# Computerized Tomography and its Applications: a Guided Tour * 

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#### Abstract

We present a review of the mathematical principles of computerized tomography. Topics treated include the role of the Radon transform and related transforms, inversion formulas, uniqueness, ill-posedness and stability, practical reconstruction algorithms, and various generalizations such as diffraction tomography. References to the most relevant literature are cited. Several applications are briefly discussed; in particular, we present a case study of a mathematical problem arising in cardiac magnetic resonance imaging.


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## 1 Introduction

The word tomography means 'reconstruction from projections', i.e. the recovery of a function from its line or (hyper)plane integrals (from the Greek $\tau o ́ \mu o \sigma$ - slice and $\gamma \rho \alpha ́ \phi \epsilon \iota \nu$-to write). In the applied sense, it is a method to reconstruct cross sections of the interior structure of an object without having to cut or damage the object. The term often occurs in the combination computerized (computed) tomography (CT) or computer-assisted tomography (CAT), since for performing the reconstructions in practice one needs the use of a digital computer. Important issues in tomography are existence, uniqueness and stability of inversion procedures, as well as the development of efficient numerical algorithms.

The internal property of the object to be reconstructed, such as a density, space-dependent attenuation coefficient, and so on, is generally referred to as the internal distribution. The physical agens or probe by which to act on this internal distribution may vary from X-rays, gamma rays, visible light, electrons or neutrons to ultrasound waves or nuclear magnetic resonance signals. When the probe is outside the object one speaks of transmission computerized tomography (TCT). In contrast with this stands emission computerized tomography (ECT), where the probe, such as a radioactive material, is inside the object. This occurs in two variants: SPECT (single particle ECT) where radiation along a half line is detected, and PET (positron emission tomography) where radiation emitted in opposite directions is detected in coincidence. Finally we mention reflection tomography, where the object is 'illuminated' by sound waves and the reflected waves are recorded to obtain line integrals of the object's reflectivity function [51, Ch. 8].

Other forms of tomography exist, such as electric impedance tomography [81, Pidcock's paper] (a nonlinear inverse problem: recovering the conductivity inside a body from electric potential measurements on the surface), biomagnetic imaging (recovering the position of electric currents from magnetic fields induced outside the body, see the paper by Louis in [42], or diffraction tomography, see [34,39,40] and also Section 2.9 below.

The emergence of the Radon transform in these practical problems can be explained by the following simplified model (in practice, many complications arise [39, 44]). If a beam of X-rays with initial intensity

[^0]$I_{0}$ passes through an object along a straight line $L$, then the intensity $I_{1}$ after having passed the object satisfies
\[

$$
\begin{equation*}
\frac{I_{1}}{I_{0}}=\exp \left\{-\int_{L} f(x) d x\right\} \tag{1}
\end{equation*}
$$

\]

where $f(x)$ denotes the X-ray attenuation coefficient of the object at the point $x$. Hence by measuring the ratio $I_{1} / I_{0}$ line integrals of the unknown distribution $f$ are obtained; these line integrals are samples of the 2D Radon transform of $f$, see Section 2.

Different modes are used in practice to sample the line integrals of the internal distribution: in parallel beam scanning parallel line integrals are determined for a fixed direction and the process is repeated for a number of different directions; in fan-beam scanning line integrals emanating from a given source point are computed for different directions, which is repeated for a certain number of source points, see Fig. 1. Instances of the use of the Radon transform in three dimensions are found in radar theory and magnetic resonance imaging, [20, 44].

One of the most prominent applications of computerized tomography occurs in diagnostic medicine, where the method is used to produce images of the interior of human organs [72]. In 1979 the Nobelprize in physiology or medicine was awarded to G.N. Hounsfield and A.M. Cormack for their fundamental work in the field. Other applications arise in radio astronomy, 3D electron microscopy, soil science, aerodynamics and geophysics, to name a few. In industry the method is used for non-destructive testing. For extensive

(a)

(b)

Figure 1: Scanning modes in 2D tomography: (a) parallel beam (b) fan-beam.
bibliographies on applications of computerized tomography the reader is referred to Deans [20] or Herman [39].

The organization of this paper is as follows. The first part is devoted to an overview of the mathematical principles of CT (Section 2). We present the Radon transform, related transforms, inversion formulas, uniqueness, the ranges, ill-posedness, sampling, reconstruction algorithms and diffraction tomography; then a number of generalizations are mentioned and we finish with some historical remarks. The second part (Section 3) contains a case study of magnetic resonance imaging, a field in which we have gained some personal experience over the last years.

## 2 The Mathematics

In this section a brief overview of the mathematics of computerized tomography is given. We restrict ourselves to an outline of the essential points. As a standard reference Natterer's book is recommended [62]. Also, the proceedings of the Oberwolfach conferences on CT [42, 44], the AMS proceedings [71] or the volume on tomography and inverse problems by Herman et al. [70] may be consulted.

### 2.1 The Radon Transform

Let $f\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a continouous function of $n$ real variables that is decreasing sufficiently fast at infinity. Let $\theta=\left(\theta_{1}, \ldots, \theta_{n}\right)$ be a unit vector in $\mathbb{R}^{n}, s$ a real number, and $\Gamma$ the hyperplane defined by

$$
\begin{equation*}
\Gamma=\left\{x \in \mathbb{R}^{n}:\langle x, \theta\rangle=s\right\} \tag{2}
\end{equation*}
$$

where $\langle$,$\rangle denotes the Euclidean inner product. The integral of f$ over the hyperplane $\Gamma$,

$$
\begin{equation*}
\mathcal{R} f(\theta, s):=\int_{\Gamma} f(x) d m(x) \tag{3}
\end{equation*}
$$

where $d m$ is the Euclidean measure on $\Gamma$, is called the Radon transform of the function $f$, see Fig. 2. The Radon operator $\mathcal{R}$ maps the Schwartz space $\mathcal{S}\left(\mathbb{R}^{n}\right)$ of rapidly decreasing $C^{\infty}$ functions on $\mathbb{R}^{n}$ to the Schwartz space $\mathcal{S}\left(Z^{n}\right)$ of rapidly decreasing $C^{\infty}$ functions on $Z^{n}$, where $Z^{n}=S^{n-1} \times \mathbb{R}=\{(\theta, s): \theta \in$ $\left.S^{n-1}, s \in \mathbb{R}\right\}$ is the unit cylinder in $\mathbb{R}^{n+1}$ and $S^{n-1}=\left\{\theta \in \mathbb{R}^{n}:|\theta|=1\right\}$ the unit sphere in $\mathbb{R}^{n}$. The integral $\mathcal{R} f(\theta, s)$, with $\theta$ and $s$ fixed, is called a projection and the function $\mathcal{R}_{\theta}: s \mapsto \mathcal{R} f(\theta, s)$ a profile, cf. Fig. 1.


Figure 2: Parameters $\theta$, s of the 2D Radon transform.

The Radon transform is connected to the Fourier transform through the Central Slice Theorem or Projection Theorem:

$$
\begin{equation*}
\tilde{f}(\alpha \theta)=\int_{-\infty}^{\infty} \mathcal{R} f(\theta, s) e^{-i \alpha s} d s, \quad \alpha \in \mathbb{R} \tag{4}
\end{equation*}
$$

where $\tilde{f}(\xi):=\int_{\mathbb{R}^{n}} e^{-i\langle x, \xi\rangle} f(x) d x$ denotes the $n$-dimensional Fourier transform of $f$. This formula says that the one-dimensional Fourier transform of the Radon transform of $f$ with respect to the radial variable $s$ equals the $n$-dimensional Fourier transform of $f$ along rays through the origin in $\mathbb{R}^{n}$.

To the Radon operator is associated a dual or backprojection operator $\mathcal{R}^{\#}: \mathcal{S}\left(Z^{n}\right) \rightarrow \mathcal{S}\left(\mathbb{R}^{n}\right)$ by

$$
\begin{equation*}
\mathcal{R}^{\#} g(x)=\int_{S^{n-1}} g(\theta,\langle x, \theta\rangle) d \theta, \quad g \in \mathcal{S}\left(Z^{n}\right) \tag{5}
\end{equation*}
$$

Whereas $\mathcal{R}$ integrates over all points in a hyperplane, $\mathcal{R} \#$ integrates over all hyperplanes through a point. Thus $\mathcal{R}, \mathcal{R}^{\#}$ form a dual pair in the sense of integral geometry [37,38]. If $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ and $g \in \mathcal{S}\left(Z^{n}\right)$, the following identity holds,

$$
\begin{equation*}
\left(\mathcal{R}^{\#} g\right) * f=\mathcal{R}^{\#}(g * \mathcal{R} f) \tag{6}
\end{equation*}
$$

where * denotes convolution on $\mathbb{R}^{n}$ and $\mathbb{R}$, respectively (the second convolution is with respect to the last argument of $g$ ). This formula forms the basis of the 'filtered backprojection' method of inverting the Radon transform, see Section 2.8.

The Radon and other transforms can be extended to $L^{p}$ spaces [70]. For example, if $\Omega^{n}$ is the unit ball in $\mathbb{R}^{n}$, the Radon transform is continuous as an operator from $L_{2}\left(\Omega^{n}\right)$ to the weighted space $L_{2}\left(Z^{n},(1-\right.$ $\left.s^{2}\right)^{(1-n) / 2}$ ). However, as a consequence of the smoothing property of the Radon transform, its inverse is not continuous as an operator between $L_{2}$-spaces. To restore continuity one may introduce Sobolev spaces [62].

### 2.2 Related Transforms

For three (or higher)-dimensional reconstruction of objects from line integrals the appropriate transform is the $X$-ray transform:

$$
\begin{equation*}
\mathcal{P} f(\theta, x)=\int_{-\infty}^{\infty} f(x+t \theta) d t, \tag{7}
\end{equation*}
$$

which is the line integral of $f$ along a line through the point $x \in \mathbb{R}^{n}$ in the direction of $\theta \in S^{n-1}$ [76]. In two dimensions the Radon and X-ray transform coincide. When using divergent beams (in two dimensions also called fan-beams) the appropriate transform is defined by

$$
\begin{equation*}
\mathcal{D} f(a, \theta)=\int_{0}^{\infty} f(a+t \theta) d t \tag{8}
\end{equation*}
$$

which is the integral of $f$ along the half-line with endpoint $a \in \mathbb{R}^{n}$ and direction $\theta \in S^{n-1}$ [36].
Relevant for SPECT is the attenuated Radon transform, which differs from the standard Radon transform by the presence of an exponential weighting factor under the integral sign. In $\mathbb{R}^{2}$, the attenuated Radon transform is defined for functions with compact support by

$$
\begin{equation*}
\mathcal{R}_{\mu} f(\theta, s)=\int_{\langle x, \theta\rangle=s} e^{-\mathcal{D} \mu\left(x, \theta^{\perp}\right)} f(x) d x \tag{9}
\end{equation*}
$$

where $\theta^{\perp}$ is the unit vector perpendicular to $\theta \in S^{1}$ for which $\operatorname{det}\left(\theta, \theta^{\perp}\right)=1, \mu$ is the attenuation function and $\mathcal{D}$ the divergent beam transform. An inversion formula exists for the case of constant attenuation $\mu$, cf. [62].

### 2.3 Inversion formulas

The inversion formula of the Radon transform takes two different forms, depending on whether the dimension is even or odd [48,67]:

$$
f(x)=\frac{1}{2}(2 \pi)^{1-n}\left\{\begin{array}{lc}
(-1)^{n-2 / 2} \int_{S^{n-1}} \mathcal{H} F^{(n-1)}(\theta,\langle x, \theta\rangle) d \theta & (n \text { even })  \tag{10}\\
(-1)^{(n-1) / 2} \int_{S^{n-1}} F^{(n-1)}(\theta,\langle x, \theta\rangle) d \theta & (n \text { odd })
\end{array}\right.
$$

Here $F=\mathcal{R} f, \mathcal{H}$ is the Hilbert transform

$$
\mathcal{H} h(s):=\frac{1}{\pi} \int_{\mathbb{R}} \frac{h(t)}{s-t} d t,
$$

and $F^{(n-1)}$ denotes the $(n-1)$-st derivative of $F$ with respect to its last (scalar) argument. An alternative form of formula (10) for $n$ even is

$$
f(x)=\frac{1}{2}(2 \pi)^{1-n}(-1)^{n / 2} \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{q} \int_{S^{n-1}} F^{(n-1)}(\theta,\langle x, \theta\rangle+q) d \theta d q \quad(n \text { even })
$$

So for $n$ odd only local information in the neighbourhood of the point $x$ is needed (computation of derivatives), whereas for even $n$ the integrals along all hyperplanes meeting the support of the function are required. The same phenomenon occurs in the solution to the Cauchy problem for the wave equation in $\mathbb{R}^{n}$, see Helgason (in $[9]$ ). In $\mathbb{R}^{2}$ the inversion formula can be rewritten in the following form, which was derived by Radon in his 1917 paper [67] ${ }^{12}$

$$
\begin{equation*}
f(x)=-\frac{1}{\pi} \int_{0}^{\infty} \frac{d \bar{F}_{x}(q)}{q} \quad(n=2) \tag{11}
\end{equation*}
$$

a formula which adorns the cover of the Journal for Computer Assisted Tomography, and in $\mathbb{R}^{3}$,

$$
\begin{equation*}
f(x)=-\frac{1}{8 \pi^{2}} \Delta_{x} \int_{S^{2}} F(\theta,\langle x, \theta\rangle) d \theta \quad(n=3) \tag{12}
\end{equation*}
$$

[^1]Here $\Delta_{x}$ is the Laplacian (acting on $x$ ) and

$$
\begin{equation*}
\bar{F}_{x}(q)=\frac{1}{\omega_{n}} \int_{S^{n-1}} F(\theta,\langle x, \theta\rangle+q) d \theta \tag{13}
\end{equation*}
$$

is the mean value of $F=\mathcal{R} f$ over the tangent hyperplanes of the sphere with radius $q$ and center $x$, where $\omega_{n}=2 \pi^{n / 2} / \Gamma(n / 2)$ is the volume of the unit hypersphere in $\mathbb{R}^{n}(\Gamma(\cdot)$ denoting the Gamma-function). The function $x \mapsto F(x,\langle x, \theta\rangle)$ occurring in (12) is a plane wave with normal $\theta$ (i.e. a function constant on each plane perpendicular to $\theta$ ) [37]. So (ignoring the Laplacian) formula (12) gives a continuous decomposition into plane waves. Since a plane wave is a function of one variable, such decompositions can be used to reduce problems in $\mathbb{R}^{n}$ to one-dimensional problems, a fact which has applications in the area of partial differential equations [37, 49, 59].

A unified form of the inversion formula for odd and even $n$ is given by [37],

$$
\begin{align*}
f & =c^{-1} \Delta_{x}^{(n-1) / 2}\left(\mathcal{R}^{\#} F\right), \quad F=\mathcal{R} f  \tag{14}\\
c & =(-4 \pi)^{(n-1) / 2} \frac{\Gamma(n / 2)}{\Gamma(1 / 2)} \tag{15}
\end{align*}
$$

The fractional powers of the Laplacian for even $n$ can be most easily defined by means of the Riesz potential $I^{\alpha}$, which is most easily defined in Fourier space:

$$
\left(I^{\alpha} F\right) \wedge(\theta, \sigma)=|s|^{-\alpha} \widehat{F}(\theta, \sigma), \quad \alpha<n
$$

where the hat denotes Fourier transformation with respect to the second argument. Then an even more general inversion formula can be written as

$$
\begin{equation*}
f=\frac{1}{2}(2 \pi)^{1-n} I^{-\alpha} \mathcal{R}^{\#} I^{\alpha-n+1} F, \quad F=\mathcal{R} f \tag{16}
\end{equation*}
$$

The formulas above arise as special cases by taking $\alpha=0$ or $\alpha=n-1$. Inversion formulas for the divergent beam transform $\mathcal{D}$ [43] and the X-ray transform $\mathcal{P}$ [75] are analogous to those for $\mathcal{R}$. Similar inversion formulas for the dual transforms exist [37].

## Series expansion methods

Other inversion formulas for the Radon transform are obtained by analytic series expansion methods. For example, when $f$ and $F=\mathcal{R} f$ are expanded in spherical harmonics,

$$
\begin{align*}
f(x) & =\sum_{\ell=0}^{\infty} \sum_{k=0}^{N(n, \ell)} f_{\ell k}(|x|) Y_{\ell k}(x /|x|),  \tag{17}\\
F(\theta, s) & =\sum_{\ell=0}^{\infty} \sum_{k=0}^{N(n, \ell)} F_{\ell k}(s) Y_{\ell k}(\theta), \tag{18}
\end{align*}
$$

where $N(n, \ell)$ denotes the number of linearly independent spherical harmonics of degree $\ell$, the following inversion formula holds for the coefficients:

$$
\begin{align*}
f_{\ell k}(r) & =c(n) r^{2-n} \int_{r}^{\infty}\left(s^{2}-r^{2}\right)^{(n-3) / 2} C_{\ell}^{(n-2) / 2}(s / r) F_{\ell k}^{(n-1)}(s) d s  \tag{19}\\
c(n) & =\frac{(-1)^{n-1}}{2 \pi^{n / 2}} \frac{\Gamma((n-2) / 2)}{\Gamma(n-2)} \tag{20}
\end{align*}
$$

where $C_{\ell}^{(n-2) / 2}$ is the Gegenbauer polynomial of degree $\ell[14,15,19]$. Various other polynomials can be used as well [20].

### 2.4 Uniqueness

From the inversion formulas it follows that $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ is uniquely determined by $\mathcal{R} f$ or $\mathcal{P} f$. If the transform is only known on a subset of the domain, as is always the case in practice, this is no longer true. Artefacts in the reconstruction are most pronounced in the vicinity of lines or planes which are tangent to curves or surfaces of discontinuity of $f$ and for which the value of $\mathcal{R} f$ is missing. Some of these incomplete data problems are ${ }^{3}$ :

## Exterior problem

This is the recovery a function in the exterior of some ball from integrals over lines or planes outside that ball. This problem is uniquely solvable provided $f$ is decaying fast enough at infinity, in view of the following 'hole theorem' [62], which is an immediate consequence of formula (19).

Theorem 1 ([62]) Let $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ and let $K$ be a compact set (the 'hole') in $\mathbb{R}^{n}$. If $\mathcal{R} f(\theta, s)=0$ for every plane $\langle x, \theta\rangle=s$ not meeting $K$, then $f=0$ outside $K$.

However, the problem is severely ill-posed, see Section 2.6 below.

## Interior problem

This is the problem of determining $f$ inside some ball from integrals inside that ball. This problem is uniquely solvable for odd dimension, due to the local character of Radon's inversion formula (10). In even dimension this is no longer true. However, the function $I^{-1} f$ is still uniquely determined by $F=\mathcal{R} f$ : using formula (16) for $n=2, \alpha=-1$ one finds

$$
I^{-1} f=\frac{1}{4 \pi} \mathcal{R}^{\#} I^{-2} F=-\frac{1}{4 \pi} \mathcal{R}^{\#} F^{\prime \prime}
$$

that is, $I^{-1} f$ can be found by applying the dual $\mathcal{R}^{\#}$ to the second derivative of $F$. This is a local operation which makes the interior problem for $I^{-1} f$ uniquely solvable in any dimension. It turns out that the function $f$ or its derivatives have discontinuities precisely where $I^{-1} f$ or its derivatives have, a fact which is exploited in local tomography [25].

## Limited angle problem

Here $\mathcal{R}_{\theta} f$ is only given for $\theta$ in a subset of a half-sphere. Uniqueness holds as long as the set of directions is such that no non-trivial homogeneous polynomial vanishes on it. The relevant theorem is the following, which is a consequence of the analyticity of the Fourier transform of functions with compact support (for the X-ray transform a similar result holds). Here $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ is the space of infinitely differentiable functions on $\mathbb{R}^{n}$ with compact support.

Theorem 2 ([75]) Let $A$ be a set of directions such that no non-trivial homogeneous polynomial vanishes on it. If $f \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ and $\mathcal{R}_{\theta} f=0$ for $\theta \in A$, then $f=0$.

## Finite number of directions

When the set of directions is finite, the solution is non-unique to a very large extent, a fact first put forward with particular force in a famous paper by Smith, Solmon and Wagner in 1977 [75]. They formulated a no-go theorem which seemed to obviate the possibility of any practical reconstruction whatsoever.

Theorem 3 ([75]) Let $\theta_{1}, \ldots, \theta_{p} \in \mathcal{S}^{n-1}$ be a finite set of directions, $K$ a compact set in $\mathbb{R}^{n}$ and $f$ an arbitrary function in $C_{0}^{\infty}(K)$. Then, for each compact set $K_{0}$ in the interior of $K$ one can find a function $f_{0} \in C_{0}^{\infty}(K)$ which coincides with $f$ on $K_{0}$ and for which all the line integrals $\mathcal{P}_{\theta_{k}} f_{0}, k=1, \ldots, p$ along the given directions are identically zero.

Here $\mathcal{P}_{\theta}$ is the operator defined by $\mathcal{P}_{\theta} f(x)=\mathcal{P} f(\theta, x)$ where $\mathcal{P}$ is the X-ray transform (7), and $C_{0}^{\infty}(K)$ is the space of infinitely differentiable functions with support in the compact set $K$. With unusual poignancy they reformulated their theorem in the following way:

[^2]Theorem3' A finite set of directions tells nothing at all.
This statement is a little paradoxical, to say the least, in view of the preceding twenty years or so of succesful tomographic reconstructions. The question is how serious the indeterminacy really is. The functions $f_{0}$ in Theorem 3, sometimes called 'ghosts', form the null-space of the transform for finitely many projections. It turns out that these functions are highly oscillatory. Hence, to resolve the indeterminacy one has to put restrictions on the variation of the function $f$ (see $[36,52]$ ). On the other hand, by increasing the number of projections, arbitrarily good approximations to the unknown function can be found [36].

Homogeneous objects An alternative way to avoid indeterminacy is to limit oneself to the class of homogeneous objects (e.g. convex sets), which are uniquely determined by a finite number of projections, thus arriving at a generalized moment problem [24,26,28,54]. A typical result is the following.

Theorem $4([\mathbf{2 8}])$ There are four directions $\theta_{1}, \ldots, \theta_{4}$ such that for each convex set $A \in \mathbb{R}^{2}$, its characteristic function $f$ is uniquely determined by $\mathcal{R}_{\theta} f, j=1, \ldots, 4$.

The related question whether a probability distribution can be inferred from its marginal distributions, was studied by Cramér and Wold [18], who discovered the Fourier method of solving Radon's problem, see below.

### 2.5 The Ranges

The range $\operatorname{Ran}(T)$ of an operator $T$ is defined by $\operatorname{Ran}(T):=\{T g: g \in \operatorname{Dom}(T)\}$ with $\operatorname{Dom}(T)$ the domain of $T$. The ranges of the Radon and X-ray transforms are governed by the Helgason-Ludwig consistency conditions. If $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$, then for $m=0,1, \ldots$, the following equations hold,

$$
\begin{align*}
\int_{\mathbb{R}} s^{m} \mathcal{R} f(\theta, s) d s & =p_{m}(\theta),  \tag{21}\\
\int_{\theta^{\perp}}\langle x, y\rangle^{m} \mathcal{P} f(\theta, x) d x & =q_{m}(y), \quad y \perp \theta \tag{22}
\end{align*}
$$

with $p_{m}, q_{m}$ homogeneous polynomials of degree $m$, and $q_{m}$ independent of $\theta$ ( $\theta^{\perp}$ is the subspace perpendicular to $\theta$ ). As an illustration of the way in which these conditions characterize the ranges, we quote the following theorem for the Radon transform:

Theorem 5 ( [37]) Let $g \in \mathcal{S}\left(Z^{n}\right)$ be even $(g(\theta, s)=g(-\theta,-s))$, and assume that for $m=0,1, \ldots$

$$
\int_{\mathbb{R}} s^{m} g(\theta, s) d s=p_{m}(\theta)
$$

is a homogeneous polynomial of degree $m$ in $\theta$. Then there is $f \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ such that $g=\mathcal{R} f$. If in addition $g(\theta, s)=0$ for $|s| \geq a$, then $f(x)=0$ for $|x| \geq a$.

In cases where the data (i.e. the values of the transform) are only partly specified, these conditions can often be exploited to restore the missing values ('data-completion').

### 2.6 Ill-Posedness

Problems in computerized tomography are (in varying degrees) ill-posed. The problem, given a bounded linear operator $A: \mathcal{H} \rightarrow \mathcal{K}$ with $\mathcal{H}$ and $\mathcal{K}$ Hilbert spaces, to find an $f \in \mathcal{H}$ such that

$$
A f=g, \quad g \in \mathcal{K}
$$

may not have a solution if the data vector $g$ is not in the range of the transform (as is often the case in practice because of noise influences), or the solution may be non-unique. These problems can be overcome by taking as solution the element $f^{\dagger}$ which is defined as the minimizer of $\|A f-g\|$ of minimal norm. The linear operator $A^{\dagger}: \operatorname{Dom}\left(A^{\dagger}\right) \rightarrow \mathcal{H}$, where $\operatorname{Dom}\left(A^{\dagger}\right):=\operatorname{Ran}(A) \oplus \operatorname{Ran}(A)^{\perp}$, which maps $g$ to $f^{\dagger}$ is known as the generalized inverse of $A$ [35]. It is well known that $f^{\dagger}=A^{\dagger} g$ is the unique element of $\mathcal{H}$ which (i) is a solution of the normal equations

$$
A^{*} A f=A^{*} g
$$

and (ii) lies in $\operatorname{Ker}(A)^{\perp}$, where $\operatorname{Ker}(A):=\{f \in \operatorname{Dom}(A): A f=0\}$ is the null space of $A$.
More seriously, stability may fail, which happens if the operator $A^{\dagger}$ is not continuous. To deal with this problem one introduces the notion of a regularization of $A^{\dagger}$, which is a family $\left\{T_{\gamma}\right\}_{\gamma}>0$ of linear continuous operators $T_{\gamma}: \mathcal{K} \rightarrow \mathcal{H}$ such that

$$
\lim _{\gamma \rightarrow 0} T_{\gamma} g=A^{\dagger} g
$$

on the domain of $A^{\dagger}$. Here $\gamma$ is called the regularization parameter. Determining the numerical value of $\gamma$ which gives good practical results is a difficult problem, although various (statistical) methods to do so have been proposed, see $[66,78]$. Examples of regularizations are

- Tikhonov-Phillips regularization, where

$$
T_{\gamma}=\left(A^{*} A+\gamma I\right)^{-1} A^{*}
$$

with $I$ the identity operator. This formula also has a statistical interpretation as the BLUE (Best Linear Unbiased Estimator) in the presence of noise, see e.g. [58].

- Iterative methods with appropriate stopping criteria, with the number of iteration steps playing the role of the regularization parameter, see below (Section 2.8).
- Truncated singular value decompositions (SVD) [23, 58, 63].

The SVD is a representation of the form

$$
A f=\sum_{k=1}^{\infty} \sigma_{k}\left\langle f, f_{k}\right\rangle g_{k}
$$

where $\left\{f_{k}\right\},\left\{g_{k}\right\}$ are systems of orthonormal eigenvectors in $\mathcal{H}, \mathcal{K}$, respectively, and $\left\{\sigma_{k}\right\}$ the singular values. Then

$$
\begin{equation*}
A^{\dagger} g=\sum_{k=1}^{\infty} \sigma_{k}^{-1}\left\langle g, g_{k}\right\rangle f_{k} . \tag{23}
\end{equation*}
$$

If $\sigma_{k} \rightarrow 0$ as $k \rightarrow \infty$, a regularization of $A^{\dagger}$ is required, for example by taking

$$
T_{\gamma} g=\sum_{k=1}^{\infty} F_{\gamma}\left(\sigma_{k}\right)\left\langle g, g_{k}\right\rangle f_{k}
$$

where $F_{\gamma}(\sigma)$ tends to zero as $\sigma \rightarrow 0$ and to $\sigma^{-1}$ as $\sigma$ becomes large. This procedure is known as digital filtering. For example one may take $k \leq \frac{1}{\gamma}$ in (23), that is, simply truncate the expansion after a finite number of terms.

The degree of ill-posedness can be inferred from the speed with which the singular values approach zero. Features in the solution for which the corresponding singular values are close to zero cannot be recovered reliably. The decay of the singular values of the Radon transform is rather slow, but limited data problems in particular can be severely ill-posed [63].

### 2.7 Sampling

In practice one has to perform reconstructions given a finite number of data. The first issue is to develop correct sampling procedures, based on Shannon's sampling theorem for equidistant sampling and its extension to arbitrary grids. The standard parallel scanning procedure is to sample $\mathcal{R} f(\theta, s)$ on a polar grid, i.e. taking $p$ directions uniformly distributed over the half-circle and $2 q+1$ equally spaced values of $s$, where $p_{\sim} \geq b$ and $q \geq b / \pi$. Here $b$ is the essential bandwidth of the function $f$, meaning that the Fourier transform $\tilde{f}(\xi)$ of $f$ is negligible for $|x i|>b$. Insufficiency of the data, either by undersampling a projection or by taking the number of projections too small, causes aliasing artefacts such as Gibbs phenomena, streaks ( $q$ too small) and Moiré patterns (display resolution too small) [51]. Other scanning geometries are used as well, for example in fan-beam scanning (see Fig. 1) which is used in positron emission tomography (PET). Even arbitrary directions can be used, cf. Grünbaum (in [44]) or [46]. Still more generally, tomography with unknown directions has been proposed $[31,32]$ in the case of identical particles with random location and orientation.

### 2.8 Reconstruction Algorithms

Methods for numerical reconstruction fall into four categories: convolution methods, Fourier methods, analytic series expansion methods and iterative algebraic methods (sometimes called 'finite series-expansion methods' [11]). The first three methods are based on analytic inversion formulas. Discretizations of equivalent inversion formulas may display very different behaviour with regard to stability, accuracy etc. This accounts for the large number of different algorithms proposed, of which only the main representatives are discussed here.

## Filtered Backprojection

The identity (6) is the basis of 'filtered backprojection'. One chooses $g$ in such a way that $\mathcal{R}^{\#} g$ is a low-pass filter approaching a delta distribution. The algorithm then becomes

$$
\begin{equation*}
f=\mathcal{R}^{\#}(g * F), \quad F=\mathcal{R} f \tag{24}
\end{equation*}
$$

that is, a convolution followed by backprojection (both operations properly discretized). Several of such windowing functions $g$ have been proposed. The general form for the Fourier transform of $g$ is

$$
\widehat{g}(\sigma)=\frac{1}{2}(2 \pi)^{\frac{1}{2}-n}|\sigma|^{n-1} \widehat{\Phi}(|\sigma| / b)
$$

where $\Phi$ is a low-pass filter with cut-off frequency $b$, that is, $0 \leq \widehat{\Phi} \leq 1$ and $\widehat{\Phi}(\sigma)=0$ for $\sigma \geq 1$ (for example, the ideal low-pass filter has $\widehat{\Phi}(\sigma)=1$ for $\sigma \leq 1)$. Variants exist where the backprojection is performed before the filtering procedure, cf. [20].

## Fourier reconstruction

The Central Slice Theorem (4)

$$
\begin{equation*}
\tilde{f}(\alpha \theta)=\int_{-\infty}^{\infty} \mathcal{R} f(\theta, s) e^{-i \alpha s} d s, \quad \alpha \in \mathbb{R} \tag{25}
\end{equation*}
$$

is the basis of Fourier reconstruction [62]. Here the projections $\mathcal{R} f(\theta, s)$ are Fourier transformed with respect to the last variable, which is followed by the inverse $n$-dimensional Fourier transform. To make use of the fast Fourier transform [65], one first has to interpolate from a polar grid to a Cartesian grid, a step which has a rather decisive influence on the numerical accuracy of the final reconstruction. If this is done appropriately one achieves a reconstruction quality comparable to that of filtered backprojection.

## Series expansion methods

Analytic series expansion methods are based upon discretization of the inversion formula (19) or related formulas for the individual expansion coefficients of the unknown function, see e.g. Deans [20].

## Discrete reconstruction methods

Here one starts from a full discretization of Radon's integral equation, usually followed by an iterative solution method ${ }^{4}$. Introduce a set of basis functions (pictures) $b_{j}(x), j=1, \ldots, N$, such that an approximation $\tilde{f}$ to the function to be reconstructed can be written as a linear combination

$$
\begin{equation*}
\tilde{f}(x)=\sum_{j=1}^{N} f_{k} b_{j}(x) \tag{26}
\end{equation*}
$$

The set of equations to be solved is

$$
\begin{equation*}
\int_{L_{i}} f(x) d x=g_{i}, \quad i=1, \ldots, M \tag{27}
\end{equation*}
$$

[^3]where the $L_{i}$ are straight lines (in fact, the method is easily adapted to the case where the lines are replaced by strips of finite width). Inserting (26) in this equation, one finds
\[

$$
\begin{equation*}
\sum_{j=1}^{N} A_{i j} f_{j}=g_{i}, \quad i=1, \ldots, M \tag{28}
\end{equation*}
$$

\]

where

$$
\begin{equation*}
A_{i j}=\int_{L_{i}} b_{j}(x) d x \tag{29}
\end{equation*}
$$

Since the discretization process involves many sources of inaccuracy (measurement errors, picture discretization, discrete approximation of the integrals (29)), it is appropriate to replace (28) by the matrix equation

$$
\begin{equation*}
A f+r=g \tag{30}
\end{equation*}
$$

where $g$ is the data vector, $r$ a residual vector -both of dimension $M$ - $f$ an $N$-dimensional object vector and $A$ a $M \times N$ projection matrix.

The residual vector is required to be orthogonal to the column space of $A$, that is,

$$
\begin{equation*}
A^{T} r=0 \tag{31}
\end{equation*}
$$

By introducing the notation

$$
x=\binom{f}{r}, \quad x=\binom{g}{0}, \quad B=\left(\begin{array}{cc}
A & I \\
0 & A^{T}
\end{array}\right)
$$

the pair of equations (30) - (31) can be written in the form

$$
\begin{equation*}
B x=y \tag{32}
\end{equation*}
$$

An often used basis set is the pixel basis, that is, the domain $D$ of the function $f$ to be reconstructed is covered by little squares $S_{j}, j=1, \ldots, N$ called pixels (picture elements), and $f$ is assumed to be constant in each pixel so that it can be represented by a $N$-dimensional vector $f$. In that case

$$
A_{i j}=\text { length of }\left(L_{i} \cap S_{j}\right)
$$

that is, $A_{i j}$ is the weight factor measuring the contribution of pixel $S_{j}$ to the $i$-th line integral, see Fig. 3 .


Figure 3: In the discrete reconstruction method one superimposes a square grid over the unknown image, where image values are assumed to be constant within each grid cell.

Special problems arise because of the fact that

1. The number of equations is very large ( $A$ may have as many as $10^{10}$ elements), and usually overdetermined $(M>N)$.
2. The matrix $A$ is sparse since a line hits only a small fraction ( $\leq 1 \%$ ) of all pixels. However, no special structure of non-zero entries can be employed. Also, the rank of $A$ will in general be smaller than the number of columns $N$.

To overcome these problems one constructs regularized least squares solutions of (30), e.g. by TikhonovPhillips regularization, see Section 2.6.

## Iterative algorithms

The standard way to implement these solution methods is to use iterative methods from numerical linear algebra. There are a number of variants which are known by the acronyms ART (algebraic reconstruction technique) [33], SIRT (simultaneous iterative reconstruction technique) and SART (simultaneous algebraic reconstruction technique). ART is a form of Gauss-Seidel iteration, and can be viewed as a generalization of the method of Kaczmarz [50]. SIRT is known as the Richardson iteration in numerical linear algebra.

Kaczmarz's method has a simple geometrical interpretation: in the 2D case, each equation in (27) defines a line in the plane and starting from an initial point one successively projects upon the onedimensional subspaces defined by these lines, see Fig. 4. Applied to the system (32), ART yields an iterative scheme of the form

$$
\begin{equation*}
x^{(i+1)}=x^{(i)}+\lambda^{(i)} \frac{\left(y_{k}-B_{k} x^{(i)}\right) B_{k}^{T}}{\left\|B_{k}\right\|^{2}}, \quad k=i \bmod (K)+1 \tag{33}
\end{equation*}
$$

Here $K=M+N, B_{k}$ is the $k$-th row of the matrix $B$, and $\lambda^{(i)}$ are relaxation parameters introduced to


Figure 4: Kaczmarz's method consists of successive orthogonal projections on the subspaces defined by the model equations (28).
speed up convergence. It is clear from this that ART is a row action method (in a single iterative step only one row of the matrix $B$ is accessed). It has been shown that the iteration (33) converges to the minimum norm solution of (32) if (i) the system is consistent, (ii) the initial vector $x^{(0)}$ lies in range $\left(A^{T}\right)$ and (iii) the relaxation parameters satisfy $0<\underline{\lim } \lambda^{(i)} \leq \overline{\lim } \lambda^{(i)}<2$. Because of the ill-posedness of the reconstruction problem one has in practice to stop the iteration after a finite number of steps (a form of regularization). The convergence behaviour of ART may be affected by reordering the equations. In the case of an inconsistent system, variants of ART have been constructed which converge to the regularized solution.

Kaczmarz's method is very versatile. For example, instead of equalities one may use inequalities in the projection process. This may be generalized even further by going to a Hilbert space formulation where the measurements and other a priori information (nonnegavity, upper and lower bounds, measurement accuracy, etc.) are modelled as convex sets in a Hilbert space and an iterative procedure is developed by successively projecting onto these sets, a method known by its acronym POCS (Projection Onto Convex

Sets), see e.g. various chapters in Stark's book [77] and references quoted therein. One also can use statistical solution concepts, see below.

In SIRT, all the equations are simultaneously processed in every iteration step. One also uses intermediate variants like block-ART, where the equations are processed in blocks before updating the values; the blocks may even vary during iteration, see Censor's contribution in [42]. Also, the CG (conjugate gradient) method has become popular in this context, see e.g. van der Sluis et al. (in [13]) or the thesis [22].

Besides additive also multiplicative algorithms have been proposed, see [10, 11, 33, 41, 47]. For a recent review, see for example the special issue [13] or the section on 'Inverse Problems and Optimization' in [42]. Efficient implementations on parallel computers exist [1, 12].

## Statistical approaches

A final class of methods, e.g. as used in emission tomography, comprises statistical approaches to image reconstruction. Especially in the context of ill-posedness and regularization, various statistical estimation procedures have been proposed with or without the use of a-priori information, often resulting in iterative algorithms $[66,77]$. Examples are the use of the maximum likelihood estimator (ML) which is computed by the EM (expectation-maximization) algorithm [73, 80], or the MAP (maximum a posteriori) estimator [30].

### 2.9 Diffraction tomography

Diffraction tomography is a special inverse scattering problem [74]. Here one starts from the Helmholtz wave equation for a plane wave $u(x)$ with varying refractive index $n(x)=\sqrt{1+f(x)}$,

$$
\begin{equation*}
\Delta u+k^{2}(1+f) u=0 \tag{34}
\end{equation*}
$$

The aim is to recover $f$ from measurements of the (scattered) wave $u$. Applying the Rytov approximation to (34), one finds

$$
\begin{equation*}
\widehat{f}\left((a(\sigma)-k) \theta+\sigma \theta^{\perp}\right)=-\frac{2}{\pi}^{\frac{1}{2}} i \frac{a(\sigma)}{k} e^{i r(k-a(\sigma))} \widehat{g}(\theta, \sigma) \tag{35}
\end{equation*}
$$

where $a(\sigma)=\sqrt{k^{2}-\sigma^{2}}$ and $\widehat{g}$ is the Fourier transform with respect to the second argument. This formula, known as the Fourier Diffraction Theorem, is the generalization of the Fourier Slice Theorem, (4). If the frequency $\sigma$ runs over $[-k, k]$ then $\left.(a(\sigma)-k) \theta+\sigma \theta^{\perp}\right)=\sqrt{\left(k^{2}-\sigma^{2}\right)} \theta+\sigma \theta^{\perp}-k \theta$ runs over the semi-circle through the origin with center $-k \theta$. Hence $f$ can be recovered from data on semi-circles through the origin $[5,6,77]$, [51, Ch.6], see Fig. 5. CT is a limiting case of this as the radiation frequency $k$ goes to infinity.


Figure 5: In diffraction tomography a function is recovered from the values of its Fourier transform on semicircles through the origin.

Reconstruction algorithms for diffraction tomography are divided in two classes, just as for the standard CT case: one, analogous to direct Fourier inversion, consists in frequency domain interpolation of the data on the semi-circles to a uniform rectangular grid, followed by Fourier inversion. Also, an analogue to filtered backprojection, called filtered backpropagation, has been developed [21].

The Rytov and Born approximations used in diffraction tomography are of limited validity. In many cases one has to go back to the original wave equation and do the reconstruction by discretizing the full forward operator and using an iterative solution method.

Applications of diffraction tomography are found in acoustic imaging, e.g. ultrasonic tomography (see Greenleaf's paper in [40] or Devaney [21]) and seismology [2]. In oceanography the method has been proposed for large scale environmental monitoring of temperature profiles in the oceans [53].

### 2.10 Generalizations

The Radon transform can be defined for generalized functions (distributions) and measures. Also, generalized transforms can be defined, where the integration is over $k$-dimensional subspaces ( $k$-plane transform or $k$-dimensional Radon transform, $[37,76]$ ), families of curves [16] or, more generally, over manifolds [ $5,6,29,37,48]$. Examples are two-point homogeneous spaces such as spaces of constant curvature, where the planes are replaced by totally geodesic submanifolds [37]. A related problem for such spaces is the Orbital integral problem, i.e. recovering a function from its integrals over (generalized) spheres [17, 37, 49]. Some inverse problems for hyperbolic partial differential equations can be reduced to a problem in integral geometry [69]. For the use of the Radon transform in the area of partial differential equations, see [37, 49, 59].

Another generalization of Radon's problem is the reconstruction of a measure from its projections, i.e. lower-dimensional measures induced by some measurable mapping (generalization of the Cramér and Wold problem) [29, 37], Fritz \& Oppel (in [44]). For the Radon transform in the complex domain, see [29] or Quinto (in [9]). Finite Radon transforms are used in various areas such as the theory of error-correcting codes (see Beth, in [44]) and in (algebraic) lattice theory, cf. [9] (Bolker's paper) or [55,56].

### 2.11 Historical Remarks

Radon derived his inversion formulas by reducing the problem to an Abel type integral equation -according to his own remarks, he first found the inversion formula by relating the inversion problem to that of the Newtonian potential [67] and using a method due to Herglotz. Previously, a similar method had been used by P. Funk (following earlier work of Minkowski who used expansions in terms of spherical functions [61]) for the elliptic case of reconstructing an even function on the sphere from the integrals along the great circles [27, Ch. 2]. In his 1917 paper, Radon also discussed the problem of determining a function on the hyperbolic plane from its integrals over all geodesics, see also [29,37].

The Dutch physicist H.A. Lorentz apparently knew the inversion formula for the 3D Radon problem at least as early as 1906, as is clear from remarks by his students Bockwinkel [7] (in a paper about light propagation in biaxial crystals) and Uhlenbeck [79] (who, on the instigation of Ehrenfest, generalized Lorentz's result to $n$ dimensions). For more information on the early history, see Cormack's paper in [71].

## 3 A case study: magnetic resonance imaging

Magnetic resonance imaging (MRI) can be used as a diagnostic technique to display cross sections of the beating heart. For diagnostic purposes a movie of the heart based upon several reconstructed heart phases will give useful information not easily obtained from static pictures. The general problem in dynamic MRI is that, because of physical limitations, the standard measurement technique is not fast enough to acquire all the data, necessary for the reconstruction of a single heart phase, in a time which is short enough that the motion of the heart is negligible.

To solve this problem a method by the name of retrospective gating was developed [8,57], which makes use of the (approximate) periodicity of the motion. That is, data corresponding to the same relative heart phase may be recorded in different heartbeats. This presupposes exact reproducability of the heart motion in successive cycles, a condition which will be violated in practice. To deal with this one assumes that there is a simple rule to map heart intervals of different duration to a standard heart interval of unit length in such a way that data are assigned to the correct heart phase. The electrocardiogram (ECG) more precisely, the times of occurrence of the so-called R-waves - is simultaneously recorded and used as a reference signal to perform synchronization of the data retrospectively, i.e. after data acquisition has been completed, this in contrast to ordinary gating or triggering, where the ECG is used for synchronization during data acquisition.

In this second part of the paper a mathematical formulation of this reconstruction problem and a corresponding solution method are sketched, as recently developed by Zwaan; see the thesis by Zwaan [84]
and also $[68,82,83]$. In Section 3.1 we first review the data acquisition process of the retrospective gating technique for cardiac imaging and explain how reconstructions at different heart phases can in principle be obtained. In Section 3.2 a mathematical problem formulation is given and a solution method described. Section 3.3 contains reconstructions from synthetic data as well as from real MRI data.

### 3.1 Cardiac magnetic resonance imaging

We start with a brief discussion of the principles of MRI for static objects [45, 60].

## Static MRI

In an MR imaging system, the object is placed in an external magnetic field with controllable field gradients $G_{x}, G_{y}, G_{z}$. The magnetic dipoles ('spins') of the hydrogen atoms in the object are excited by external radio pulses, after which the system is allowed to return to equilibrium. During this process the excited spins give up energy in the form of radiofrequecy radiation, which is recorded (the FID-free induction decay-signal). By turning on a slice selection gradient $G_{z}$ during excitation by the radio pulse, a very narrow slice perpendicular to the $z$-direction is excited, which is the reason why MRI is an tomographic technique. To provide spatial encoding within the selected plane, one subsequently turns on a phaseencoding gradient $G_{y}$ in the $y$-direction during a time $t_{y}$. Finally, one applies a readout gradient $G_{x}$ in the $x$-direction, during which the FID signal is recorded and sampled every $\delta t$ seconds at times $t_{k_{x}}=$ $\left(-\frac{1}{2} k_{x}^{\mathrm{m}}+k_{x}\right) \delta t, k_{x}=0,1, \ldots, k_{x}^{\mathrm{m}}-1$, where $k_{x}^{\mathrm{m}}$ (assumed to be even) is the number of sampling points. Similarly, $G_{y}$ is varied in steps of size $\Delta_{y}: G_{y}=\left(-\frac{1}{2} k_{y}^{\mathrm{m}}+k_{y}\right) \Delta_{y}, k_{y}=0,1, \ldots, k_{y}^{\mathrm{m}}-1$, with $t_{y}$ fixed. The FID signal $S\left(k_{x}, k_{y}\right)$ which is recorded for a given value of $k_{x}$ and $k_{y}$ is approximately equal to the Fourier transform of the spin density $f$ :

$$
\begin{equation*}
\left.S\left(k_{x}, k_{y}\right)=\int f(x, y) e^{-2 \pi i\left(\frac{k_{x}}{L_{x}} x+\frac{k_{y}}{L_{y}} y\right.}\right) d x d y \tag{36}
\end{equation*}
$$

where $L_{x}$ and $L_{y}$ are the fields of view in the $x$ - and $y$-direction, respectively. The sequence of measurements $\left\{S\left(k_{x}, k_{y}\right): k_{x}=0, . ., k_{x}^{\mathrm{m}}-1\right\}$, with $k_{y}$ fixed, gives the Fourier transform for a discrete number of points on a horizontal line in the Fourier plane; such a horizontal trajectory is called a profile. Note that it takes a finite amount of time to measure a trajectory, since successive samples of the FID signal are taken at intervals $\delta t$.

To avoid aliasing by undersampling, both $\delta t$ and $\Delta_{y}$ should be chosen in accordance with the Nyquist relation for sampling bandlimited functions:

$$
\begin{equation*}
\delta t=2 \pi /\left(\gamma G_{x} L_{x}\right) \quad \Delta_{y}=2 \pi /\left(\gamma t_{y} L_{y}\right) \tag{37}
\end{equation*}
$$

An estimate $\tilde{f}(x, y)$ of the proton density $f(x, y)$ of the selected slice is then constructed by performing the discrete inverse Fourier transform:

$$
\begin{equation*}
\tilde{f}(x, y)=\left[L_{x} L_{y}\right]^{-1} \sum_{k_{x}=-\frac{1}{2} k_{x}^{\mathrm{m}}}^{\frac{1}{2} k_{x}^{\mathrm{m}}-1} \sum_{k_{y}=-\frac{1}{2} k_{y}^{\mathrm{m}}}^{\frac{1}{2} k_{y}^{\mathrm{m}}-1} S\left(\frac{2 \pi}{L_{x}} k_{x}, \frac{2 \pi}{L_{y}} k_{y}\right) e^{i\left(\frac{2 \pi}{L_{x}} k_{x} x+\frac{2 \pi}{L_{y}} k_{y} y\right)} \tag{38}
\end{equation*}
$$

which can be efficiently computed by the Fast Fourier Transform.

## Dynamic cardiac MRI by retrospective gating

The aim of cardiac MRI is to reconstruct images of a cross section of the beating heart at a number of phases during the heart cycle. One therefore needs to measure a complete set of $k_{x}^{\mathrm{m}} \times k_{y}^{\mathrm{m}}$ Fourier coefficients for every phase at which a reconstruction is desired. However, as we have seen above, acquiring Fourier data requires a certain amount of time. The measurement of a single profile - one line in Fourier space - at a certain phase $\phi$ takes in the order of 10 ms , which is small enough for the heart motion to be negligible. However, before the next profile (with the value of $k_{y}$ increased) can be measured, gradient fields have to be altered and also the spin system has to relax to equilibrium. This implies that the next profile will be measured at a different heart phase $\phi^{\prime}$. But clearly it is necessary to measure all lines in Fourier space at phase $\phi$ before the proton density at that phase can be obtained by Fourier inversion. Here the periodicity of the heart motion comes to the rescue: simply measure the second profile in the next heartbeat-with $k_{y}$
increased-at the same heart phase $\phi$; the third profile in the heart beat after that, and so on. Of course we want to obtain pictures at a number, say $N_{p r}$, of heart phases. Therefore a whole series of profiles is measured in a single heart cycle with a fixed interval time $T_{r e p}$ between measurements and a fixed value of $k_{y}$, so that at all phases the same line in $\mathbf{k}$-space is measured. To obtain the necessary data for all heart phases one has to sample during a number of $k_{y}^{\mathrm{m}}$ heartbeats.

In reality the situation is more complicated, since heart beats are not all of the same duration. In the retrospective gating method one assumes that all heart beats are again identical after they have been rescaled to a common duration. Then the same acquisition procedure as above can be applied, that is, one uses an uninterrupted sequence of profile measurements with a fixed repetition time $T_{\text {rep }}$ and stepping up the value of $k_{y}$ after $N_{p r}$ profiles have been measured. Simultaneously, but independently of the profile measurements, the ECG is recorded to enable a-posteriori assignment of the data to the correct heart phase; see Fig. 6, where each small vertical bar represents a complete profile (i.e. $k_{x}^{\mathrm{m}}$ measurements).


Figure 6: Acquisition method of retrospective gating. Large vertical bar: $R$-wave. Small vertical bar: a profile.

Summarizing, the data acquisition process contains the following steps:

1. Initialize the phase encoding gradient: $k_{y}=-\frac{1}{2} k_{y}^{\mathrm{m}}$.
2. Measure profiles with a repetition time $T_{\text {rep }}$ until $N_{p r}$ profiles have been recorded. Each profile consists of $k_{x}^{\mathrm{m}}$ measurements of the Fourier transform of the cross section, with $k_{y}$ fixed.
3. Increase the phase-encoding gradient: $k_{y} \rightarrow k_{y}+1$; go to (ii). If $k_{y}=\frac{1}{2} k_{y}^{\mathrm{m}}-1$, stop.
4. Simultaneously measure the times $R_{k}$ of occurrence of the R-waves.

By the assumption made above there exists a fixed interval $I:=[0,1]$, referred to as the standard heart interval, on which data recorded during heartbeats of different duration are mapped. We refer to original measurement time as 'time' $\tau$, and to relative time on the standard heart interval as 'phase' $t$.

Let $D$ denote a domain in the plane. The proton density is a function $F: D \times \mathbb{R} \rightarrow \mathbb{R}$. We want to reconstruct the proton density $f: D \times I \rightarrow \mathbb{R}$ on the standard heart interval. Each measurement of $F$ at time $\tau$ corresponds to a measurement of $f$ at a converted time $t(\tau)$ :

$$
\begin{equation*}
F(x, y, \tau)=f(x, y, t(\tau)), \quad t \in[0,1] \tag{39}
\end{equation*}
$$

We use 'linear stretching' to relate $\tau$ and $t$ :

$$
\begin{equation*}
t(\tau)=\frac{\tau-R_{k}}{R_{k+1}-R_{k}}, \quad \tau \in\left[R_{k}, R_{k+1}\right) \tag{40}
\end{equation*}
$$

where $R_{k}$ is the time at which the $k$ th R -wave occurs, for $k=1,2, \ldots$.
As a result of the time-to-phase conversion, the data are reordered, see Fig. 7. The converted time $t$ of the $k_{x}$ th sample of the $i$ th profile at the $k_{y}$ th phase encoding step is a function of both $k_{x}$ and $k_{y}$ : $t=t_{i}\left(k_{x}, k_{y}\right), i=1, \ldots, N_{p r}$. The collection of measured data thus consists of the numbers

$$
g_{i}\left(k_{x}, k_{y}\right):=\widehat{f}\left(k_{x}, k_{y}, t_{i}\left(k_{x}, k_{y}\right)\right)
$$

for $k_{x}=-\frac{1}{2} k_{x}^{\mathrm{m}}, \ldots, \frac{1}{2} k_{x}^{\mathrm{m}}-1 ; k_{y}=-\frac{1}{2} k_{y}^{\mathrm{m}}, \ldots, \frac{1}{2} k_{y}^{\mathrm{m}}-1$ and $i=0, \ldots, N_{p r}-1$. The next problem is that the reordered phases do not match with the wanted phases, which usually are a number of equally spaced values on the standard heart interval. Also, for each value of $k_{y}$ the pattern of reordered phases will be different.


Figure 7: Reordering of the time markers by the time-to-phase conversion.

### 3.2 The mixed Fourier-interpolation problem

Let

$$
\begin{aligned}
\mathbf{k} & :=\left(k_{x}, k_{y}\right), g_{\mathbf{k}, i}:=g_{i}\left(k_{x}, k_{y}\right), t_{\mathbf{k}, i}:=t_{i}\left(k_{x}, k_{y}\right) \\
\mathbb{K} & :=\left\{\left(k_{x}, k_{y}\right): k_{x}=-\frac{1}{2} k_{x}^{\mathrm{m}}, \ldots, \frac{1}{2} k_{x}^{\mathrm{m}}-1 ; k_{y}=-\frac{1}{2} k_{y}^{\mathrm{m}}, \ldots, \frac{1}{2} k_{y}^{\mathrm{m}}-1\right\}, \\
\mathbb{I} & :=\left\{i: i=0, \ldots, N_{p r}-1\right\} .
\end{aligned}
$$

Then the problem to be solved reads as follows:
Problem formulation Given a sequence of real numbers $\left\{t_{\mathbf{k}, i}\right\}$ and a sequence of complex numbers $\left\{g_{\mathbf{k}, i}\right\}$, find a function $f: D \times I \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
\widehat{f}\left(\mathbf{k}, t_{\mathbf{k}, i}\right)=g_{\mathbf{k}, i}, \quad \mathbf{k} \in \mathbb{K}, i \in \mathbb{I} \tag{41}
\end{equation*}
$$

It is assumed that for each $\mathbf{k}$ the sequence $\left\{t_{i}(\mathbf{k})\right\}_{i \in \mathbb{I}}$ consists of distinct real numbers. The problem here is both the $k_{y}$-dependence and non-uniform sampling of the time points $\left\{t_{i}\right\}$. Since the solution of this problem involves Fourier inversion in the spatial domain and interpolation in the time domain, we will refer to (41) as the mixed Fourier-interpolation problem.

The interpolation problem (41) has a natural decomposition into a space-dependent and a timedependent component. Assume that for each $\mathbf{r}:=(x, y) \in D$, the function $f(\mathbf{r}): t \rightarrow f(\mathbf{r}, t)$ is an element of a certain Hilbert space $\mathcal{H}$. The inner product on $\mathcal{H}$ is denoted by angular brackets $\langle., .\rangle_{\mathcal{H}}$. The function $f$ itself is assumed to be an element of a larger Hilbert space $\mathcal{W}$ with inner product

$$
\begin{equation*}
\langle f, g\rangle_{\mathcal{W}}:=\int_{D}\langle f(\mathbf{r}), g(\mathbf{r})\rangle_{\mathcal{H}} d \mathbf{r} \tag{42}
\end{equation*}
$$

Let $\left\{h_{i}\right\}_{i \in \mathbb{I}}$ be an orthonormal basis for $\mathcal{H}$ and $\left\{e_{\mathbf{k}}\right\}_{\mathbf{k} \in \mathbb{K}}$ be the (orthonormal) Fourier basis for $L^{2}(D)$, $e_{\mathbf{k}}(\mathbf{r})=(2 \pi)^{-1} e^{i \mathbf{k} \cdot \mathbf{r}}$, where without loss of generality we have taken $D=[-\pi, \pi]^{2}$. It is assumed that $\mathcal{H}$ is a Reproducing Kernel Hilbert Space (RKHS) [3,4]: for all $t \in \mathbb{R}$, there exists an element $Q_{t} \in \mathcal{H}$ such that point evaluations can be written as inner products:

$$
h(t)=\left\langle h, Q_{t}\right\rangle_{\mathcal{H}}, \quad \text { for each } h \in \mathcal{H}
$$

The function $Q: \mathbb{R}^{2} \rightarrow \mathbb{C}$ with $Q(t, s):=Q_{t}(s)=\left\langle Q_{t}, Q_{s}\right\rangle_{\mathcal{H}}$ is called the reproducing kernel.


Figure 8: Original chest phantom. From left to right, top to bottom: phases 0-7.

For the mixed problem we can perform a similar reformulation: define $\varphi_{\mathbf{k}, i}(t)=Q\left(t_{\mathbf{k}, i}, t\right), \mathbf{k} \in \mathbb{K}, i \in \mathbb{I}$. Then

$$
\begin{equation*}
\widehat{f}\left(\mathbf{k}, t_{\mathbf{k}, i}\right)=\left\langle\widehat{f}(\mathbf{k}), \varphi_{\mathbf{k}, i}\right\rangle_{\mathcal{H}}, \tag{43}
\end{equation*}
$$

where $\widehat{f}(\mathbf{k})$ denotes the (generalized) Fourier transform of $f$. Since, for arbitrary $h \in \mathcal{H},\left\langle f, e_{\mathbf{k}} h\right\rangle_{\mathcal{W}}=$ $\langle\widehat{f}(\mathbf{k}), h\rangle_{\mathcal{H}},(41)$ can be reformulated as a moment problem:

$$
\begin{equation*}
\left\langle f, e_{\mathbf{k}} \varphi_{\mathbf{k}, i}\right\rangle_{\mathcal{W}}=g_{\mathbf{k}, i}, \quad \forall \mathbf{k} \in \mathbb{K}, \forall i \in \mathbb{I} \tag{44}
\end{equation*}
$$

In the case of a finite number of measurements (44) will not have a unique solution so that it is appropriate to consider the minimum norm solution. We assume that, for all $\mathbf{k} \in \mathbb{K}$, the system $\left\{\varphi_{\mathbf{k}, i}\right\}_{i \in \mathbb{I}}$ is linearly independent in $\mathcal{H}$ as long as the sequence $\left\{t_{\mathbf{k}, i}\right\}_{i \in \mathbb{I}}$ consists of distinct real numbers. Then the system $\left\{e_{\mathbf{k}} \varphi_{\mathbf{k}, i}\right\}_{i \in \mathbb{I}, \mathbf{k} \in \mathbb{K}}$ is a linearly independent system of vectors spanning a linear subspace of $\mathcal{W}$, denoted by $\mathcal{W}_{n}$.

## Reconstruction

The solution of minimal norm to problem (44) is unique, lies in $\mathcal{W}_{n}$ and is given by

$$
\begin{equation*}
f=\sum_{\mathbf{k} \in \mathbb{K}} c_{\mathbf{k}} e_{\mathbf{k}} \tag{45}
\end{equation*}
$$

where $c_{\mathbf{k}}$ is defined by

$$
\begin{equation*}
c_{\mathbf{k}}=\sum_{i \in \mathbb{I}} g_{\mathbf{k}, i} \psi_{\mathbf{k}, i} . \tag{46}
\end{equation*}
$$

Here $\psi_{\mathbf{k}, i}$, which is orthogonal to $\varphi_{\mathbf{k}, j}, j \neq i$, is given by

$$
\begin{equation*}
\psi_{\mathbf{k}, i}=\sum_{j \in \mathbb{I}} \overline{\left(G^{-1}(\mathbf{k})\right)_{i j}} \varphi_{\mathbf{k}, j} \tag{47}
\end{equation*}
$$

where, for each $\mathbf{k} \in \mathbb{K}, G(\mathbf{k})$ is the Gram matrix, defined by

$$
\begin{equation*}
(G(\mathbf{k}))_{i j}=\left\langle\varphi_{\mathbf{k}, j}, \varphi_{\mathbf{k}, i}\right\rangle_{\mathcal{H}}=Q\left(t_{\mathbf{k}, j}, t_{\mathbf{k}, i}\right), \quad \forall i, j \in \mathbb{I} . \tag{48}
\end{equation*}
$$

So for each fixed frequency vector $\mathbf{k}$, a solution $c_{\mathbf{k}}$ to the interpolation problem (41) is computed and then the inverse Fourier transform is taken:

$$
\begin{equation*}
f(\mathbf{r}, t)=\sum_{\mathbf{k} \in \mathbb{K}} c_{\mathbf{k}}(t) e_{\mathbf{k}}(\mathbf{r})=\sum_{\mathbf{k} \in \mathbb{K}} e_{\mathbf{k}}(\mathbf{r}) \sum_{i, j \in \mathbb{I}} g_{\mathbf{k}, i} \overline{\left(G^{-1}(\mathbf{k})\right)_{i j}} \varphi_{j}(\mathbf{k}, t) . \tag{49}
\end{equation*}
$$

For $\mathcal{H}$ two cases are considered:

- $\mathcal{H}$ is the Paley-Wiener space $\mathbb{P}_{r}$ of bandwidth $r$ defined by $\mathbb{P}_{r}:=\left\{f \in L^{2}(\mathbb{R})\right.$ : supp $\left.\widehat{f} \subset[-r, r]\right\}$, where 'supp' denotes the support of a function. The reproducing kernel for a given bandwidth $r$ is given by $Q(t, s)=\sqrt{r / \pi} \operatorname{sinc}_{r}(s-t \pi / r)$. Here $\operatorname{sinc}_{r}$ denotes the sinc-function which is defined by $\operatorname{sinc}_{r}(t):=[\sin (r t)] / r t$ for $t \neq 0$ and $\operatorname{sinc}_{r}(0)=1$.
- $\mathcal{H}$ is the space $\mathrm{K}^{2 n-1}$ of splines of odd degree $2 n-1$. The reproducing kernel is

$$
\begin{aligned}
Q(t, s) & :=\sum_{k=0}^{n-1} \frac{(t-a)^{k}(s-a)^{k}}{(k!)^{2}}+\sum_{k=0}^{n-1}(-1)^{n+k+1} \frac{(t-a)^{2 n-k-1}(s-a)^{k}}{(2 n-k-1)!k!} \\
& +\frac{(-1)^{n}}{(2 n-1)!}(t-s)_{+}^{2 n-1}
\end{aligned}
$$

where $t_{+}^{k}:=t^{k}$ for $t \geq 0$ and zero for $t<0$.
Stability By computing error estimates it turns out that problem (41) is stable for perturbation of the data and time points $[68,83]$. However it is ill-conditioned when the norm of the Gram matrix comes close to 1 , which happens when the time points $\left\{t_{\mathbf{k}, i}\right\}_{i \in \mathbb{I}}$ (for a certain $\mathbf{k} \in \mathbb{K}$ ) are lying close to each other. In that case we use Tychonov-Phillips regularization, see Section 2.6, which for the mixed problem has the form

$$
\begin{equation*}
\left(\mathcal{T}^{\gamma} g\right)(\mathbf{r}, t)=\sum_{\mathbf{k} \in \mathbb{K}} e_{\mathbf{k}}(\mathbf{r}) \sum_{i, j \in \mathbb{I}} g_{\mathbf{k}, i}\left((G(\mathbf{k})+\gamma I)^{-1}\right)_{i j} \varphi_{\mathbf{k}, j}(t) \tag{50}
\end{equation*}
$$

for $g:=\left\{g_{\mathbf{k}, i}\right\}_{\mathbf{k} \in \mathbb{K}, i \in \mathbb{I}} \in \ell^{2}(\mathbb{K} \times \mathbb{I})$, and $(G(\mathbf{k}))_{i j}=\left\langle\varphi_{\mathbf{k}, j}, \varphi_{\mathbf{k}, i}\right\rangle_{\mathcal{H}}$.
An important observation is that the regularized solution decomposes into a sum of solutions to smaller moment problems, one for each wave vector $\mathbf{k}$. For practical computer implementation this has the advantage of reduced data storage requirements. Also it opens up the possibility of parallel computation.

### 3.3 Results

We have performed reconstructions of synthetic images ('chest phantoms'), as well as of real MRI data. The chest phantom is defined in terms of several (solid) ellipses, some of which are changing as a function of time, see Fig. 8. In the case of synthetic images, we first generate times $R_{k}$ of R-waves, measurement times $t_{\mathbf{k}, i}$ and corresponding Fourier coefficients $g_{\mathbf{k}, i}$, which then serve as input to the reconstruction procedures (in the case of real data these numbers are produced by the MRI scanner). The times $R_{k}$ of Rwaves are chosen as random samples from a uniform distribution on the interval $\left[\bar{T}_{R R}(1-\epsilon), \bar{T}_{R R}(1+\epsilon)\right]$. Here $\epsilon$ is the crucial model parameter determining the relative variation of the interval times of the 'phantom heart'. The case $\epsilon=0$ corresponds to a perfectly regular heartbeat. Measurement times are defined as $\tau_{i}\left(k_{x}, k_{y}\right)=\left(k_{y} N_{p r}+i\right) T_{r e p}+k_{x} \delta t$, where $\left(k_{x}, k_{y}\right) \in \mathbb{K}, i \in \mathbb{I}$ and where $N_{p r}$ is the number of profiles, $T_{\text {rep }}$ the repetition time, and $\delta t$ the time between successive samples in a profile. Finally, Fourier coefficients are generated as follows. Let a single heartbeat of the chest phantom be denoted by $f(\mathbf{r}, t)$, where $\mathbf{r} \in D, t \in[0,1]$. We extend this function to a larger interval $\left[0, R_{J}\right) \subset$ operator by 'inverting' the time-to-phase conversion of Section 2.2. That is, given the sequence of times $R_{k}$ of R-waves, $0=R_{1}<R_{2}<R_{3}<\ldots<R_{J}$, we define $F: D \times$ operator $\rightarrow$ operator by

$$
F(\mathbf{r}, \tau):=f(\mathbf{r}, t(\tau)), \quad \text { for } \tau \in\left[R_{j}, R_{j+1}\right)
$$

where

$$
t(\tau)=\frac{\tau-R_{j}}{R_{j+1}-R_{j}}
$$

From (3.3) the Fourier coefficients of this function $F$ with respect to the spatial parameters $\mathbf{r}=(x, y)$, are given by $\widehat{F}\left(\mathbf{k}, \tau_{i}(\mathbf{k})\right)=\widehat{f}\left(\mathbf{k}, t\left(\tau_{i}(\mathbf{k})\right)\right)$. Since $f$ is the known synthetic image, the corresponding Fourier coefficient can be computed as soon as the rescaled times $t\left(\tau_{i}(\mathbf{k})\right)$ have been determined. This has to be done for all $\mathbf{k} \in \mathbb{K}, i \in \mathbb{I}$.

The reconstructions are performed by first interpolating in the time-domain for each fixed Fourier frequency $\mathbf{k}$, followed by Fourier inversion at a number $\Phi_{\mathrm{m}}$ of equidistant phases $\phi_{i}=i / \Phi_{\mathrm{m}}, \quad i=$ $0,1, \ldots, \Phi_{\mathrm{m}}-1$. For the time-interpolation first or third order spline interpolation and sinc interpolation
are used, referred to as order 1 reconstruction, sinc reconstruction etc. We also implemented the original technique of Bohning [8], referred to as order 0 reconstruction. Here the interpolated value at phase $\phi_{i}$ is taken to be the average of the data in the interval $\left[\phi_{i}, \phi_{i+1}\right)$, where a value zero is assigned if no data fall in this interval. This method leads to severe aliasing artefacts if the number of profiles is small. Also regularized sinc-interpolation by the Tychonov-Phillips method was used (referred to as reg.sinc below) with a value $\gamma=0.01$ for the regularization parameter.

Synthetic data A relative variation of $25 \%$ in the length of the (simulated) heart beats was taken. The reconstructed images are of size $128 \times 128$ and whereas the value $\Phi_{\mathrm{m}}=8$ was used.

To test stability the measurement times were perturbed by adding a uniform perturbation in the interval $[-0.08,0.08]$. The reconstructions corresponding to the perturbed measurement times are shown in Fig. 9 for $N_{p r}=15$. In each case the first row displays order 0 reconstruction, the second row order 1 , the third one order 3 , the fourth one sinc-reconstruction and the fifth regularized sinc-reconstruction. In each row we show phases 0,3 and 6 from left to right.

By comparing these reconstructions to the originals at the corresponding phases in Fig. 8, one observes that the sinc, order 1 and order 3 reconstruction don't behave well under perturbation of the measurement times. Also small white 'clouds' appear in order $0-3$ reconstruction. The sinc-reconstruction is very bad, but (except for the initial phase) the regularized sinc-reconstruction is the best of all reconstruction algorithms. For a quantitative comparison (using the $L^{2}$ error norm), see [68], where similar comparisons have been made for unperturbed data and for perturbations of the measured Fourier coefficients.

Real MRI data Reconstructions were made from real MRI data obtained from Philips Medical Systems Division (Best, The Netherlands). The parameters in the reconstructions are: $k_{x}^{\mathrm{m}}=k_{y}^{\mathrm{m}}=128, N_{p r}=50$. Results, at phase 2 of the heart cycle, are shown in Fig. 10. The two ellipse-like structures on the left and the right side are cross sections of the arms. The big circular part is the bone of the chest. In the picture the heart is located at the top. The grey part of the heart is the muscle tissue and the lighter parts are the heart-chambers.

Important is how well the heart muscle (in particular, its boundary) is reconstructed, and how little noise is contained in those parts of the image which contain moving heart structures. In the order 0 reconstruction the contours of the heart muscle are somewhat vague. The regularized sinc-reconstruction behaves better than sinc reconstruction, but in this case is not a real improvement compared to order 0 reconstruction. This may be due to the fairly large number of profiles used, which tends to diminish the differences between the various interpolation methods. For a fuller discussion we refer to [68].


Figure 9: Reconstructions from perturbed time markers at phases 0, 3 and 6, from left to right. ( $N_{p r}=15$ ). First row: order 0; second row: order 1; third row: order 3; fourth row: sinc; fifth row: reg.sinc.


Figure 10: Reconstructions of $M R$ images at phase 2. Top left: order 0; Top right: order 1; Middle left: order 3; Middle right: sinc; Bottom: reg.sinc.

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[^0]:    *Original version in: Nieuw Archief voor Wiskunde 10 (3), November 1992, pp. 277-308.

[^1]:    ${ }^{1}$ Reprinted in [71]; for an English translation, see [20] or [37].
    ${ }^{2}$ For an elementary derivation see [64].

[^2]:    ${ }^{3}$ A similar classification exists for inverse problems in Potential Theory.

[^3]:    ${ }^{4}$ This is the method employed by the earliest CT scanners developed by Hounsfield [47].

