{(0)} = \psi_A^{\downarrow j}(f_j)$, cf. (**??**), where the partial reconstruction f_j is computed according to the recursion (**??**). For a pyramid with L levels, $\hat{f}_j^{(0)}$ only takes the highest approximation signal f_L, as well as the detail signals d_m with $m = L-1, L-2, \ldots, j$ into account. In all cases, we computed the detail signals and reconstructions by using ordinary addition and subtraction in (**??**) and (**??**).

Second, we looked at the entropy of the detail signals, which is a measure for the amount of data compression which is achievable. Both measures are essential quality indicators for the case of volume rendering of three-dimensional data which is the motivation for this work (see the introduction).