

Solving the empty space problem in robot path planning by mathematical morphology¹

J.B.T.M. Roerdink

Dept. of Computing Science, University of Groningen

P.O. Box 800, 9700 AV Groningen, The Netherlands

Tel. +31-50-3633931; Fax +31-50-3633800; Email roe@cs.rug.nl

Abstract— In this paper we formulate a morphological approach to path planning problems, in particular with respect to the empty-space problem, that is, the question of finding the allowed states for an object, moving in a space with obstacles. Our approach is based upon a recent generalization of mathematical morphology to spaces with noncommutative invariance groups.

I. INTRODUCTION

The problem of path planning is to find a path for an object, say a robot or a car, moving in a space (called ‘work space’) with obstacles. The problem falls apart into two distinct subproblems [3]. First, the *empty-space problem*: find the allowed states of the robot¹. Any possible configuration of the robot is represented as a point in a *configuration space* \mathcal{C} , whose dimensionality equals the number of degrees of freedom of the robot [2]. Points in \mathcal{C} such that a robot in that configuration would collide with any of the obstacles in work space are ‘forbidden’. The set of allowed points of \mathcal{C} is called ‘empty-space’. The second problem to be solved is the *find-path problem*: find a trajectory in the empty space, where the definition of ‘trajectory’ has to specify which transitions between allowed states are permissible. This may involve certain additional constraints, such as connectedness or continuity. Kinematic constraints also fall in this category; for example, a car cannot move sideways, etc.

This paper, which uses an approach based on mathematical morphology, is only concerned with the empty-space problem. That is, we do not deal with the *kinematics*, nor the *dynamics* of robot motion.

First consider *mobile* robots whose location is not fixed. If only translations of the robot are possible the solution is simple. Since a robot has a finite size, one can find allowed positions of the (arbitrarily chosen) center of the robot by an ordinary erosion of the space outside the obstacles, where the structuring element B is the robot itself. Equivalently, one may perform the dilation by the reflected

set $\overset{\vee}{B}$ of the set of obstacles to find the *forbidden* positions of the center of the robot. The second alternative is more efficient when the obstacle space is smaller than the space outside the obstacles.

If the robot has rotational degrees of freedom the problem is more difficult. Now one has to perform dilations with all rotated versions of the robot. This case was briefly discussed already in [7]. The problem becomes even more difficult when the robot has internal degrees of freedom. This happens, for example, for a robot with several rotating joints.

The goal of this paper is to outline a general solution of the empty space problem in terms of morphological operations, thus generalizing the existing solution for robots with translational degrees of freedom. To this end we use a general construction of morphological operators on spaces with transitive transformation groups developed recently by the author [5]. Here *transitive* means that for any pair of points in a set \mathcal{X} acted upon by a group Γ , there exists a transformation $g \in \Gamma$ which maps one point on the other. In the classical case, the group Γ is given by the translation group \mathbf{T} , which acts on the plane $\mathcal{X} = \mathbb{R}^2$. When also rotations are allowed, the group Γ becomes the Euclidean motion group \mathbf{M} . For robots with several joints the problem can be decomposed into subproblems for each joint, with associated motion groups Γ , as will be shown below. The examples given here are limited to planar robots, but the results can easily be extended to higher dimensional cases.

Another class of problems arises for manipulator robots, that is, robots — with one or more joints — *fixed* in one point. Then the group of allowed motions, e.g. rotations of the joints, is no longer transitive on the points of the plane, and a modification of the framework of transitive transform-

¹From now on, we use the word ‘robot’ as a generic term for a moving object.

ation groups is necessary. It turns out that even then a morphological approach to the empty-space problem is possible. This case, requiring the study of intransitive group actions, will be treated in a forthcoming paper [6].

II. PRELIMINARIES

A. Generalized Minkowski operators

The classical Minkowski addition and subtraction for subsets X, A of \mathbb{R}^n are given by

$$X \oplus A = \bigcup_{a \in A} X_a, \quad (1)$$

$$X \ominus A = \bigcap_{a \in A} X_{-a}, \quad (2)$$

where

$$X_a = \tau_a(X) = \{x + a : x \in X\},$$

is the translate of X over the vector $a \in \mathbb{R}^n$, $x + y$ is the sum of x and y , and $-x$ the reflection of x . It can be shown that

$$X \oplus A = \{h \in \mathbb{R}^n : \overset{\vee}{A}_h \uparrow X\}, \quad (3)$$

where $\overset{\vee}{A} = \{-a : a \in A\}$ is the *reflection* of A and $A \uparrow B$ (A ‘hits’ B) is a general notation for $A \cap B \neq \emptyset$.

On any group Γ one can define generalizations of the Minkowski operations [5]. Recall that a *dilation* (*erosion*) is a mapping commuting with unions (intersections). For any subsets G, H of Γ define the dilation

$$(G) := G \overset{\Gamma}{\oplus} H := \bigcup_{h \in H} Gh = \bigcup_{g \in G} gH, \quad (4)$$

which generalizes the Minkowski addition to non-commutative groups. Here

$$gH := \{gh : h \in H\}, \quad Gh := \{gh : g \in G\},$$

with gh the group product of g and h . Similarly, define the erosion (h^{-1} is the group inverse of h)

$$\varepsilon(G) := G \overset{\lambda}{\ominus} H := \bigcap_{h \in H} Gh^{-1},$$

which generalizes the Minkowski subtraction. Both mappings are *left-invariant*, e.g.

$$(gG) = g(G), \quad \forall g \in \Gamma.$$

This is the reason for the superscript ‘ λ ’ on the ‘ \ominus ’ symbol. For later use we also define the *inverted* set G^{-1} of G by

$$G^{-1} = \{g^{-1} : g \in G\}. \quad (5)$$

Duality by complementation is expressed by the formula $(G \overset{\Gamma}{\oplus} H)^c = G^c \overset{\lambda}{\ominus} H^{-1}$.

B. Group actions

Let \mathcal{X} be a non-empty set, Γ a transformation group on \mathcal{X} , that is, each element $g \in \Gamma$ is a mapping $g : \mathcal{X} \rightarrow \mathcal{X}$, satisfying

$$(i) \quad gh(x) = g(h(x)) \quad (ii) \quad e(x) = x,$$

where e is the unit element of Γ , and gh denotes the product of two group elements g and h . The *inverse* of an element $g \in \Gamma$ will be denoted by g^{-1} . Instead of $g(x)$ we will also write gx . We say that Γ is a *group action* on \mathcal{X} [1,4,8].

The group Γ is called *transitive* on \mathcal{X} if for each $x, y \in \mathcal{X}$ there is a $g \in \Gamma$ such that $gx = y$, and *simply transitive* when this element g is unique. A *homogeneous space* is a pair (Γ, \mathcal{X}) where Γ is a group acting transitively on \mathcal{X} . Any transitive abelian permutation group Γ is simply transitive. If Γ acts on \mathcal{X} , the *stabilizer* of $x \in \mathcal{X}$ is the subgroup $\Gamma_x := \{g \in \Gamma : gx = x\}$. Let \emptyset be an arbitrary but fixed point of \mathcal{X} , henceforth called the *origin*. The stabilizer Γ_\emptyset will be denoted by Σ from now on:

$$\Sigma := \Gamma_\emptyset = \{g \in \Gamma : g\emptyset = \emptyset\}. \quad (6)$$

The set of group elements which map \emptyset to a given point x is called a *left coset* and denoted by

$$g_x \Sigma := \{g_x s : s \in \Sigma\}. \quad (7)$$

Here g_x is a representative (an arbitrary element) of this coset.

In the following we present two examples, as we will need them in what follows. In each case Γ denotes the group and \mathcal{X} the corresponding set.

Example 1: $\mathcal{X} = \text{Euclidean space } \mathbb{R}^n$, Γ = the Euclidean translation group \mathbf{T} . \mathbf{T} is abelian. Elements of \mathbf{T} can be parametrized by vectors $h \in \mathbb{R}^n$, with τ_h the translation over the vector h :

$$\tau_h x = x + h, \quad h \in \mathbf{T}, x \in \mathbb{R}^n. \quad (8)$$

Example 2: $\mathcal{X} = \text{Euclidean space } \mathbb{R}^n$ ($n \geq 2$), Γ = the Euclidean motion group $\mathbf{M} := E^+(3)$ (proper Euclidean group, group of rigid motions), i.e. the group generated by translations and rotations (see [7]). The subgroup leaving a point p fixed is the set of all rotations around that point. \mathbf{M} is not abelian. The collection of translations forms the Euclidean translation group \mathbf{T} . The stabilizer, denoted by \mathbf{R} , equals the circle group S^1 (also commutative) of rotations around the origin. Let τ_h denote the translation over the vector $h \in \mathbb{R}^2$ and ρ_ϕ^p the rotation over an angle ϕ around the point p .

The following relations, whose proof is left to the reader, are needed in the sequel:

$$\rho_\phi^p = \tau_p \rho_\phi^0 \tau_{-p} = \tau_{p-\rho_\phi^0 p} \rho_\phi^0, \quad (9)$$

$$\rho_\phi^p \tau_h = \tau_{\rho_\phi^0 h} \rho_\phi^p. \quad (10)$$

Let $\gamma_{h,\phi}$ denote a rotation around the origin followed by a translation:

$$\gamma_{h,\phi} = \tau_h \rho_\phi^0, \quad h \in \mathbb{R}^2, \phi \in S^1. \quad (11)$$

Any element of \mathbf{M} can be written in this form. Using the rules (9)–(10) one finds the multiplication rule

$$\gamma_{h,\phi} \gamma_{h',\phi'} = \gamma_{h+\rho_\phi^0 h', \phi+\phi'}. \quad (12)$$

B.1 Geometrical representation

The following representation is useful in this case [5,7]. Attach a set of unit vectors \vec{v} with direction varying over the unit circle to each a point in the plane. We call $\mathbf{p} := (x, \vec{v})$ a *pointer*. Given any pointer $\mathbf{p} = (x, \vec{v})$, there is a unique element of the group $\mathbf{\Gamma}$ which maps a fixed pointer $\mathbf{b} := (\phi, \vec{e}_1)$ to \mathbf{p} , where ϕ is the origin and \vec{e}_1 a unit vector in the x -direction. So the pointer $\mathbf{p} = (x, \vec{v})$, where $\vec{v} = (\cos \phi, \sin \phi)$, represents the motion $\gamma_{x,\phi}$. In this representation, the rotation group \mathbf{R} is the set of unit vectors attached to the origin and \mathbf{T} is represented by the collection of horizontal unit vectors attached to points of \mathbb{R}^2 . In the discrete case we will use a hexagonal grid and replace \mathbf{R} by a finite group \mathbf{H} consisting of rotations over multiples of 60° . The coset $\tau_x \mathbf{R}$ is represented on the hexagonal grid by the six unit vectors attached to the point x [5].

B.2 Morphological operations

One can construct morphological operations on an arbitrary homogeneous space \mathcal{X} as follows. Define the ‘origin’ ϕ to be an arbitrarily chosen point of \mathcal{X} . To each subset X of \mathcal{X} we associate all elements of the group which map the origin ϕ to an element of X . We also can go back from the group $\mathbf{\Gamma}$ to the space \mathcal{X} by associating to each subset G of $\mathbf{\Gamma}$ the collection of all points $g\phi$ where g ranges over G . This is summarized in the following definition. Here and below, we use the symbol $\mathcal{P}(A)$ for the set of all subsets of A , A an arbitrary set.

Definition 3: The *lift* $\vartheta : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathbf{\Gamma})$ and *canonical projection* $\pi : \mathcal{P}(\mathbf{\Gamma}) \rightarrow \mathcal{P}(\mathcal{X})$ are defined by

$$\begin{aligned} \vartheta(X) &= \{g \in \mathbf{\Gamma} : g\phi \in X\}, \quad X \subseteq \mathcal{X} \\ \pi(G) &= \{g\phi : g \in G\}, \quad G \subseteq \mathbf{\Gamma}. \end{aligned}$$

For the case of Example 2, these formulas specialize to

$$\vartheta(X) = \bigcup_{x \in X} \tau_x \mathbf{R} = \tau(X) \overset{\mathbf{M}}{\oplus} \mathbf{R}, \quad (13)$$

where

$$\tau(X) := \{\tau_x : x \in X\}. \quad (14)$$

In [5,7] a construction was performed of various morphological operators between the distinct lattices $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathbf{\Gamma})$. Here we restrict ourselves to dilations. That is, consider the mapping — referred to as the *hitting function* below — which associates to a subset X of \mathcal{X} the set of group elements $g \in \mathbf{\Gamma}$ for which the translated set $gB := \{gb : b \in B\}$ hits X (cf. (3)):

$$\mathcal{Q}_B(X) := \{g \in \mathbf{\Gamma} : gB \uparrow X\}. \quad (15)$$

Then it was shown in [5] that

$$\begin{aligned} \mathcal{Q}_B(X) &= \{g \in \mathbf{\Gamma} : g\vartheta(B) \uparrow \vartheta(X)\} \\ &= \vartheta(X) \overset{\mathbf{\Gamma}}{\oplus} \vartheta^{-1}(B). \end{aligned} \quad (16)$$

Here $\vartheta(X) \overset{\mathbf{\Gamma}}{\oplus} \vartheta^{-1}(B)$ is a generalized Minkowski operation on subsets of $\mathbf{\Gamma}$ as defined in (4), with $\vartheta^{-1}(B)$ the inverted set of $\vartheta(B)$, cf. (5). This mapping is

- a dilation $\mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathbf{\Gamma})$;
- invariant under $\mathbf{\Gamma}$, that is $\mathcal{Q}_B(g(X)) := g\mathcal{Q}_B(X)$, $g \in \mathbf{\Gamma}$.

More generally, if A is a subset of $\mathbf{\Gamma}$ and $x \mapsto g_x A$ a function from \mathcal{X} to $\mathbf{\Gamma}$, the mapping

$$\delta_A^\Gamma(X) := \bigcup_{x \in X} g_x A, \quad (17)$$

is a dilation $\mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathbf{\Gamma})$ which is $\mathbf{\Gamma}$ -invariant. For this reason we speak sometimes of *group dilations*, or $\mathbf{\Gamma}$ -*dilations*. Dilations (and erosions) from $\mathcal{P}(\mathbf{\Gamma})$ to $\mathcal{P}(\mathcal{X})$ or from $\mathcal{P}(\mathcal{X})$ to $\mathcal{P}(\mathcal{X})$ can be constructed similarly but are not needed in the sequel.

III. PATH PLANNING

In this section we give a solution of the empty space problem in path planning by morphological techniques. Only the case of *mobile* robots is discussed here. First the standard case of a robot with translational degrees of freedom is reviewed. Next we allow rotations, and give the solution already outlined in [7]. Finally we consider robots with two fully rotational joints, where the rotation angle runs from 0 to 2π . Once more it is emphasized that the find-path problem does not concern us here:

when we speak of an ‘allowed motion’ g below, we merely mean that the state resulting from the action of g on an allowed initial state is allowed as well. This does not imply any statement whatsoever about the permissibility of intermediate states during the motion.

A. Mobile robots with one joint

In this subsection the allowed motion group equals either the translation group or the translation-rotation group.

A.1 Translations only

Consider a robot moving in the plane \mathbb{R}^2 with obstacles. The robot corresponds to a subset B of the plane and the obstacles to another subset, say X . The problem is to find the set of forbidden configurations. The state of the robot is parametrized by the location h of an arbitrary point of the robot B , initially at the origin; hence the configuration space \mathcal{C} is identical to \mathbb{R}^2 . The allowed motions form the translation group \mathbf{T} which can be identified with \mathbb{R}^2 , see Example 1 above. The forbidden points can be identified with the set

$$\mathcal{Q}_B(X) = \{h \in \mathbb{R}^2 : \tau_h B \uparrow X\}. \quad (18)$$

One immediately recognizes this (cf. (3)) as the Euclidean dilation of X by \check{B} . This leads to the first result.

Proposition 4: *If $B \in \mathbb{R}^2$ is a robot with translational degrees of freedom, then the hitting function $\mathcal{Q}_B : \mathcal{P}(\mathbb{R}^2) \rightarrow \mathcal{P}(\mathcal{C})$ is given by*

$$\begin{aligned} \mathcal{Q}_B(X) &= X \oplus^{\mathbf{T}} \check{B} \\ &=: \delta_B^{\mathbf{T}}(X) = \bigcup_{x \in X} \tau_x \check{B}. \end{aligned} \quad (19)$$

For clarity the dependence of the Minkowski sum on the Euclidean translation group \mathbf{T} is explicitly indicated in (19). An example can be found in Fig. 1.

A.2 Translations and rotations

Next the case of a mobile robot with translational and rotational degrees of freedom is considered. The appropriate group is \mathbf{M} , the Euclidean motion group, see Example 2. To parametrize the state of the robot, choose two distinct points, say P_1 and P_2 inside the robot. The configuration space \mathcal{C} is 3-dimensional in this case,

$$\mathcal{C} := \{(h, \phi) : h \in \mathbb{R}^2, \phi \in S^1\}, \quad (20)$$

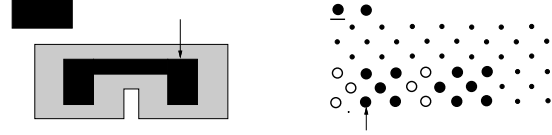


Figure 1: The forbidden set $\mathcal{Q}_B(X)$ for a robot B with translational degrees of freedom. (a) continuous case: dark area X : the obstacle; hatched area (plus the enclosed dark area): the dilated set. The origin ϕ is in the center of the rectangle B . (b) discrete case: heavy dots: the obstacle X ; heavy plus open dots: the dilated set. The underlining in B indicates the location of the origin.

with h the location of point P_1 and ϕ the angle of the line segment $[P_1 P_2]$ with respect to the x -axis. Note that \mathcal{C} can be identified with the parameter space of the Euclidean motion group \mathbf{M} : to each (h, ϕ) in \mathcal{C} corresponds a unique motion $\gamma_{h, \phi} = \tau_h \rho_\phi^0$ and vice versa. This identification will tacitly be made below without further comment. An alternative way to represent \mathcal{C} is by means of pointers; see Sect. B. Assuming the initial state to be equal to $\{h = (0, 0), \phi = 0\}$, the hitting function in this case becomes

$$\mathcal{Q}_B(X) = \{(h, \phi) \in \mathcal{C} : \gamma_{h, \phi} B \uparrow X\}. \quad (21)$$

Now the results of Sect. B are applicable. From the general result (16) together with (13)–(14) it follows that

$$\begin{aligned} \mathcal{Q}_B(X) &= \vartheta(X) \overset{\mathbf{M}}{\oplus} \vartheta^{-1}(B) \\ &= \vartheta(X) \overset{\mathbf{M}}{\oplus} (\mathbf{R} \overset{\mathbf{M}}{\oplus} \tau^{-1}(B)) \end{aligned} \quad (22)$$

where (cf. (14))

$$\tau^{-1}(B) = \{\tau_b^{-1} : b \in B\} = \tau(\check{B}), \quad (23)$$

with \check{B} the reflected set of B . From (22) one obtains using the following result.

Proposition 5: *If $B \in \mathbb{R}^2$ is a robot with translational and rotational degrees of freedom, then the hitting function $\mathcal{Q}_B : \mathcal{P}(\mathbb{R}^2) \rightarrow \mathcal{P}(\mathcal{C})$ is given by*

$$\mathcal{Q}_B(X) = \bigcup_{x \in X} \tau_x (\mathbf{R} \overset{\mathbf{M}}{\oplus} \tau^{-1}(B)) \quad (24)$$

$$= \bigcup_{\phi \in S^1} \bigcup_{x \in X} \tau_x \rho_\phi^0 \tau(\check{B}) \quad (25)$$

The equality (25) expresses the fact, which is obvious from (21), that $\mathcal{Q}_B(X)$ can be found by doing,

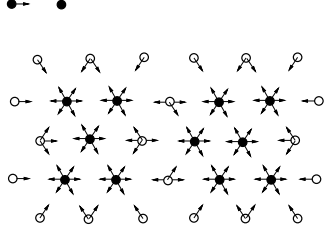


Figure 2: The forbidden set $\mathcal{Q}_B(X)$ for a robot B with translational and rotational degrees of freedom. Heavy dots: the obstacle space X ; b : base pointer. Arrows attached to heavy and open dots: the forbidden set.

for each $\phi \in S^1$ an ordinary dilation with a rotated version $\rho_\phi^0 B$ of the structuring element. Equation (24) says that $\mathcal{Q}_B(X)$ can also be found as a union of translates, that is, by a dilation

$$\delta_B^T(X) := \bigcup_{x \in X} \tau_x \tilde{B}, \quad (26)$$

where

$$\tilde{B} := \mathbf{R} \oplus \tau^{-1}(B). \quad (27)$$

Eq. (26) differs from a usual \mathbf{T} -dilation through the fact that the structuring element (27) is not a planar, but a 3D subset of \mathcal{C} . The construction of \tilde{B} is straightforward:

- take the reflection \check{B} of B ;
- lift \check{B} to \mathcal{C} by applying τ ;
- construct rotated copies of $\tau(\check{B})$ in \mathcal{C} .

These results can be nicely presented geometrically using the representation by pointers (cf. Example 2), see Fig. 2. Alternatively, one may use a 3D representation in configuration space \mathcal{C} .

B. Mobile robots with several joints

The case of mobile robots with a finite number of joints can be solved by a combination of cases presented above. This is demonstrated here for a two-joint robot. This case is atypical in the sense that it can be completely decomposed into two 3D problems. The case of three joints is considerably more involved; lack of space does not permit us to treat it here. The interested reader is referred to the full paper [6].

So consider a robot B consisting of two joints B_1 and B_2 , which are connected in one point P . Allowed motions are translation of the complete robot

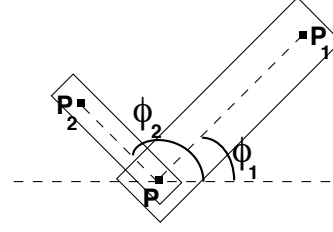


Figure 3: Mobile robot with two freely rotating joints.

B and free rotations of B_1 , B_2 around P . To parametrize the state of the robot, choose a point P_1 inside B_1 and a point P_2 inside B_2 — both distinct from P . As state vector we now take (h, ϕ_1, ϕ_2) , where h is the location of the point P in the plane, and ϕ_1, ϕ_2 are the angles of the two joints (i.e., of $[PP_1]$ and $[PP_2]$, respectively) with respect to the x -axis; cf. Fig. 3. The configuration space \mathcal{C} is 4-dimensional in this case,

$$\mathcal{C} := \{(h, \phi_1, \phi_2) : h \in \mathbb{R}^2, \phi_1 \in S^1, \phi_2 \in S^1\}. \quad (28)$$

The hitting function is now

$$\mathcal{Q}_B(X) = \{(h, \phi_1, \phi_2) \in \mathcal{C} : \tau_h(\rho_{\phi_1}^0 B_1 \cup \rho_{\phi_2}^0 B_2) \uparrow X\}, \quad (29)$$

where it is assumed that the initial state is $(0, 0, 0)$. Using the fact that τ_h commutes with unions, and realizing that the union of two sets hits a third set X if and only if one of these sets hits X , (29) can be written as a union of two \mathbf{M} -dilations:

$$\mathcal{Q}_B(X) = \mathcal{Q}_{B_1}(X) \cup \mathcal{Q}_{B_2}(X) \quad (30)$$

where, for $i = 1, 2$,

$$\mathcal{Q}_{B_i}(X) = \{(h, \phi_1, \phi_2) \in \mathcal{C} : (h, \phi_i) \in \delta_{B_i}^M(X)\} \quad (31)$$

Here $\delta_{B_i}^M : \mathcal{P}(\mathcal{C}_i) \rightarrow \mathcal{P}(\mathcal{C}_i)$, where

$$\mathcal{C}_i := \{(h, \phi_i) : h \in \mathbb{R}^2, \phi_i \in S^1\}, \quad (32)$$

are two \mathbf{M} -dilations as defined in (26).

This formula shows that the empty space problem is decomposed into two independent parts, one for each part of the robot. Moreover, each of the two subproblems is essentially a 3D problem. It can be shown that (30) is again expressible as a \mathbf{T} -dilation, with a structuring element in \mathcal{C} ; cf. [6].

REFERENCES

- [1] Berger, M., *Geometry I*. Springer, Berlin, 1987.
- [2] Dorst, L., Mandhyan, I. and Trovato, K., "The geometrical representation of path planning problems," *Robotics and Autonomous Systems*, 7, pp. 181–195, 1991.
- [3] Gouzènes, L., "Strategies for solving collision-free trajectories problems for mobile and manipulator robots," *Intern. J. Robotics Res.*, 3, pp. 51–65, 1984.
- [4] Robinson, D. J. S., *A Course in the Theory of Groups*. Springer, Berlin, 1982.
- [5] Roerdink, J. B. T. M., "Mathematical morphology with non-commutative symmetry groups," in *Mathematical Morphology in Image Processing* (Chapter 7), Dougherty, E. R., Ed. New York, NY: Marcel Dekker, pp. 205–254, 1993.
- [6] Roerdink, J. B. T. M., On the empty space problem in robot path planning: a solution by mathematical morphology, 1993, in preparation.
- [7] Roerdink, J. B. T. M., "On the construction of translation and rotation invariant morphological operators," in *Mathematical Morphology: Theory and Hardware*, Haralick, R. M., Ed. Oxford Univ. Press, To appear (Report AM-R9025, Centre for Mathematics and Computer Science, Amsterdam, 1990).
- [8] Suzuki, M., *Group Theory*. Springer, Berlin, 1982.